Pion Updates

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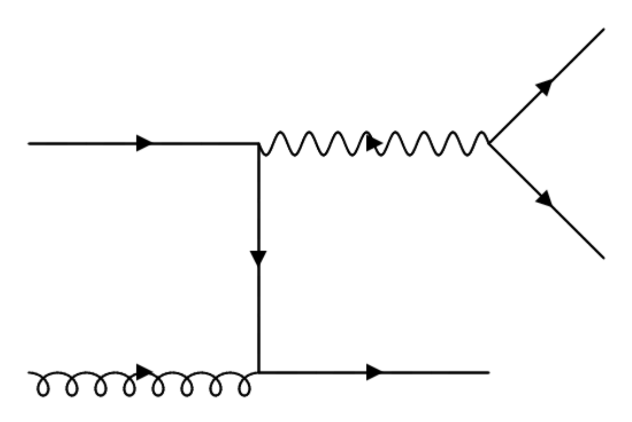
Group Meeting 4/24/2020

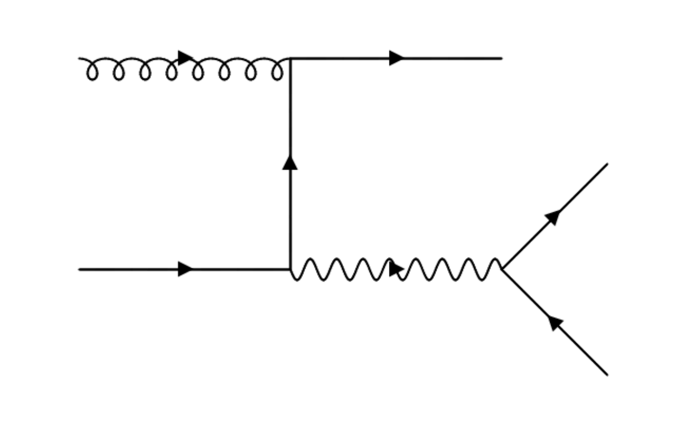
Outline

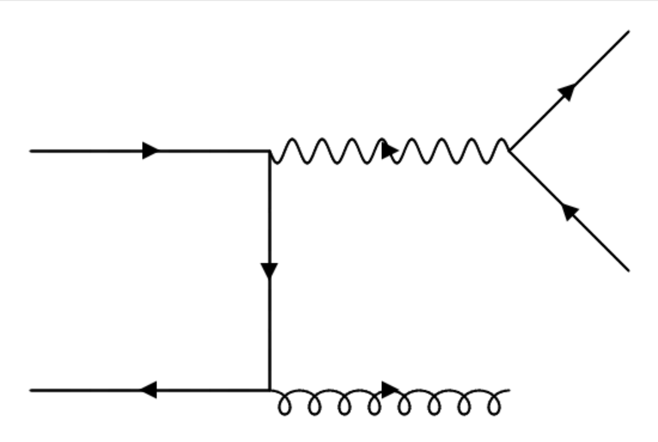
1. p_T-dependent DY review
2. Stability of PDFs
3. Pion vs Proton Structure
4. TMDs

p_T-dependent DY

Leading Order Diagrams







Drell-Yan (DY)

- p_T dependent DY

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \sum_{ab} \int_{x_{a,\min}}^{1} dx_a \frac{x_a x_b}{x_a - 1} f_{a/A}(x_a, \mu^2) f_{b/B}(x_b, \mu^2) \frac{d\hat{\sigma}_{ab}}{dQ^2 d\hat{t}}$$

Here, y is the rapidity, Q^2 is the invariant mass squared of the virtual photon, q_T is the transverse momentum of the virtual photon

Ambiguity of Scale

- In Collinear Factorization, one needs a hard scale that is $\mu \gg \Lambda$, where μ is a hard, partonic scale, and Λ is a scale associated with soft, non-perturbative physics
- In DIS, for instance, one hard scale exists, Q^2, which is the invariant mass of the virtual photon
- In DY, again, only one hard scale exists, Q^2
- However, in the q_T-dependent DY, two scales exist
  - The invariant mass of the dilepton pair, Q^2 is measured, but also the transverse momentum of the dilepton pair, p_T
  - Which scale is appropriate?

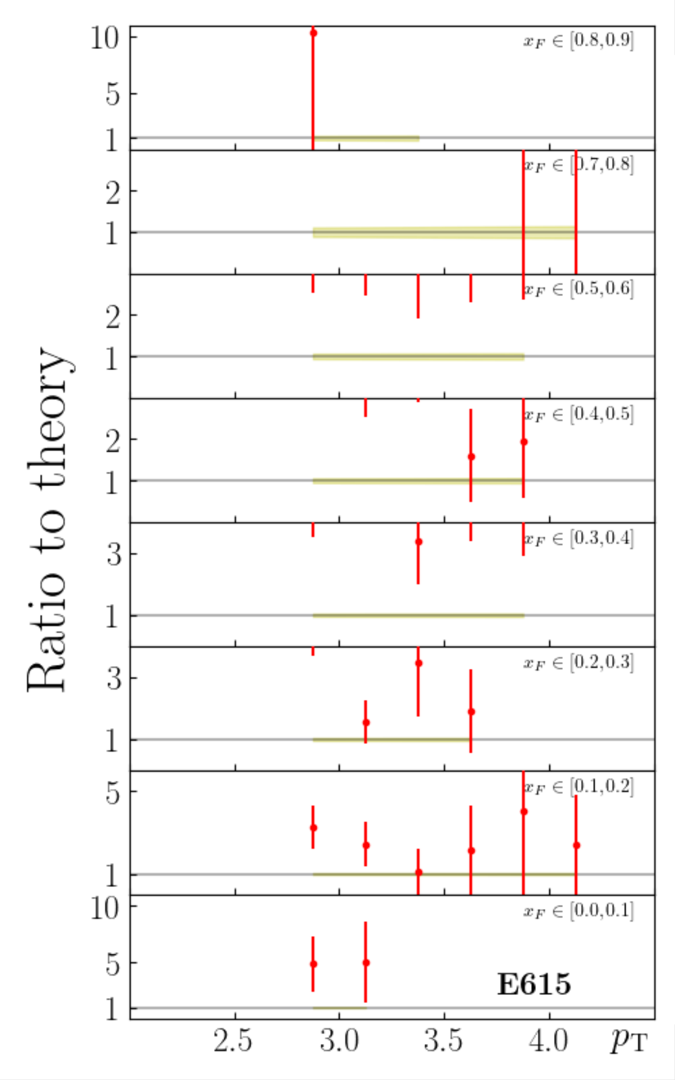
Exploration of Scale

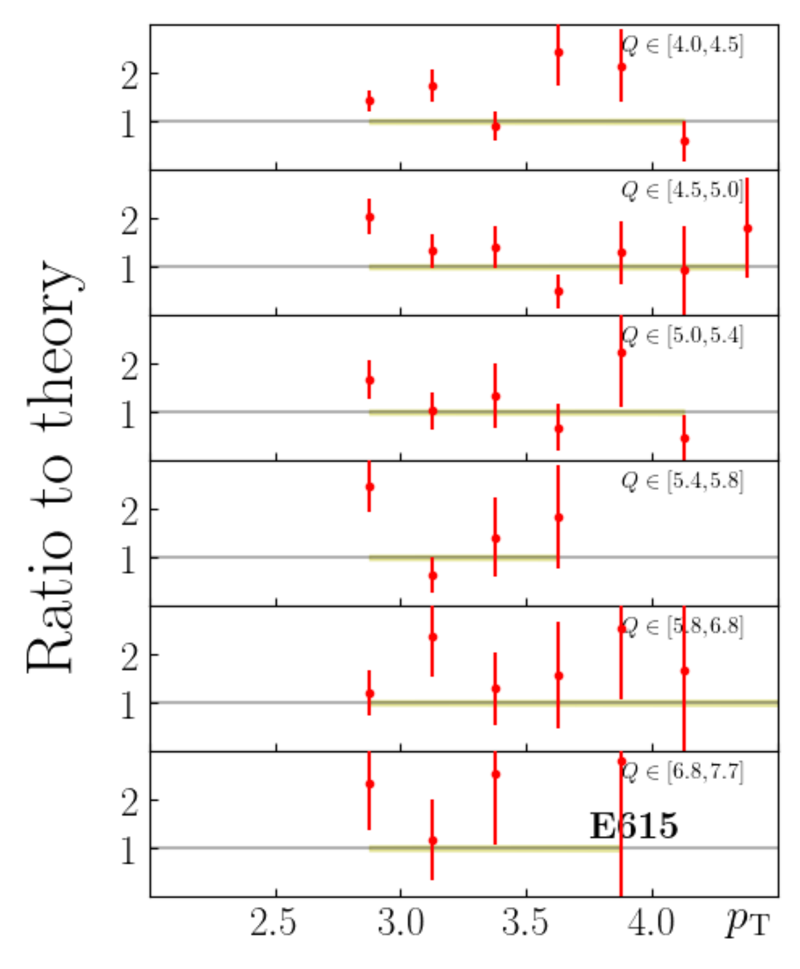
- We performed fits with $\mu = Q$, and had trouble fitting the q_T-dependent data

number of replicas = 250

summary of resultsFinal/step13a

prediction	reaction	idx	col	npts	chi2	chi2/npts	norm
False	ln	1000	H1	58	24.43	0.42	1.27
False	ln	2000	ZEUS	50	78.91	1.58	0.94
False	dy-pion	10001	E615	42	37.21	0.89	1.11
False	dy-pion	10002	NA10	34	20.57	0.61	0.91
False	dy-pion	10003	NA10	20	14.13	0.71	0.85
False	pion_qT	1001	E615	34	51.91	1.53	0.50
False	pion_qT	1002	E615	49	172.45	3.52	0.50
				287	399.61	1.39	





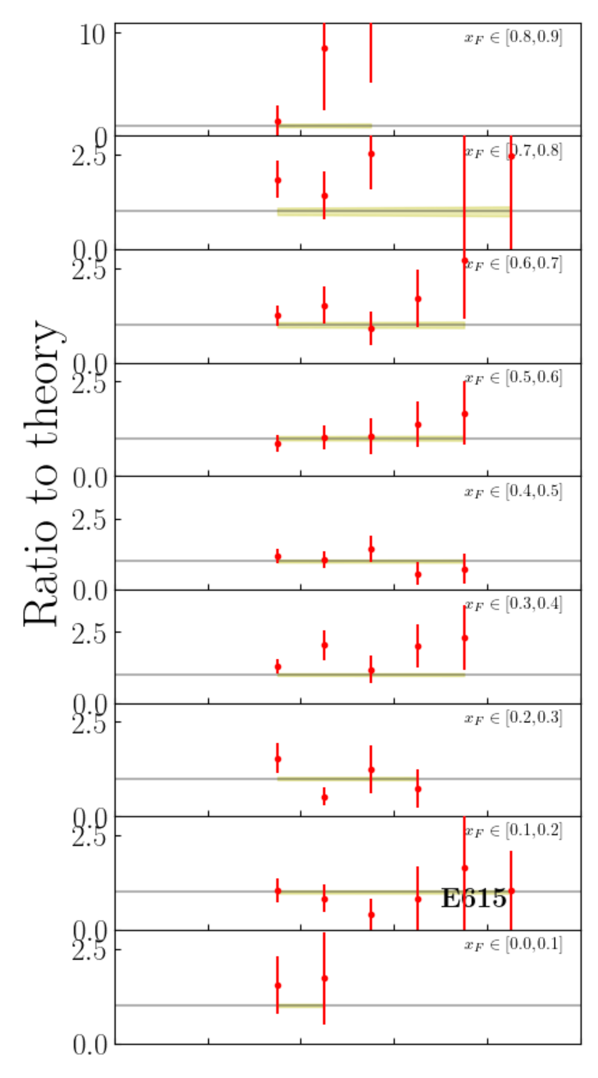
Exploration of Scale

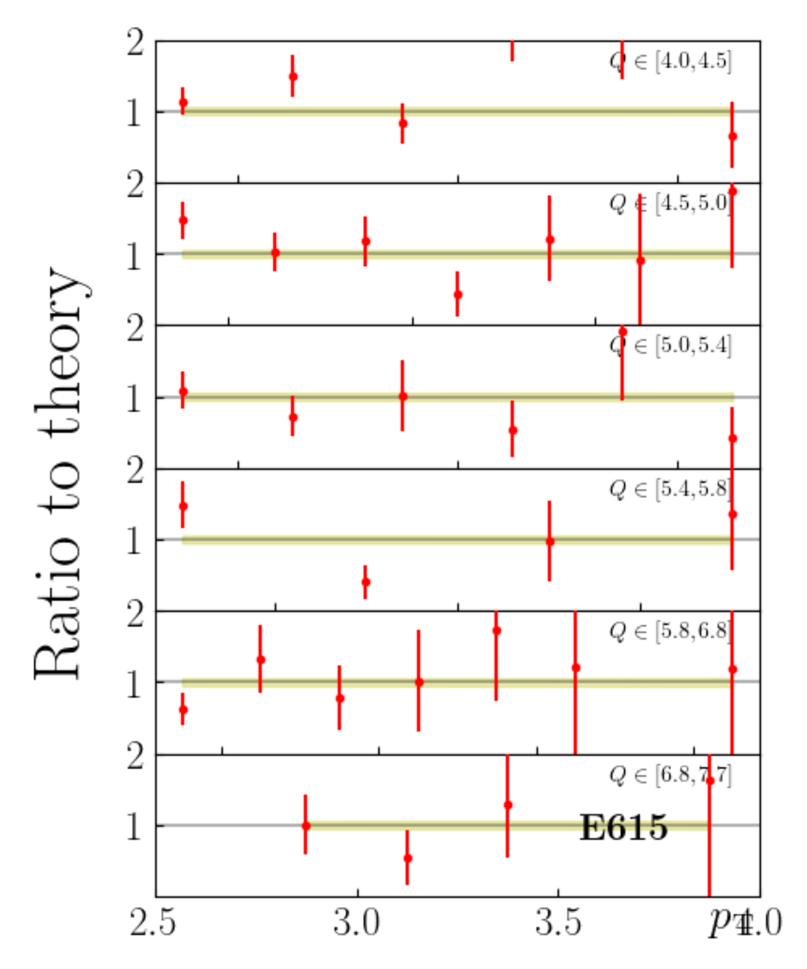
- We performed fits with $\mu = q_T/2$, and had much better success

number of replicas = 999

summary of resultsFinal/step13b

prediction	reaction	idx	col	npts	chi2	chi2/npts	norm
False	ln	1000	H1	58	22.52	0.39	1.26
False	ln	2000	ZEUS	50	73.22	1.46	0.95
False	dy-pion	10001	E615	42	37.03	0.88	1.05
False	dy-pion	10002	NA10	34	18.53	0.54	0.86
False	dy-pion	10003	NA10	20	14.11	0.71	0.80
False	pion_qT	1001	E615	34	37.86	1.11	1.11
False	pion_qT	1002	E615	49	40.49	0.83	0.51
				287	243.74	0.85	





Why $\boldsymbol{q_T/2}$?

