Threshold Resummation in Drell-Yan with Applications to Pion PDFs

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Group Meeting 5/15/2020

Outline

1. Introduction and Motivation
2. Resummation Formalism
3. Minimal Prescription
4. Borel Prescription
5. Preliminary Results
6. Future Work and Summary

Introduction/Motivation

Motivation

- QCD allows us to study the structure of protons in terms of partons (quarks, antiquarks, and gluons)
- Use factorization theorems to separate hard partonic physics out of soft, non-perturbative objects to quantify structure

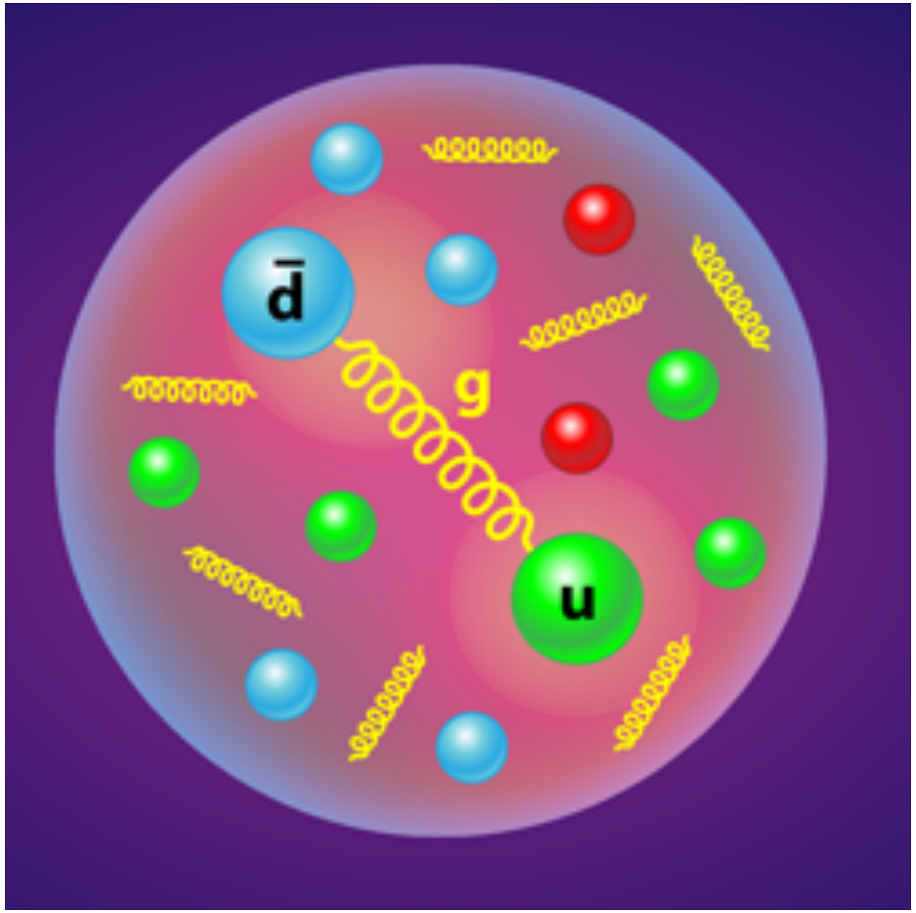
Motivation

What to do:

- Define a structure of nucleons in terms of quantum field theories
- Identify theoretical observables that factorize into non-perturbative objects and perturbatively calculable physics
- Perform global QCD analysis as structures are universal and are the same in all subprocesses

Pions

- Pion is the Goldstone boson associated with chiral symmetry breaking
- Lightest hadron as $\frac{m_\pi}{M_N} \ll 1$ and dictates the nature of hadronic interactions at low energies
- Simultaneously a $q\bar{q}$ bound state