- We see that the χ^2 for the q_T-dependent DY datasets are considerably lower for the choice $\mu = q_T/2$ than $\mu = Q$
- Recall that data is underpredicted by over a factor of 2 when $\mu = Q$
- Such a large normalization correction is unsettling and could point to the need for higher order terms
- However, when $\mu = q_T/2$, the normalization for the $\frac{d\sigma}{dQdq_T}$ data is within the reported normalization uncertainty ($\frac{d\sigma}{dQdq_T}$ still has norm=0.51)

Why $q_T/2$?

- Recall Dave Soper's lectures
- The full Δ does *not* depend on μ
- In a perturbative analysis, one wants to suppress higher order corrections to get closer to the full Δ
- A choice of $\mu = Q$ here eliminates the logs, but other constant terms may still be present
- Perhaps in our case, there exist higher order terms with logs of μ^2/q_T^2

The choice of scale

- Our example: $e^+e^- \to \gamma^* \to$ hadrons

$$\sigma_{\text{tot}} = \frac{4\pi\alpha^2}{Q^2}\left(\sum e_f^2\right)[1+\Delta]$$

$$\Delta(\mu) =$$

$$\frac{\alpha_s(\mu)}{\pi} + \left[1.4092 + 1.9167\ \log\left(\mu^2/Q^2\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$

$$+\left[-12.805 + 7.8179\ \log\left(\mu^2/Q^2\right) + 3.674\ \log^2\left(\mu^2/Q^2\right)\right]$$

$$\times\left(\frac{\alpha_s(\mu)}{\pi}\right)^3$$

$$+\cdots$$

Is q_T a safe scale?

- A scale should be related to some hard scale in the measurements in order to suppress $\mathcal{O}(\frac{\Lambda^2}{Q^2})$ terms and make factorization
- Following renormalization group equations, as in α_S, one should keep Lorentz invariance as much as possible
- Examples of Lorentz invariants are 4-momenta squared, or dot products of 4-momenta (like Q^2 or Mandelstam s, t, and u)
- However, q_T is dependent on reference frame!
  - If change from the hadron-hadron COM, transverse momentum takes on a different meaning
  - Looking at the photon rest frame, there is no transverse momentum!
- In the hadron-hadron COM frame, however, q_T^2 *is* an invariant quantity

Kinematics of q_T^2

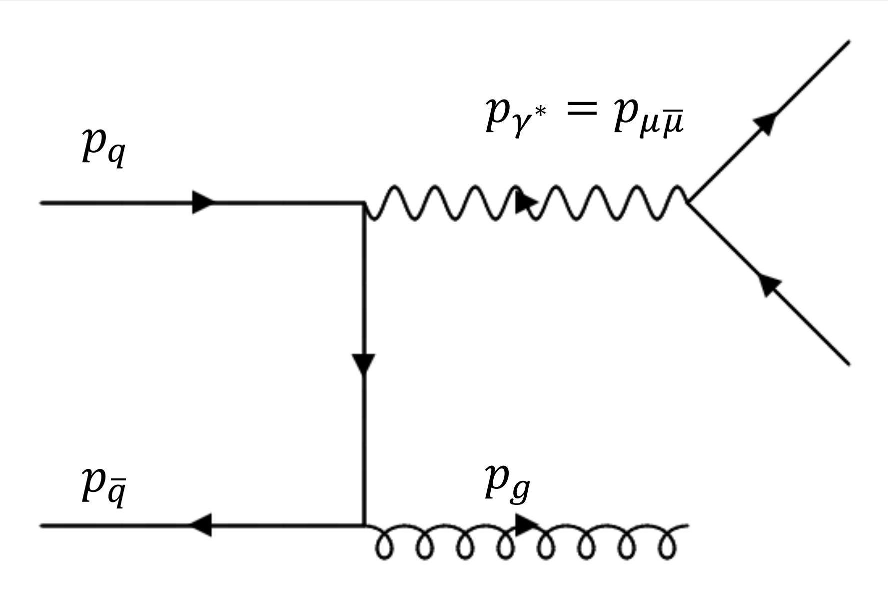
- We can describe all of momentum squared and dot products of the 4 particle momenta that are involved

$$p_a = x_a P_A = \left(x_a \frac{\sqrt{s}}{2}, \vec{0}_\perp, x_a \frac{\sqrt{s}}{2}\right)$$

$$p_b = x_b P_B = \left(x_b \frac{\sqrt{s}}{2}, \vec{0}_\perp, -x_b \frac{\sqrt{s}}{2}\right)$$

$$p_{\gamma^*} = p_{\mu\bar{\mu}} = (E, \vec{p}_T, p_L)$$

$$E = \sqrt{Q^2 + p_T^2 + p_L}$$



Kinematics of q_T^2

- We can describe all of momentum squared and dot products of the 4 particle momenta that are involved

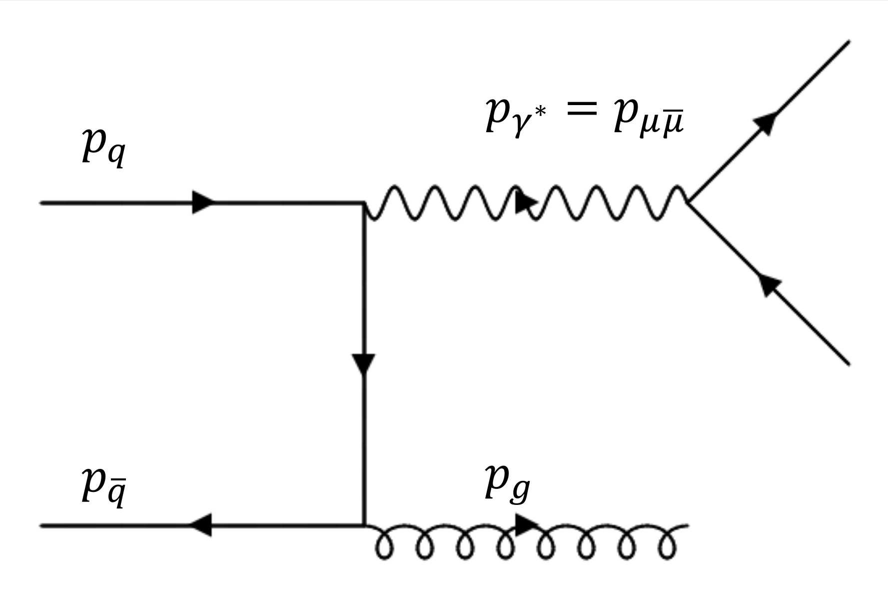
$$x_E = \frac{2E}{\sqrt{s}} = x_1 + x_2$$

$$x_T = \frac{2p_T}{\sqrt{s}}$$

$$x_L = x_F = \frac{2p_L}{\sqrt{s}} = x_1 - x_2$$

$$x_1 = -(u - Q^2)/s = \frac{1}{2}(x_T^2 + 4\tau)^{1/2} e^{Y}$$

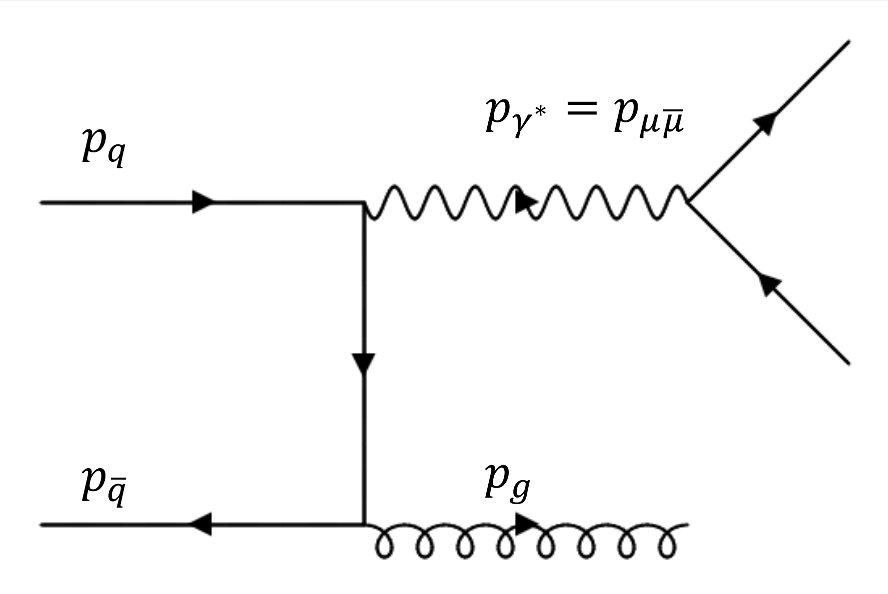
$$x_2 = -(t - Q^2)/s = \frac{1}{2}(x_T^2 + 4\tau)^{1/2} e^{-Y}$$



Kinematics of q_T^2

- We can describe all of momentum squared and dot products of the 4 particle momenta that are involved

$$
\begin{aligned}
p_g &= p_a + p_b - p_{\mu\bar{\mu}} \\
&= \left((x_a + x_b)\frac{\sqrt{s}}{2} - E, -\vec{p}_T, (x_a - x_b)\frac{\sqrt{s}}{2} - p_L\right) \\
&= \frac{\sqrt{s}}{2}(x_a + x_b - x_E, -\vec{x}_T, x_a - x_b - x_L) \\
&= \frac{\sqrt{s}}{2}(x_a + x_b - x_1 - x_2, -\vec{x}_T, x_a - x_b - x_1 + x_2)
\end{aligned}
$$



Invariant momenta and dot products

$$p_a^2 = p_b^2 = p_g^2 = 0$$

$$p_{\mu\bar{\mu}} = Q^2$$

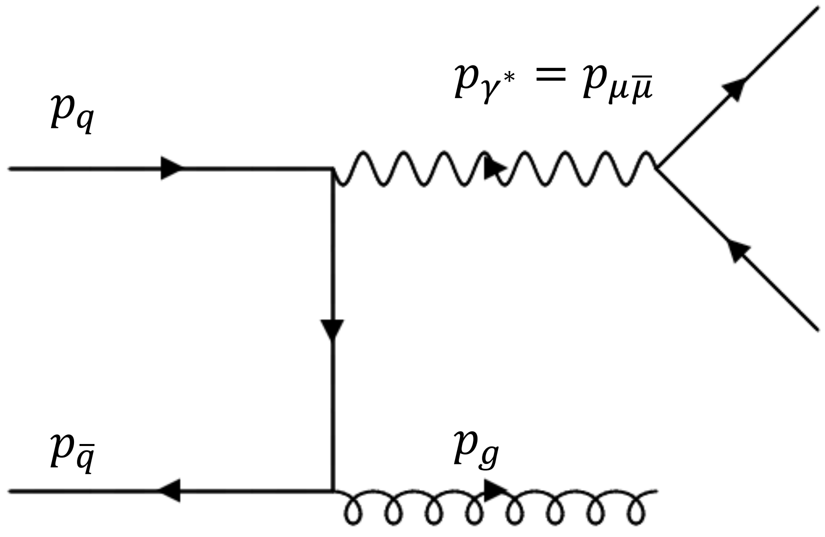
$$p_q \cdot p_{\bar{q}} = x_a x_b \frac{s}{2}$$

$$p_q \cdot p_{\mu\bar{\mu}} = x_a x_2 \frac{s}{2}$$

$$p_{\bar{q}} \cdot p_{\mu\bar{\mu}} = x_b x_1 \frac{s}{2}$$

$$p_q \cdot p_g = x_a(x_b - x_2)\frac{s}{2}$$

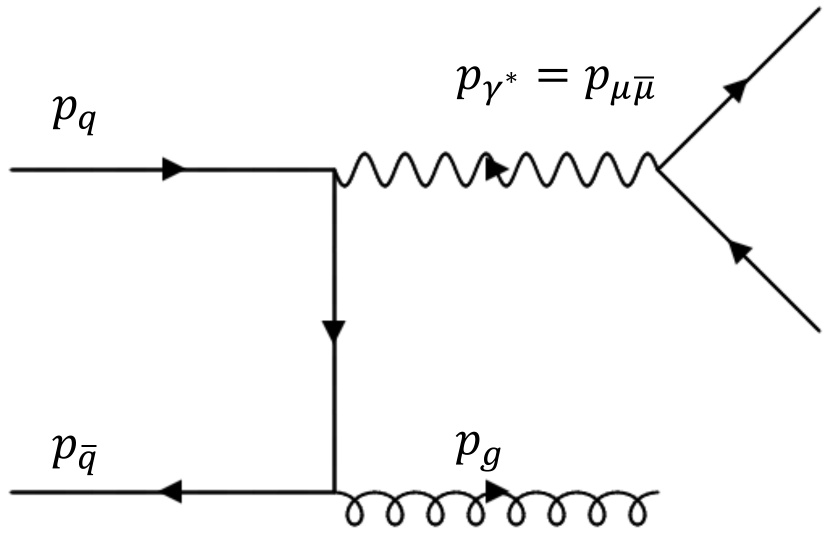
$$p_{\bar{q}} \cdot p_g = x_b(x_a - x_1)\frac{s}{2}$$



Invariant q_T^2

$$
\begin{aligned}
p_g \cdot p_{\mu\bar{\mu}} &= \frac{s}{4}(x_E, \vec{x}_T, x_L) \cdot (x_a + x_b - x_1 - x_2, -\vec{x}_T, x_a - x_b - x_1 + x_2) \\
&= \frac{s}{4}(x_a x_1 + x_b x_1 - x_1^2 - x_1 x_2 + x_a x_2 + x_b x_2 - x_1 x_2 - x_2^2 + x_T^2 \\
&- x_a x_1 + x_b x_1 + x_1^2 - x_1 x_2 + x_a x_2 - x_b x_2 - x_1 x_2 + x_2^2) \\
&= \frac{s}{2}(x_1(x_b - x_2) + x_2(x_a - x_1) + \frac{x_T^2}{2})
\end{aligned}
$$

A clue at q_T^2



Invariant q_T^2

$$(p_a + p_b)^2 = \hat{s} = (p_{\mu\bar{\mu}} + p_g)^2$$

$$\begin{aligned} p_{\mu\bar{\mu}} \cdot p_g &= \frac{\hat{s} - Q^2}{2} \\ &= \frac{s}{2}\Big[\frac{Q^2 - u}{s}\Big(\frac{Q^2 - \hat{u}}{Q^2 - u} - \frac{Q^2 - t}{s}\Big) + \frac{Q^2 - t}{s}\Big(\frac{Q^2 - \hat{t}}{Q^2 - t} - \frac{Q^2 - u}{s}\Big) + \frac{x_T^2}{2}\Big] \end{aligned}$$

$$\begin{aligned} p_T^2 &= \frac{x_T^2 s}{4} \\ &= \frac{\hat{s} - Q^2}{2} - \frac{s}{2}\Big[\frac{Q^2 - \hat{u}}{s} - \frac{(Q^2 - u)(Q^2 - t)}{s^2} + \frac{Q^2 - \hat{t}}{s} - \frac{(Q^2 - t)(Q^2 - u)}{s^2}\Big] \\ &= \frac{1}{2}\Big[(\hat{s} + \hat{t} + \hat{u} - Q^2) - 2Q^2 + \frac{2}{s}(Q^4 - uQ^2 - tQ^2 + ut)\Big] \\ &= \frac{1}{s}(-Q^2(s + u + t) + Q^4 + ut) \\ &= \frac{1}{s}(-Q^2(Q^2 + p_X^2) + Q^4 + ut) \\ &= \frac{ut - Q^2 p_X^2}{s} \end{aligned}$$

Assumed here are:

$\hat{s} + \hat{t} + \hat{u} = Q^2$

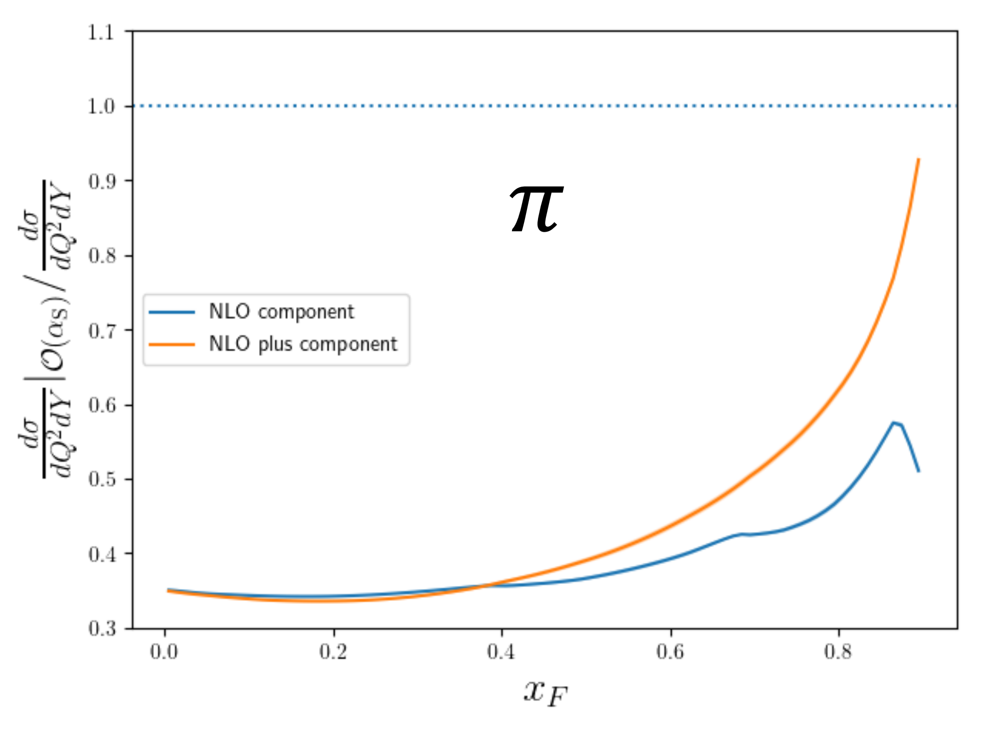
$s + t + u = Q^2 + p_X^2$

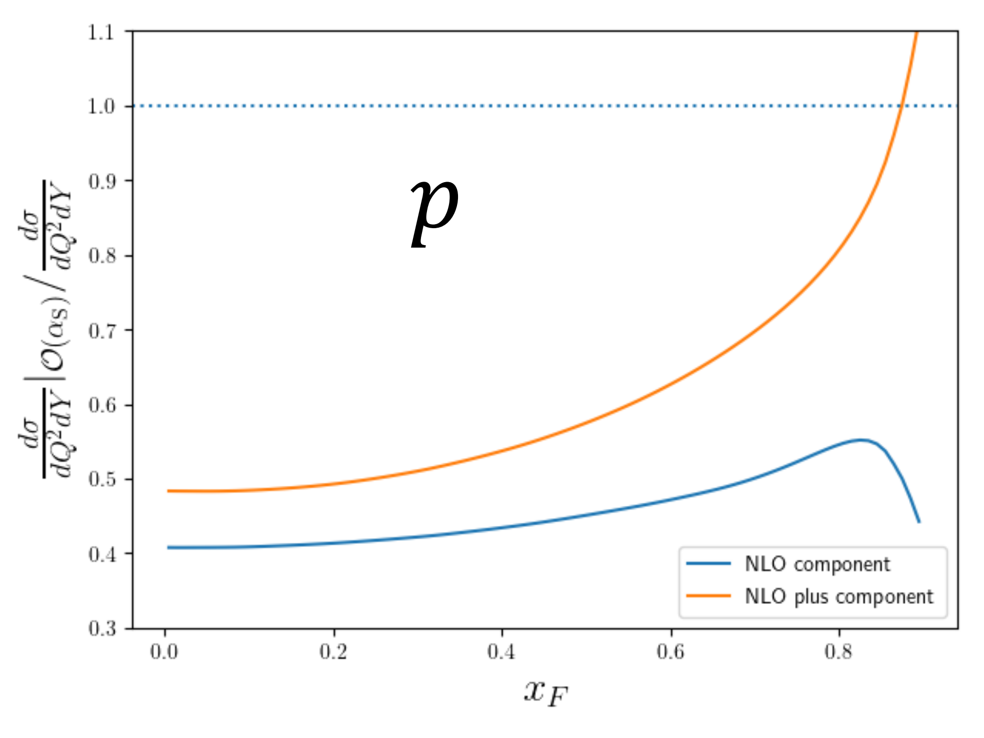
Form of p_T^2

- The result $p_T^2 = \frac{ut - Q^2 p_X^2}{s}$ can be connected with prompt photon and the exclusive process
- In prompt photon, the emitted photon in hadron-hadron collisions is real and measured, and thus $Q^2 = 0$
- In the exclusive process, no other hadrons are emitted, meaning $p_X^2 = 0$

A note on the x_F cut

- We choose the maximum $x_F = 0.6$ so that we don't run into a region where the $\mathcal{O}(\alpha_S)$ terms are a major contributor of the cross-section

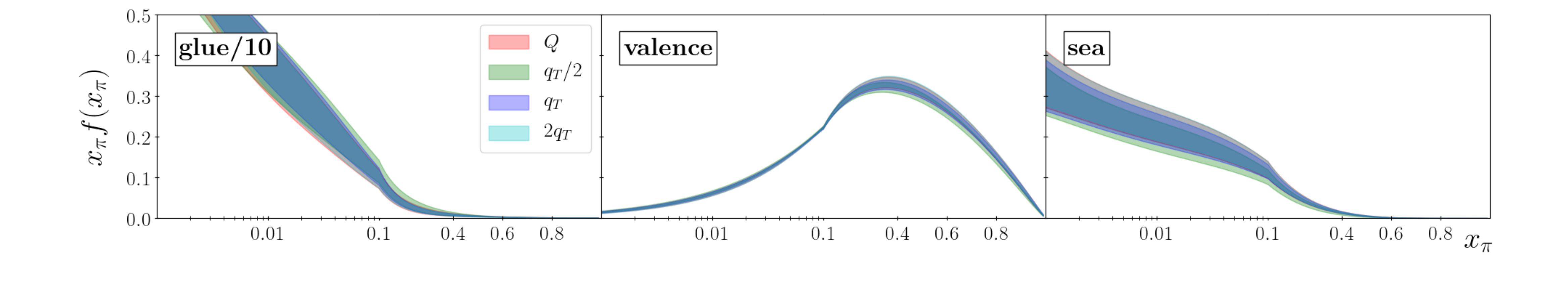




Stability of the PDFs

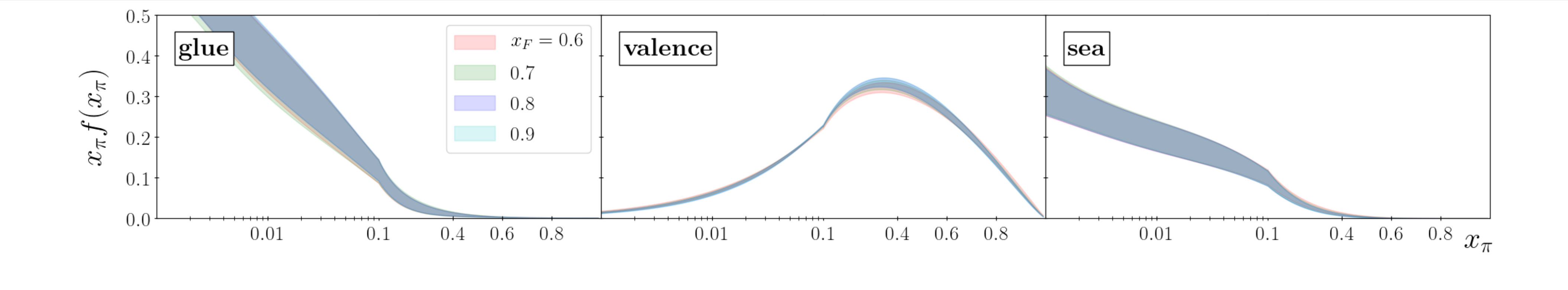
Stability with respect to scale

- We find that regardless of the scale dependence you use to fit the q_T-dependent DY data, the PDFs remain the same



Stability with respect to cuts on x_F

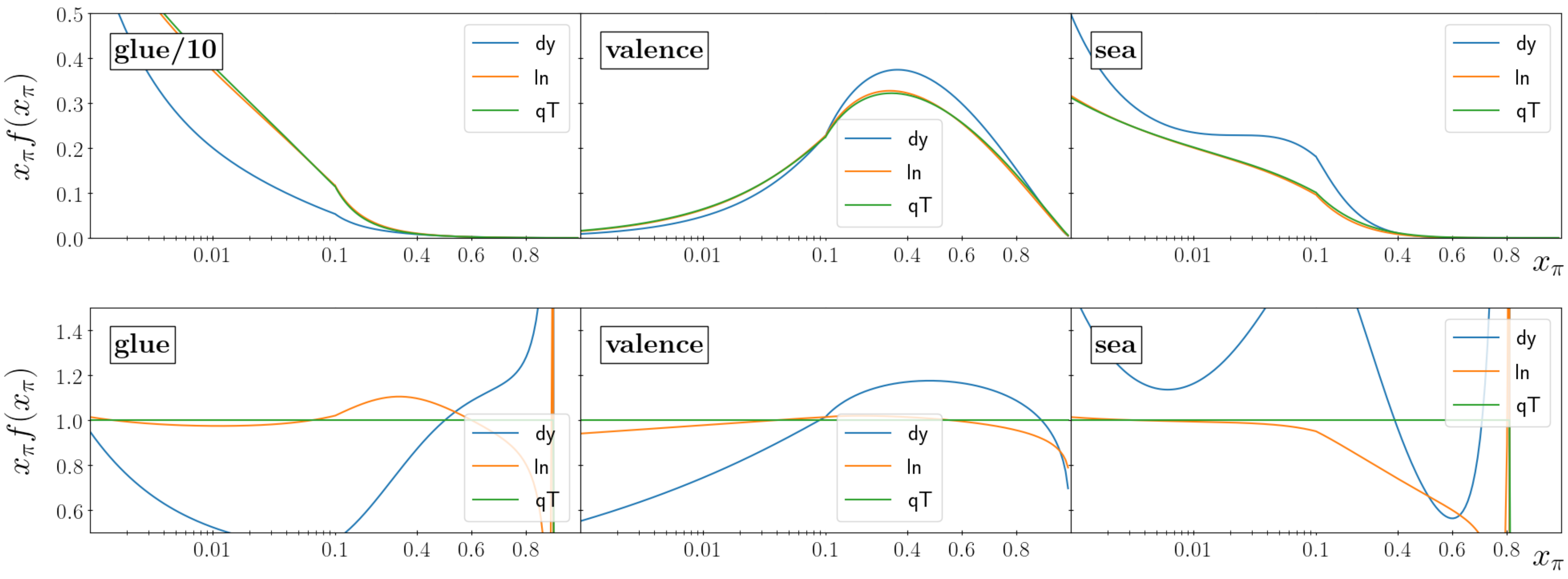
- Even though we want to avoid problematic regions in x_F, we can explore the stability of the PDFs



- That the PDFs don't change as a function of the cut on x_F is reassuring and we keep $x_{F,max} = 0.6$

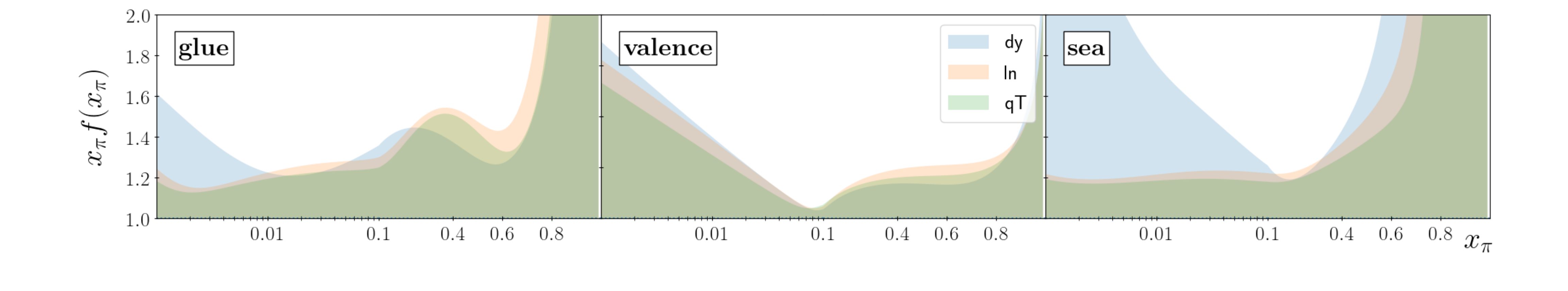
Does the q_T-dependent data affect the PDFs?

- We would like to see the impact of the q_T-data on the PDFs
- We see the central values change considerably with the inclusion of LN data, but when we add the q_T-data, barely anything changes



How about the uncertainties?

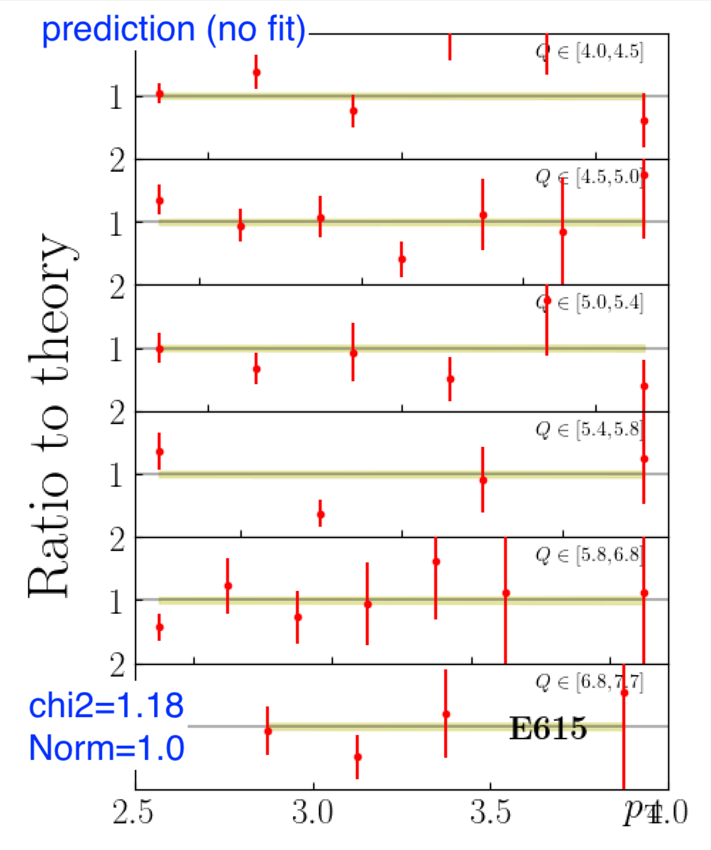
- Looking at the uncertainties of the PDFs, we can see how much of an effect the q_T-dependent DY has

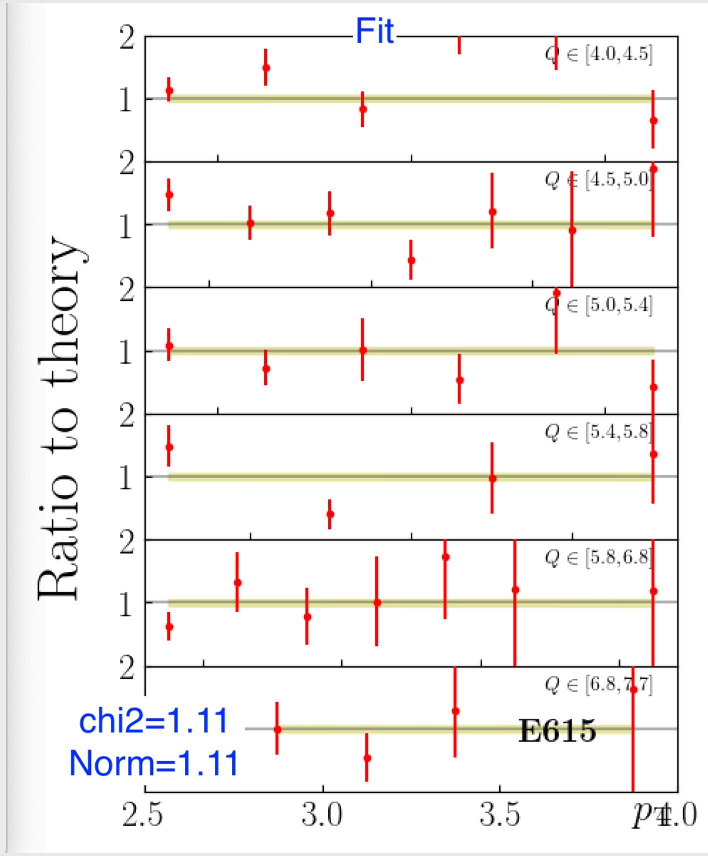


- Uncertainties don't change much either

If not much changes, how about making predictions?

- We can use the PDFs extracted from DY+LN data only to attempt to describe the q_T-dependent DY data
- The same cannot be said for the x_F-dependent data as the normalization for the fit is 0.51
- However, if we use 0.51 for the normalization for the prediction, we end up with a good description ($\chi^2 = 0.91$ vs $\chi^2 = 0.83$ for full fit)



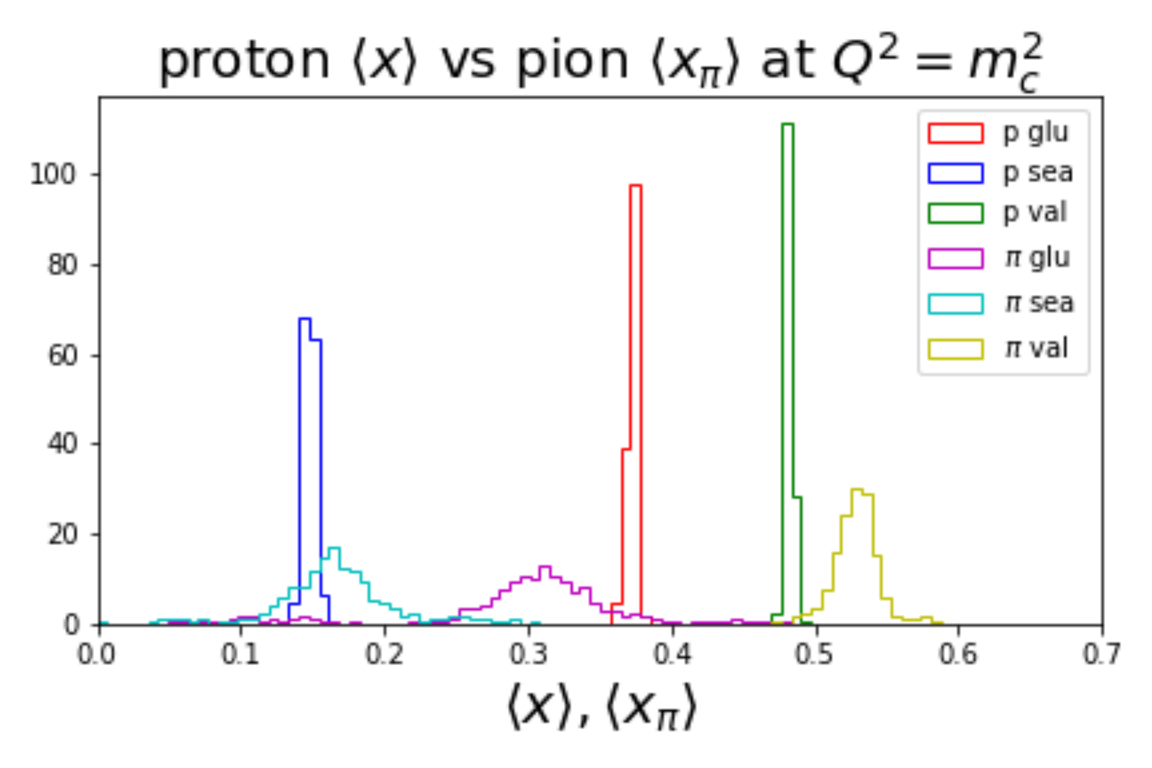


Pion vs Proton Structure

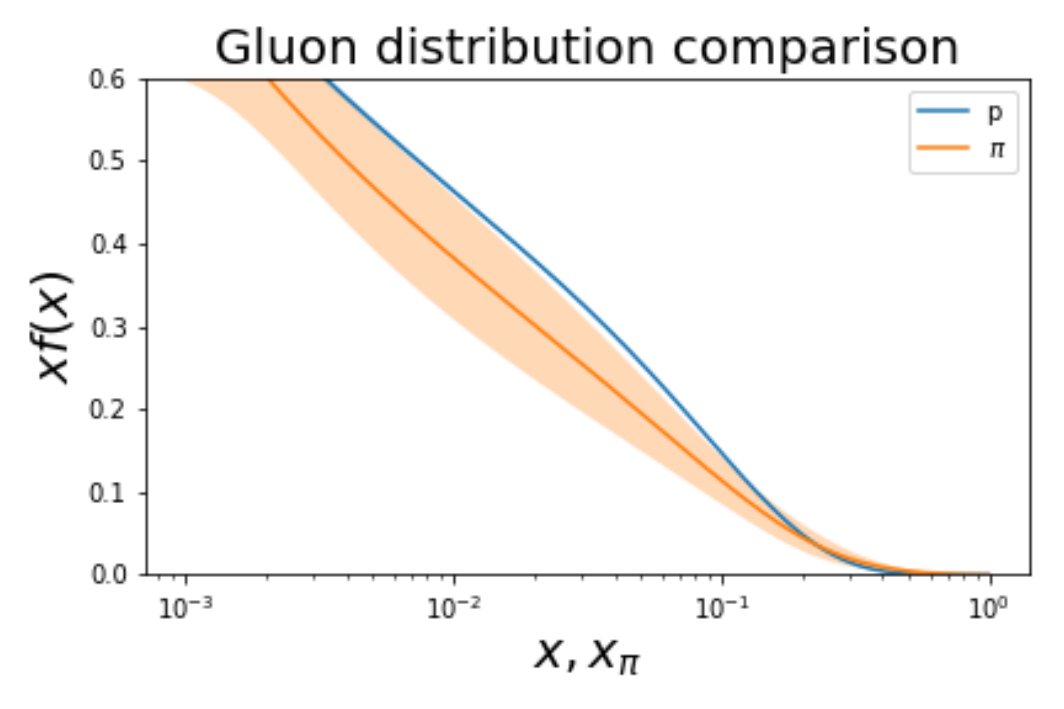
Consider the momentum fraction

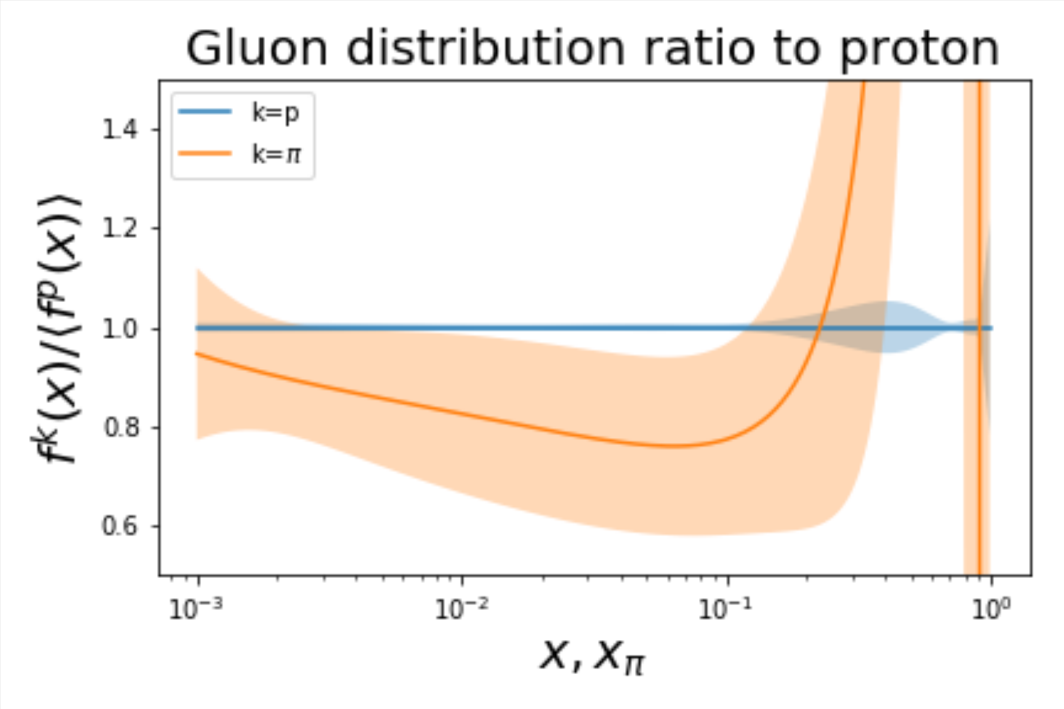
- How are the partons in the hadrons distributed?

Flavor	Pion $\langle x_\pi \rangle_{\text{flavor}}$	Proton $\langle x \rangle_{\text{flavor}}$
valence	0.528 ± 0.0161	0.481 ± 0.0026
sea	0.165 ± 0.0429	0.147 ± 0.0038
gluon	0.299 ± 0.0712	0.372 ± 0.0032



Gluon distributions





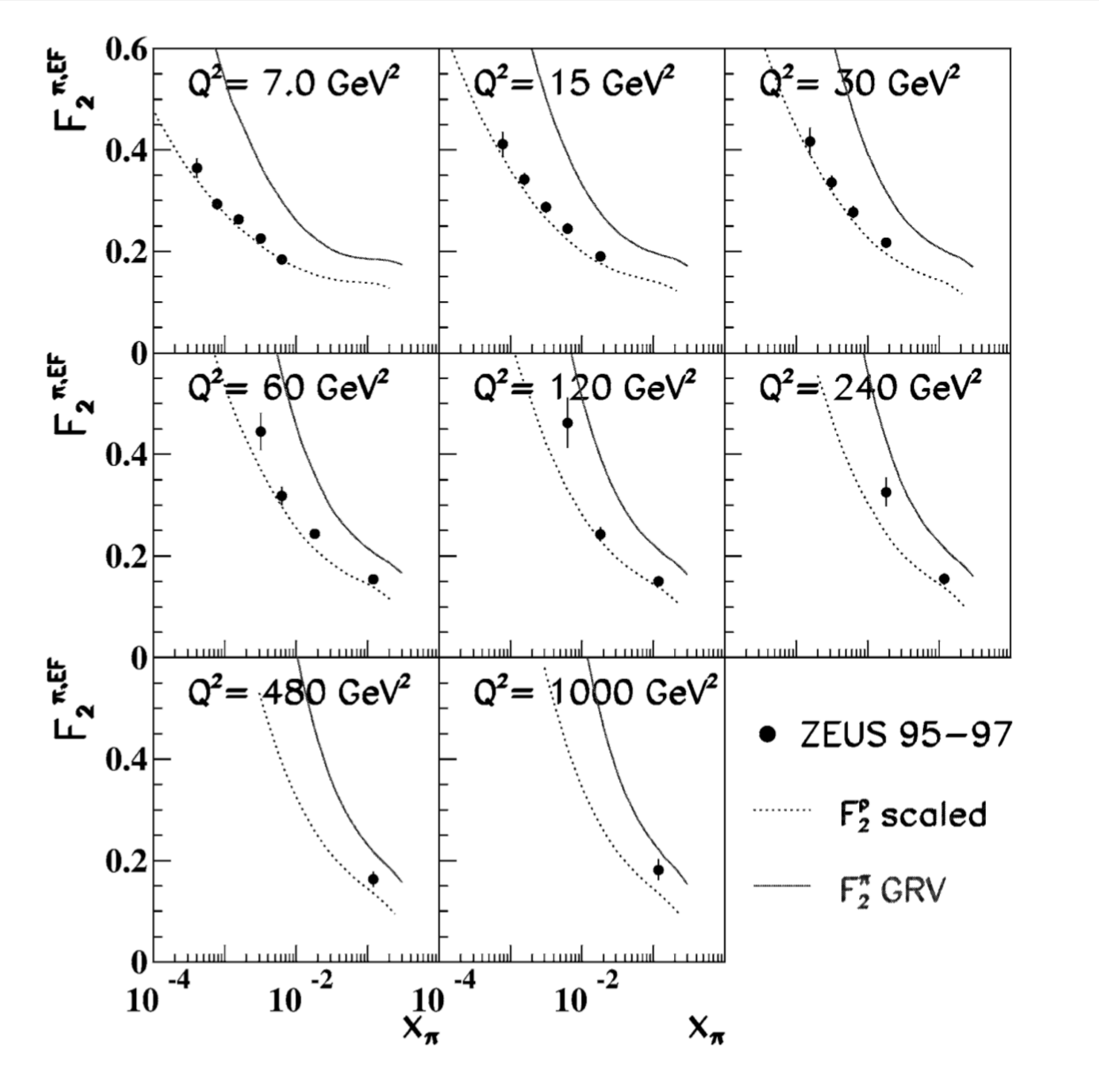
- This question was asked at my prelim!

High $\boldsymbol{x}$-region doesn’t agree

- The slope of the gluon is rather consistent whether it’s pion or proton
- The DGLAP equations of evolution is the same – why the generation of gluons in likely similar
- However, the large momentum fraction range isn’t good agreement

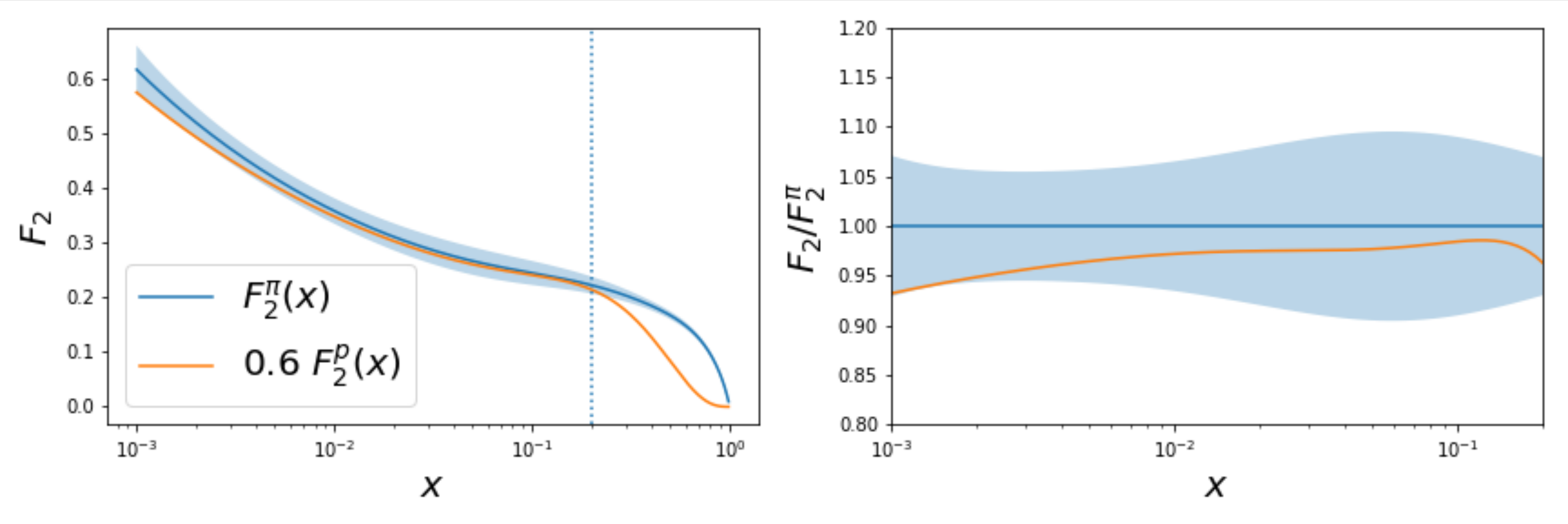
F_2 Structure Functions

- Specifically in the ZEUS paper, we can see some attempts to equate F_2^p with F_2^π
- Here, the GRV pion PDFs are used to construct F_2^π
  - This is problematic, because GRV only parameterizes the PDF using high-x_π data (DY and prompt photon)
- F_2^p is scaled by 0.361



The ZEUS method - normalization

- If we multiply the F_2^p (generated by JAM19 PDFs) by 0.6, we get the following plot



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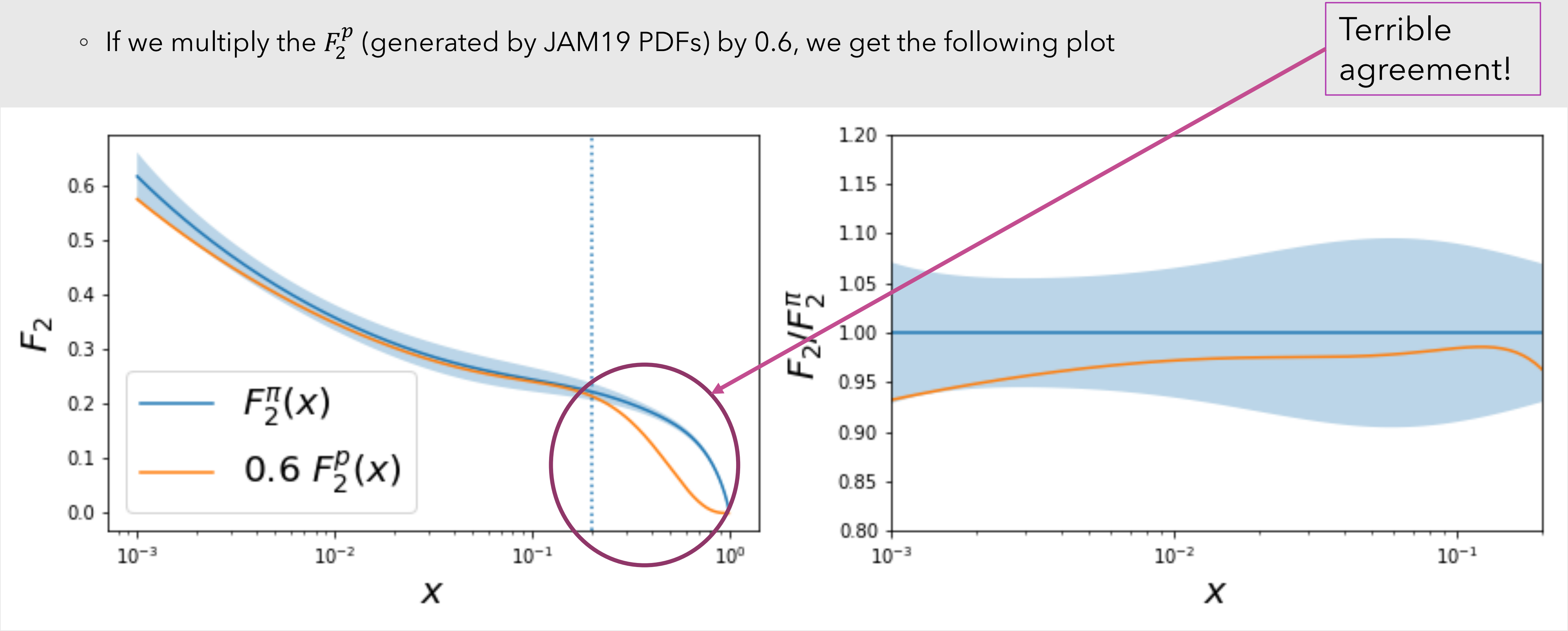
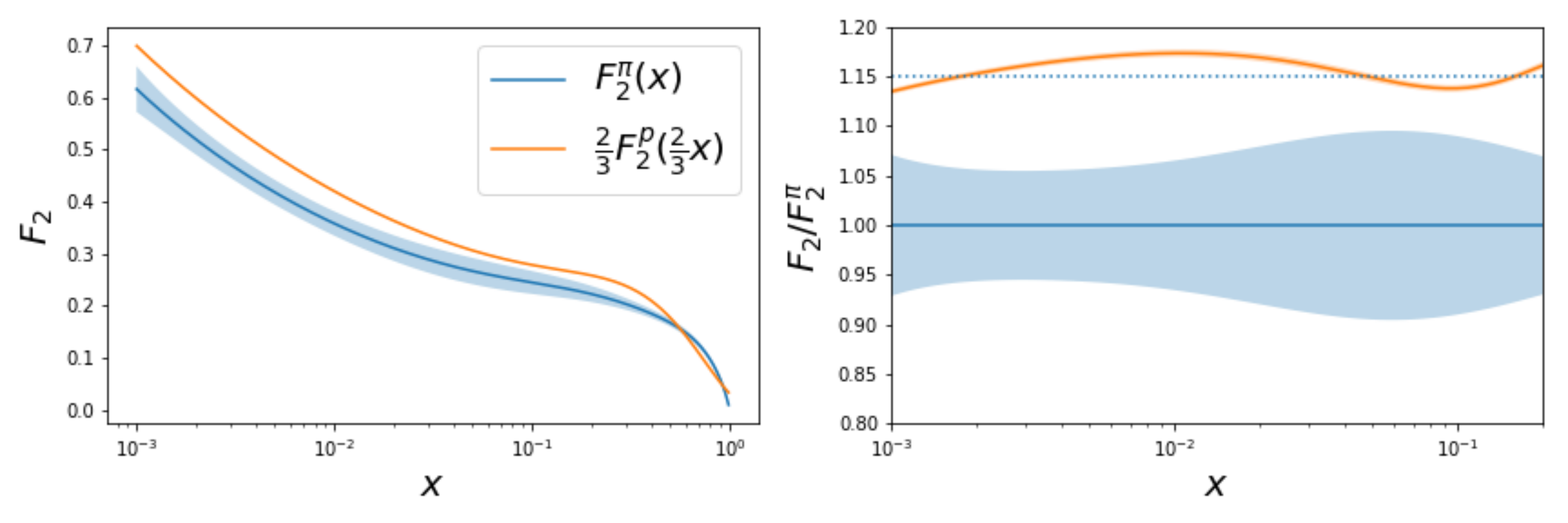
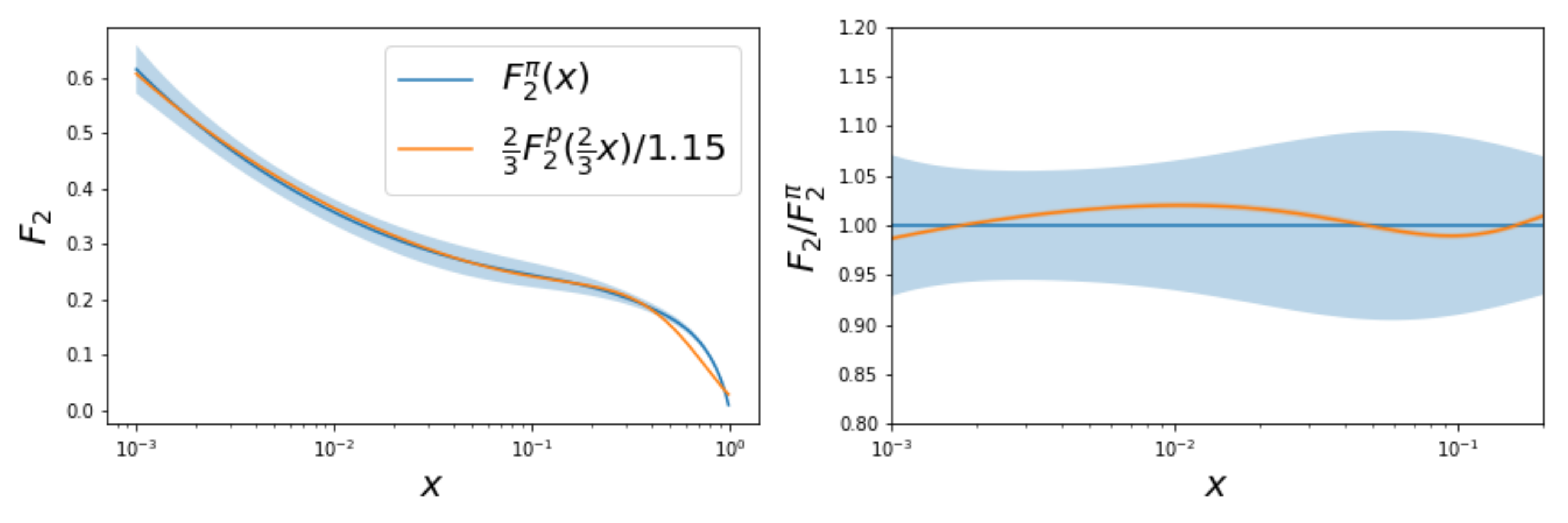


The Nikolaev, Speth, and Zoller (NSZ) method

- A parameterization of the proton's structure function in relation to the pion's structure function as $F_2^p(x, Q^2) = \frac{2}{3} F_2^\pi\left(\frac{2}{3}x, Q^2\right)$
- Used a **color-dipole BFKL-Regge expansion**



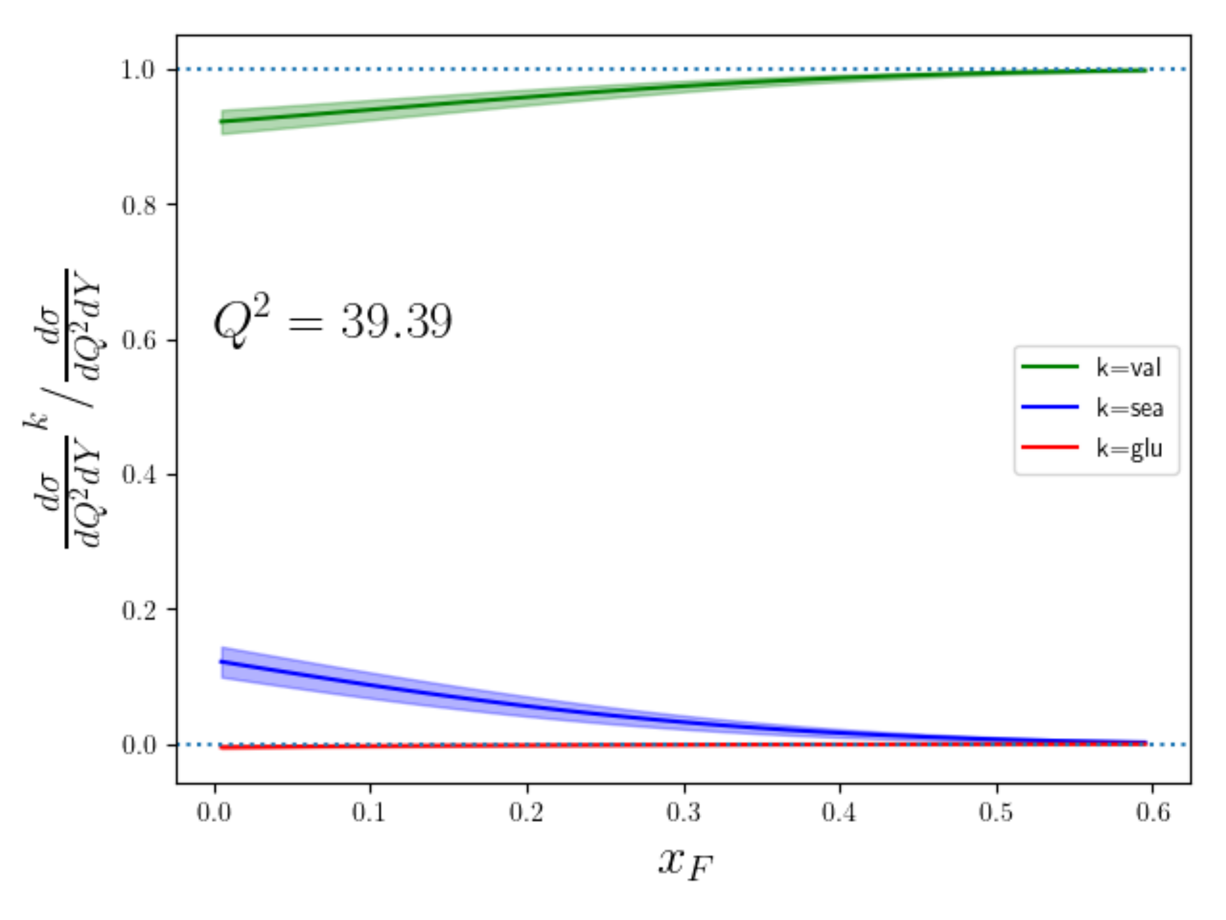
Normalized NSZ



Channel-by-channel

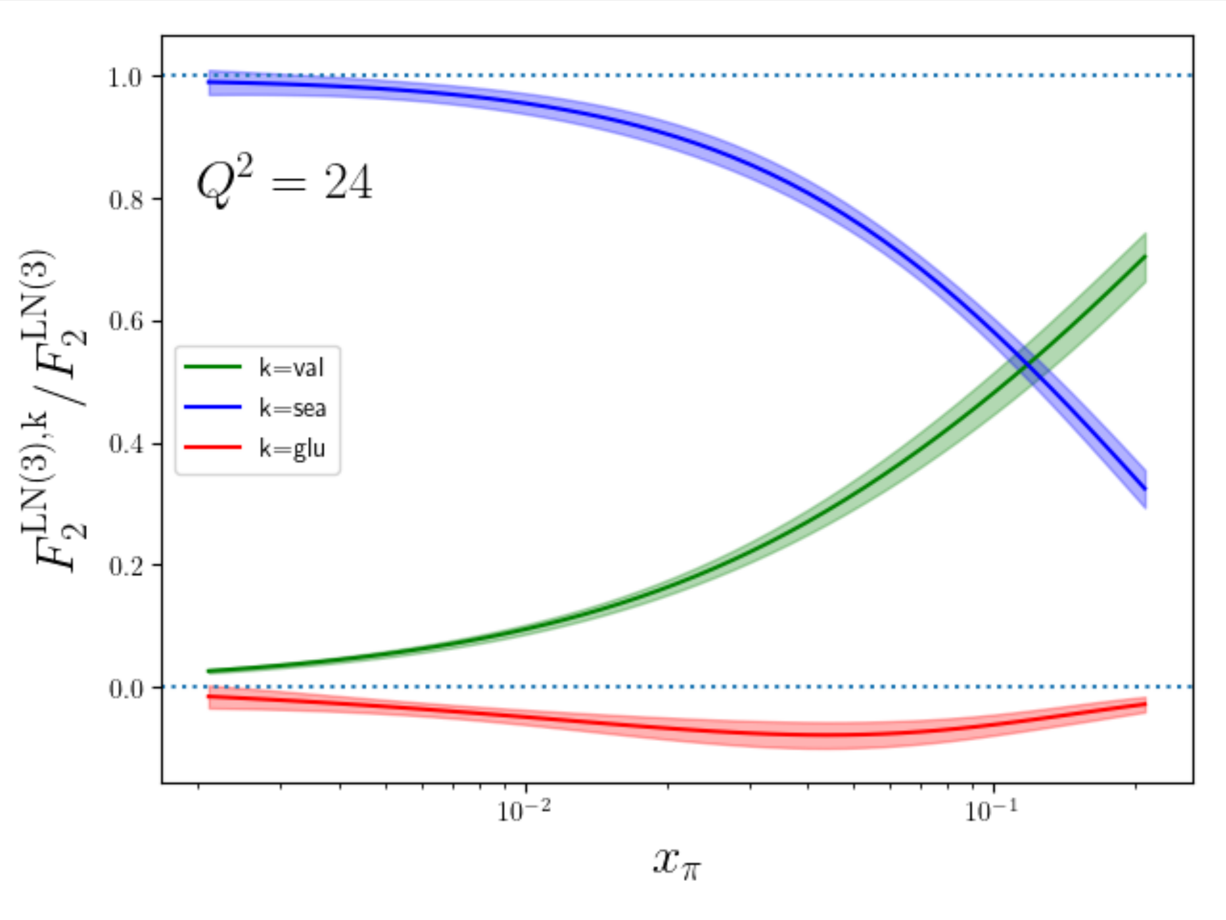
Channel-by-channel contributions - DY

- I show the contributions to the observable in terms of the degrees of freedom of the fit, *i.e.* the valence quark, sea quark, and gluon distributions
- Gluon is negligible



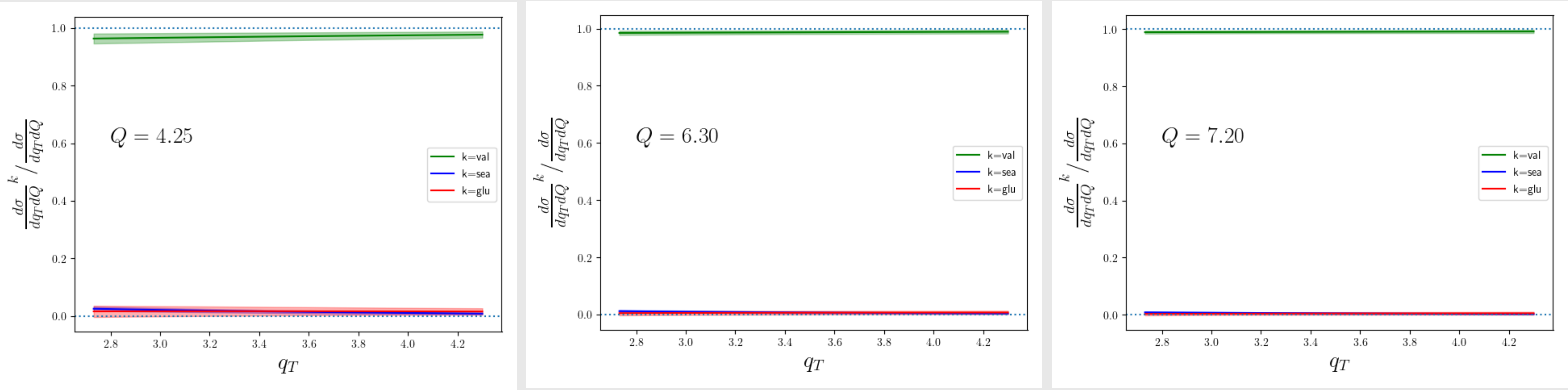
Channel-by-channel contribution - LN

- Sea quark and gluon distributions are much larger



Channel-by-channel contribution DY-q_T

- The hypothesis was that by including the q_T-dependent DY data, we constrain better the gluon in the large-x_π region



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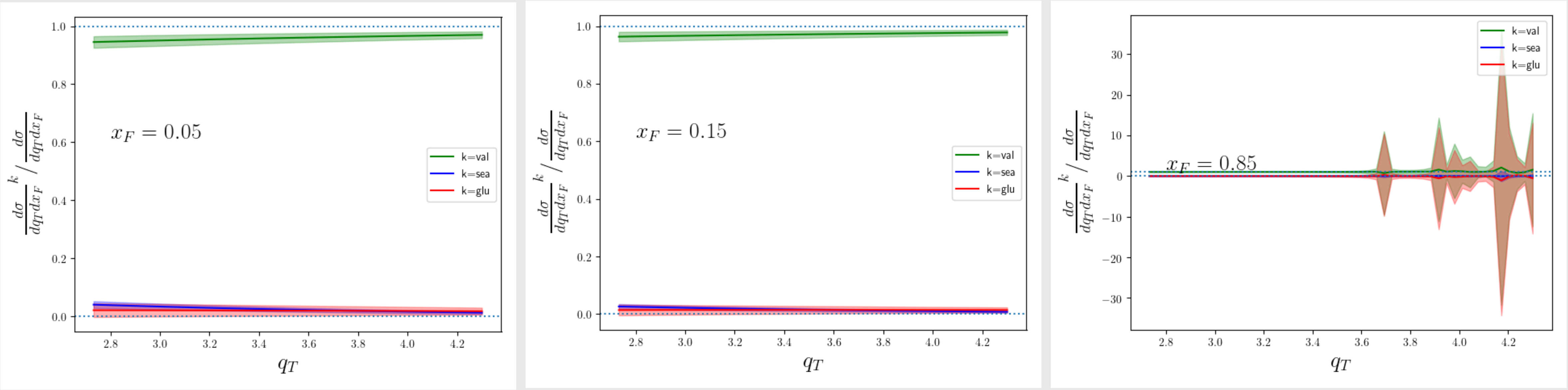
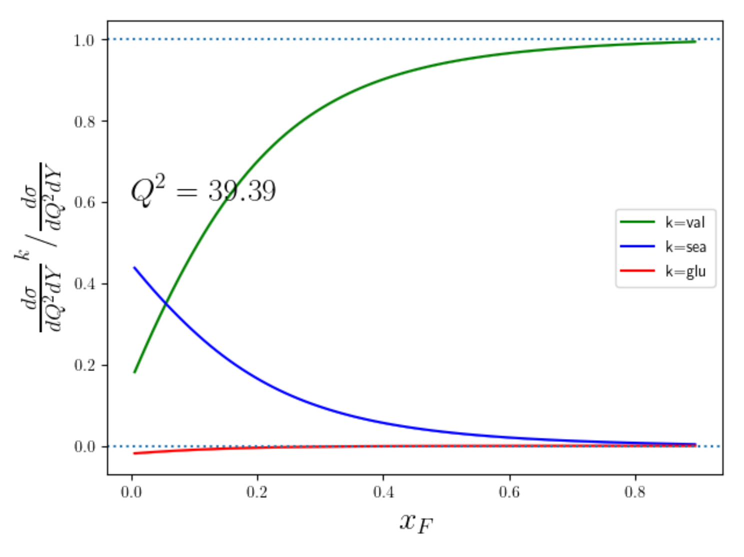
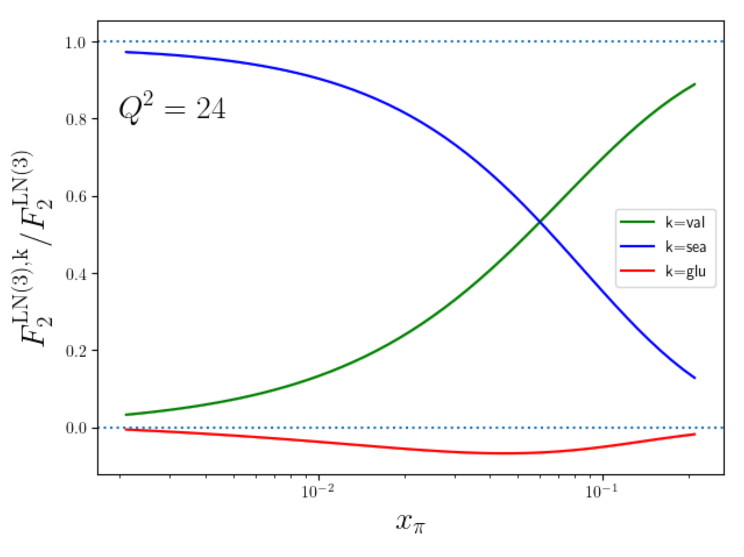
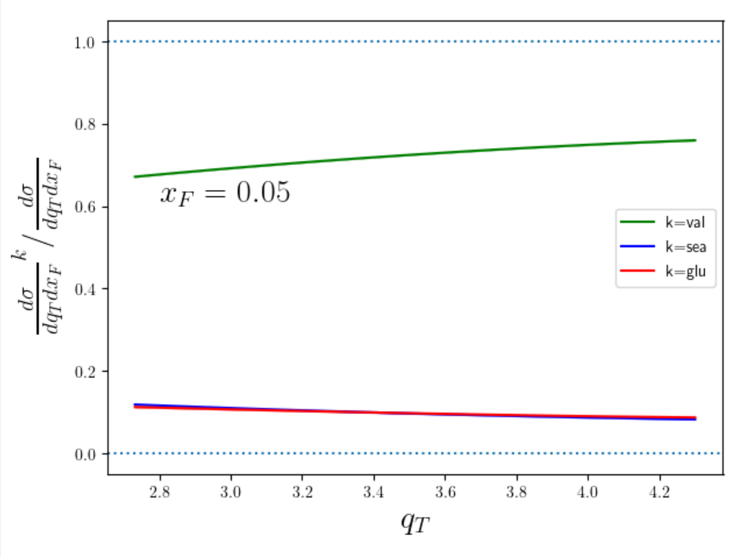


If we had $\boldsymbol{pW}$ data in the same way as $\boldsymbol{\pi W}$

- Can use kinematics and tungsten PDF to determine predictions for channel-by channel in the Proton





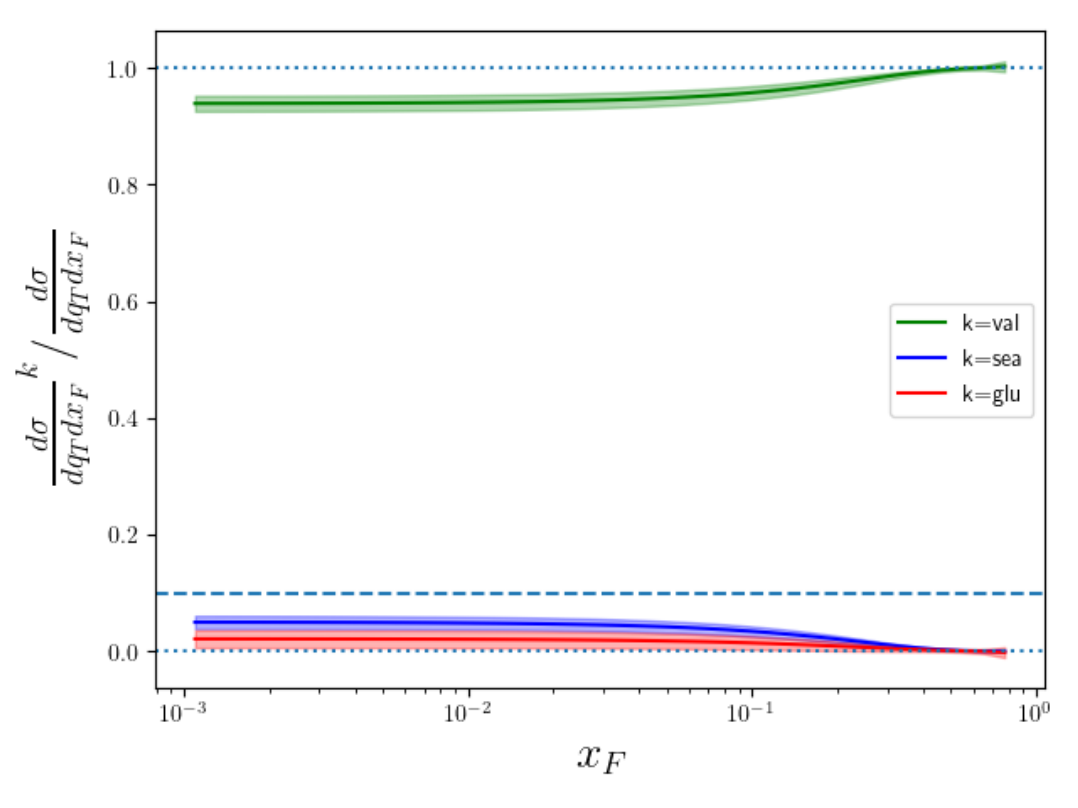


Is low-$\boldsymbol{x_F}$ suitable to constrain $\boldsymbol{\pi}$ gluon?

Recall $x_F = 2\sqrt{\tau}\sinh Y$, where Y is the rapidity.

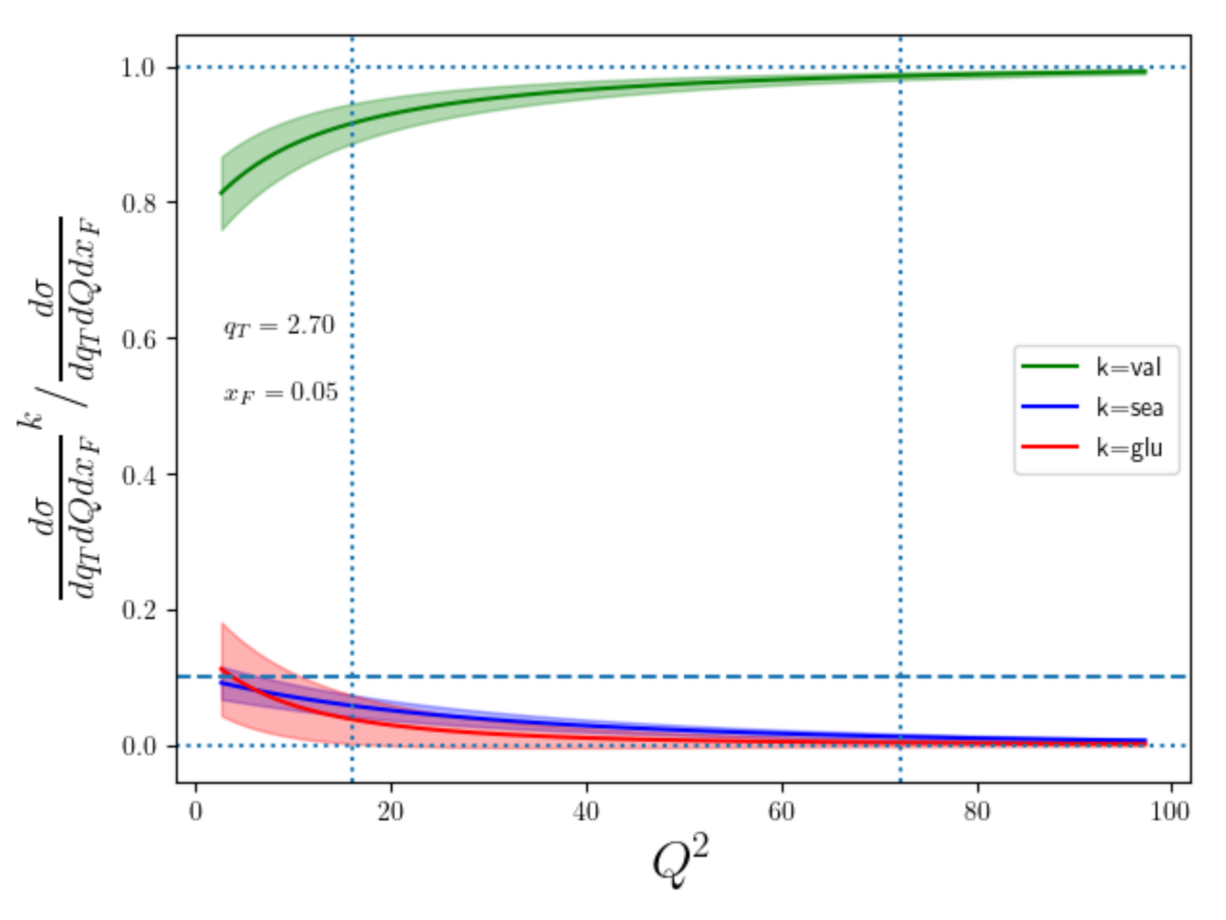
If $x_F \to 0$, then $Y \to 0$.

We already know that the cross section levels off near $Y = 0$, so very small x_F will not change the constraints!



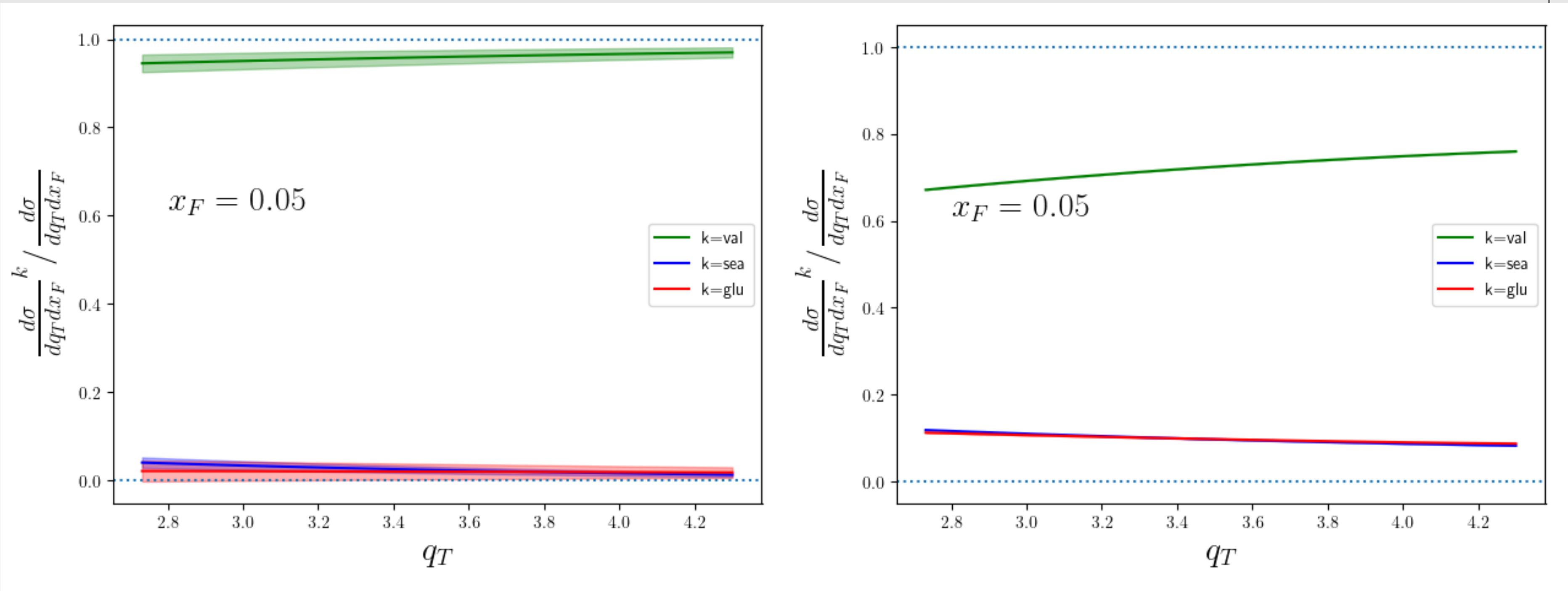
Are there *any* kinematic regions?

- We want to be able to constrain the gluon well at any kinematics, but is this possible?
- We make a theoretical triply differential plot as a function of Q^2 at the lowest q_T that we feel comfortable in the FO regime
- Vertical dotted lines are experimental bookends
- Need to go to very low Q^2 to get 10% contribution of the cross section from the gluon
- **Hopeless** to constrain the gluon with q_T-dependent DY



Still Hope for the proton!

- Looking at the contribution for the proton's gluon at large x, we have hope by examining the q_T-dependent DY data



Pion

Proton

There is still merit to studying pion-induced q_T-dependent DY data

TMD Factorization

Much of the following has been taken from Aybat, Rogers, Phys. Rev. **D83** :114042, (2011).

Field’s book

- Field’s book has some *primordial* motion inside the incoming hadrons to describe the low-q_T data
- The incoming partons here have some initial transverse momentum
- Primordial terms were usually shown as some Gaussian, such as $f(k_T^2) = \frac{1}{4\pi\sigma_q^2}\exp\left(-\frac{k_T^2}{4\sigma_q^2}\right)$

Asymptotic term

- As first shown in Collins, Soper, Sterman, Nuclear Phys. **B250** (1985) 199-224, there was a term, Y, different from the purely TMD $\widetilde{W}$

$$\frac{d\sigma}{dQ^2\,dy\,dQ_T^2} \sim \frac{4\pi^2\alpha^2}{9Q^2 s}\left\{(2\pi)^{-2}\int d^2b\, e^{iQ_T\cdot b}\,\widetilde{W}(b;Q,x_A,x_B) + Y(Q_T;Q,x_A,x_B)\right\}.$$

The W term describes low-q_T physics; Associated with TMDs

- Y is the fixed order term minus the asymptotic term. $Y = FO - AY$
- At very low q_T, the fixed order and asymptotic terms should *cancel*, leaving only the TMD physics
- At very high q_T, the TMD terms and asymptotic terms should *cancel*, leaving only the fixed-order physics

Asymptotic Term Form

- This is what the asymptotic term looks like

$$AY(q_T, Q, x_a, x_b) = \sum_{a,b} \int_{x_a}^{1} \frac{d\xi_a}{\xi_a} \int_{x_b}^{1} \frac{d\xi_b}{\xi_b} \sum_{N=1}^{\infty} \left(\frac{\alpha_S(\mu)}{\pi}\right)^N \times R_{ab}^{(N)} f_{a/A}(\xi_a, \mu) f_{b/B}(\xi_b, \mu)$$

Hard physics that follows the limit of the fixed order as $q_T \to 0$

Asymptotic Term

$$R^{(1)}_{q\bar{q}} = R^{(1)}_{\bar{q}q} = \frac{2e_q^2}{3\pi q_T^2}\Big[2\delta(1-z_a)\delta(1-z_b)[\log(\frac{Q^2}{q_T^2}) - \frac{3}{2}] + \delta(1-z_a)\big[\frac{1+z_b^2}{1-z_b}\big]_+ + \delta(1-z_b)\big[\frac{1+z_a^2}{1-z_a}\big]_+\Big]$$

$$R^{(1)}_{qg} = R^{(1)}_{\bar{q}g} = \frac{e_q^2}{4\pi q_T^2}\big[z_b^2 + (1-z_b)^2\big]\delta(1-z_a)$$

$$R^{(1)}_{gq} = R^{(1)}_{g\bar{q}} = \frac{e_q^2}{4\pi q_T^2}\big[z_a^2 + (1-z_a)^2\big]\delta(1-z_b)$$

$$z_a = \frac{x_a}{\xi_a}, z_b = \frac{x_b}{\xi_b}$$

Asymptotic – plus term

- Have to treat the plus term correctly with the lower bound $\neq 0$

$$\int_{x_b}^{1} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big) = \int_0^1 \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big) - \int_0^{x_b} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big)$$

$$\int_0^1 \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big) = \int_0^1 dz_b \frac{1+z_b^2}{1-z_b} f_a(x_a)\left(\frac{f_b(\frac{x_b}{z_b})}{z_b} - f_b(x_b)\right)$$

$$\int_0^{x_b} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big) = \int_0^{x_b} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big] f_a(x_a) f_b\Big(\frac{x_b}{z_b}\Big)$$

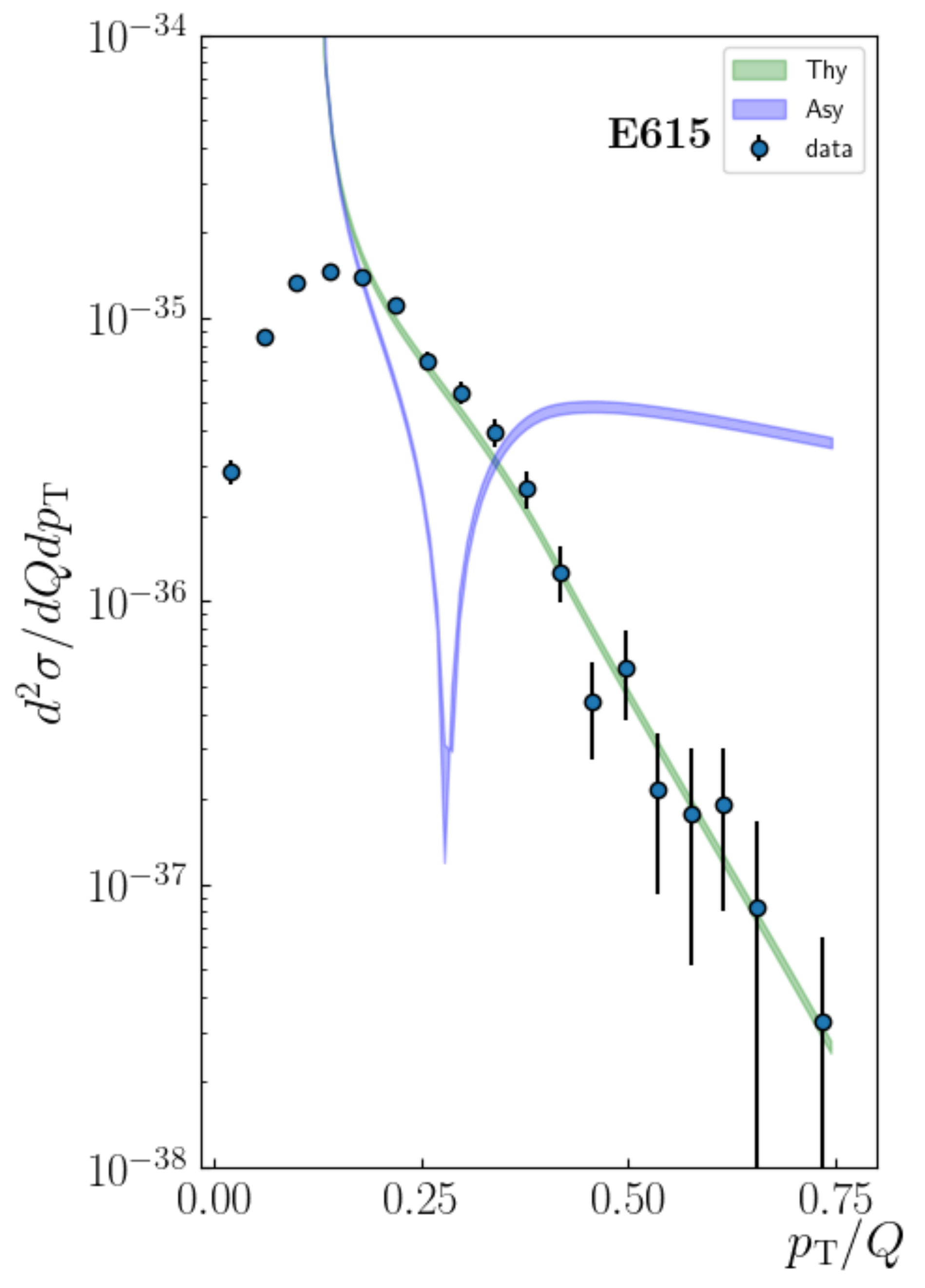
Asymptotic – plus term

$$\int_{x_b}^{1} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b(\frac{x_b}{z_b}) = \int_0^{x_b} dz_b \frac{1+z_b^2}{1-z_b} f_a(x_a)\Big(\frac{f_b(\frac{x_b}{z_b})}{z_b} - f_b(x_b)\Big)$$
$$+ \int_{x_b}^{1} dz_b \frac{1+z_b^2}{1-z_b} f_a(x_a)\Big(\frac{f_b(\frac{x_b}{z_b})}{z_b} - f_b(x_b)\Big)$$
$$- \int_0^{x_b} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big] f_a(x_a) f_b(\frac{x_b}{z_b})$$

$$\int_{x_b}^{1} \frac{dz_b}{z_b}\Big[\frac{1+z_b^2}{1-z_b}\Big]_+ f_a(x_a) f_b(\frac{x_b}{z_b}) = \int_{x_b}^{1} dz_b \frac{1+z_b^2}{1-z_b} f_a(x_a)\Big(\frac{f_b(\frac{x_b}{z_b})}{z_b} - f_b(x_b)\Big)$$
$$- f_a(x_a) f_b(x_b) \int_0^{x_b} dz_b \frac{1+z_b^2}{1-z_b}$$
$$= \int_{x_b}^{1} dz_b \frac{1+z_b^2}{1-z_b} f_a(x_a)\Big(\frac{f_b(\frac{x_b}{z_b})}{z_b} - f_b(x_b)\Big)$$
$$+ f_a(x_a) f_b(x_b)\Big[\frac{1}{2} x_b(2+x_b) + 2\log(1-x_b)\Big]$$

Comparison with data

- At low q_T, we can see that the asymptotic and fixed order curves match exactly
- The asymptotic term does not do a good job of describing the data
- Mostly to negate the fixed order or W term at the ends of the p_T spectrum



TMD factorization setup

- We can look to SIDIS (Semi-Inclusive Deep Inelastic Scattering) as a natural language for TMD physics

$$W^{\mu\nu} = \sum_f |\mathcal{H}_f(Q;\mu)^2|^{\mu\nu} \times$$
$$\int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T}\, F_{f/p}(x, \mathbf{k}_{1T}; \mu; \zeta_F)\, D_{h/f}(z, z\mathbf{k}_{2T}; \mu; \zeta_D) \times$$
$$\times\, \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T})$$

- Here, $\mathcal{H}_f$ are the hard parts, and the integration is for the soft parts over the external k_T
- $F_{f/p}$ is the TMDPDF of the proton
- $D_{h/f}$ is the TMDFF for detected hadrons
- Note the two scales μ-for renormalization group equations, and ζ – for solving CSS equations (for rapidity evolution)

A Brief Word on Divergences

- Wilson lines are buried in the definition of the TMDs
- TMD correlation functions contain light-cone divergences when Wilson lines point in exactly light-like directions

$$W(\infty, x; n) = P \exp\left[-ig_0 \int_0^\infty ds\, n \cdot A_0^a(x + sn) t^a\right]$$

- Can point the n vector off the light cone a little bit to avoid divergence

$$n_\mathrm{A} = (1, -e^{-2y_\mathrm{A}}, \mathbf{0}_t) \qquad n_\mathrm{B} = (-e^{2y_\mathrm{B}}, 1, \mathbf{0}_t).$$

- They tilted Wilson line directions are space-like: $n^2 < 0$

TMDs in $\boldsymbol{b}$-space

- While the interpretation of TMDs are in momentum-space (intrinsic transverse momentum of partons), the TMD evolution equations and factorization are more appropriate in b_T-space
- b_T is the Fourier conjugate of k_T

$$
\begin{aligned}
W^{\mu\nu} = & \\
\sum_f |\mathcal{H}_f(Q;\mu)^2|^{\mu\nu} & \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T} \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{q}_T - \mathbf{k}_{2T}) \\
& \times F_{f/p}(x, \mathbf{k}_{1T}; \mu; \zeta_F)\, D_{h/f}(z, z\mathbf{k}_{2T}; \mu; \zeta_D) \\
& = \sum_f |\mathcal{H}_f(Q;\mu)^2|^{\mu\nu} \int \frac{d^2\mathbf{b}_T}{(2\pi)^2}\, e^{-i\mathbf{q}_T\cdot\mathbf{b}_T} \\
& \quad \times \tilde{F}_{f/p}(x, \mathbf{b}_T; \mu; \zeta_F)\, \tilde{D}_{h/f}(z, \mathbf{b}_T; \mu; \zeta_D). \qquad (9)
\end{aligned}
$$

- Will be working with $\tilde{F}$ and $\widetilde{D}$ in b space

Evolution

- We see the evolution with respect to the "rapidity" scale ζ

$$\frac{\partial \ln \tilde{F}(x, \mathbf{b}_T; \mu, \zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(\mathbf{b}_T; \mu)$$

- Calculable in perturbation theory

$$\tilde{K}(\mu, b_T) = -\frac{\alpha_S(\mu) C_F}{\pi} \left[\log(\mu^2 b_T^2) - \log(4) + 2\gamma_E \right] = -\frac{\alpha_S(\mu) C_F}{\pi} \left[\log \left(\frac{\mu b_T}{b_0} \right)^2 \right]$$

$$b_0 = \frac{2}{e^{\gamma_E}}$$

- Reason in the 2nd form is that it's common to choose scale $\mu = \frac{b_0}{b_T}$ such that $\tilde{K} = 0$

Small-$\boldsymbol{b_T}$

- At small b_T (large k_T), one can write the TMD PDF in terms of *collinear* PDFs
- A convolution of the PDF and a perturbatively calculable coefficient function
- This expansion is known as (operator product expansion) OPE

$$\tilde{F}_{f/P}(x, \mathbf{b}_T; \mu, \zeta_F) = \\ = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_T; \zeta_F, \mu, g(\mu)) f_{j/P}(\hat{x}; \mu) \\ + \mathcal{O}((\Lambda_{\mathrm{QCD}} b_T)^a).$$

- When b_T gets large ($b_T \gtrsim \Lambda_{QCD}^{-1}$), expansion breaks down
- Would need to incorporate intrinsic k_T behavior in the hadron non-perturbatively
- Large b_T-dependence cannot be calculated by pQCD, but the scale dependence can be handled

b_*

- Recall, however, that the W term is a Fourier transform, and one must integrate **all $\boldsymbol{b_T}$**
- To combat it, a prescription is adopted,

$$\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}.$$

- Where b_* replaces b_T in the OPE expansion such that when b_T is small, b_* behaves as b_T
- But when b_T grows large, b_* approaches a maximum value chosen to limit the large-b_T spoiling OPE

- Additionally, in the calculation of the hard coefficient $\tilde{C}$, the appropriate scale is determined by the size of b_*

$$\mu_b = \frac{C_1}{b_*(\mathbf{b}_T)}.$$

- Where C_1 is commonly $2e^{-\gamma_E}$

Full Evolution and OPE

- The evolution occurs in exponentials multiplied by the OPE piece

$$\tilde{F}_{f/P}(x, \mathbf{b}_T; \mu, \zeta_F) = \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b)}^{\text{A}}$$

$$\overbrace{\times \exp\left\{\ln \frac{\sqrt{\zeta_F}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta_F}}{\mu'} \gamma_K(g(\mu'))\right]\right\}}^{\text{B}} \times \overbrace{\exp\left\{g_{j/P}(x, b_T) + g_K(b_T) \ln \frac{\sqrt{\zeta_F}}{\sqrt{\zeta_{F,0}}}\right\}}^{\text{C}} .$$

- The **A** term represents the OPE
- The **B** term represents the perturbatively calculable evolution of the OPE
- The **C** term represents the non-perturbative physics of the TMD (usually Gaussians), that need to be parameterized in a fit

Coefficient Functions

$$\tilde{C}_{j'/j}(x, \mathbf{b}_T; \mu; \zeta_F/\mu^2) = \delta_{j'j}\delta(1-x) + \delta_{j'j}\frac{\alpha_s C_\mathrm{F}}{2\pi}\left\{2\left[\ln\left(\frac{2}{\mu b_\mathrm{T}}\right) - \gamma_\mathrm{E}\right]\left[\left(\frac{2}{1-x}\right)_+ - 1 - x\right] + 1 - x + \right.$$
$$\left. + \delta(1-x)\left[-\frac{1}{2}\left[\ln\left(b_\mathrm{T}^2\mu^2\right) - 2(\ln 2 - \gamma_\mathrm{E})\right]^2 - \left[\ln(b_\mathrm{T}^2\mu^2) - 2(\ln 2 - \gamma_\mathrm{E})\right]\ln\left(\frac{\zeta_F}{\mu^2}\right)\right]\right\} + \mathcal{O}(\alpha_s^2).$$

$$\tilde{C}_{j'/g}(x, \mathbf{b}_T; \mu; \zeta_F/\mu^2) = \frac{\alpha_s T_\mathrm{f}}{2\pi}\left(2\left[1 - 2x(1-x)\right]\left[\ln\left(\frac{2}{b_\mathrm{T}\mu}\right) - \gamma_\mathrm{E}\right] + 2x(1-x)\right) + \mathcal{O}(\alpha_s^2)$$

Anomalous Dimensions

- The anomalous dimensions appear in the **B** term

$$\gamma_{\mathrm{F}}(\mu;\zeta_F/\mu^2)=\alpha_s\frac{C_{\mathrm{F}}}{\pi}\left(\frac{3}{2}-\ln\left(\frac{\zeta_F}{\mu^2}\right)\right)+\mathcal{O}(\alpha_s^2).$$

$$\gamma_K(\mu)=2\frac{\alpha_s C_F}{\pi}+\mathcal{O}(\alpha_s^2).$$

When $\boldsymbol{b_T}$ is too small

- There is not only a problem when b_T goes too large (which is fixed by the b_* prescription)

$$\overbrace{\times \exp\left\{\ln\frac{\sqrt{\zeta_F}}{\mu_b}\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(g(\mu');1) - \ln\frac{\sqrt{\zeta_F}}{\mu'}\gamma_K(g(\mu'))\right]\right\}}^{\mathrm{B}}$$

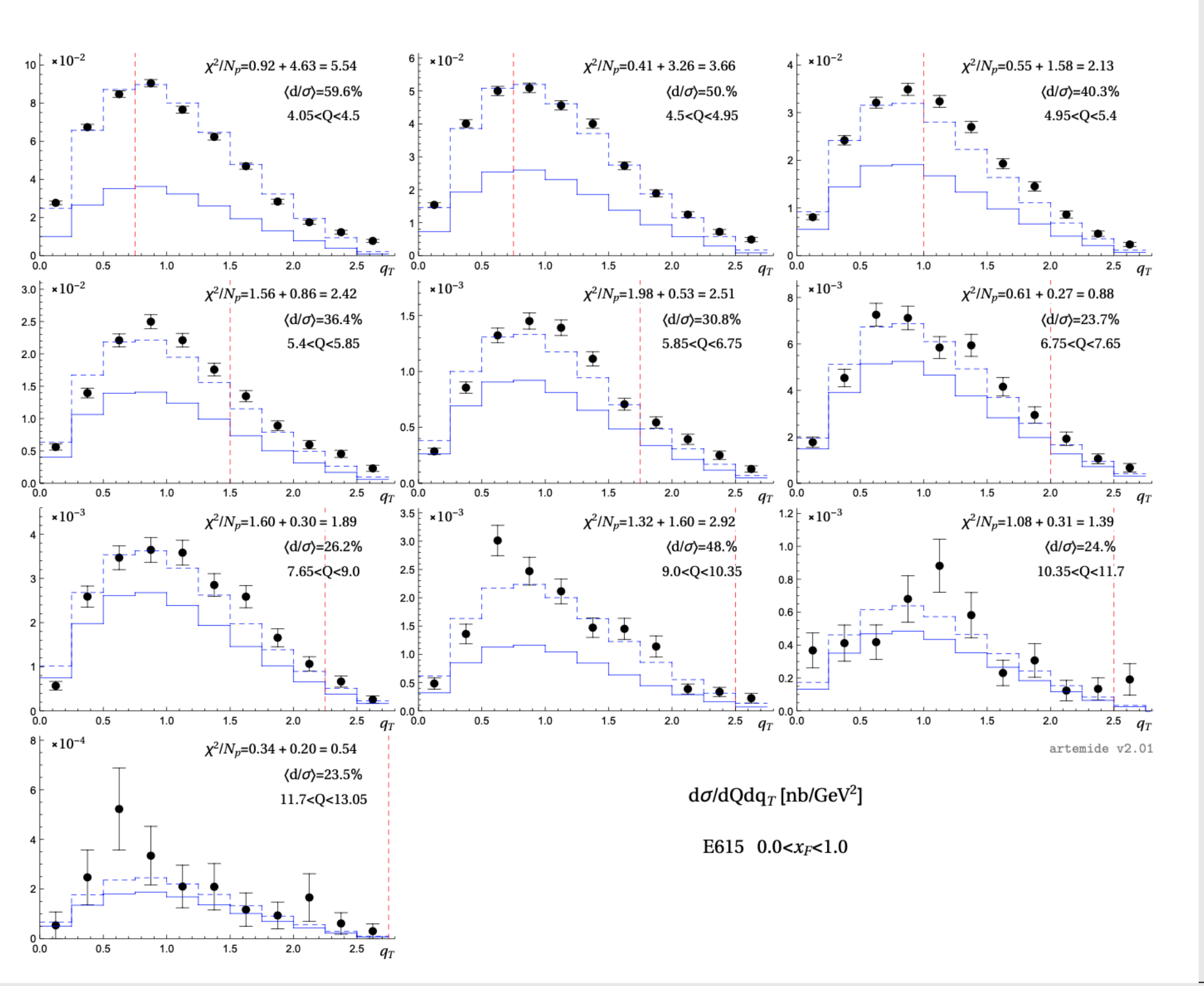
- One can see the limits of integration on the μ' integral
- If b_T is too small, μ_b will grow as $\mu_b \sim 1/b_T$
- The limits of integration will flip for a given μ, and the sign will change in the exponent – NOT GOOD
- Use the b_c prescription, where $b_c(b_\mathrm{T}) = \sqrt{b_\mathrm{T}^2 + b_0^2/(C_5 Q)^2}\,.$
- At a very small b_T, there will be some b_{min} that it will approach

$$b_*(b_c(b_\mathrm{T})) \longrightarrow \begin{cases} b_\mathrm{min} & b_\mathrm{T} \ll b_\mathrm{min} \\ b_\mathrm{T} & b_\mathrm{min} \ll b_\mathrm{T} \ll b_\mathrm{max} \\ b_\mathrm{max} & b_\mathrm{T} \gg b_\mathrm{max}\,. \end{cases}$$

- Combine them

Some Fitting Success by Vladimirov

- Can describe the low-q_T data well using TMD formulation
- πW DY (E615)
- Dashed lines are with systematic shift



Success is Difficult to describe both regions

- Previous attempts have been made to describe both high- and low-q_T data
- Vladimirov was able to describe low-q_T, we are able to describe large q_T
- SIDIS has been described with the low-q_T, but the fixed order is not described well:
- Lots of matching must occur. *i.e.* the large and small b_T regions, and the FO to the W at intermediate q_T
- The π-induced DY has hope to be able to describe both regions because there has been individual success in the both regions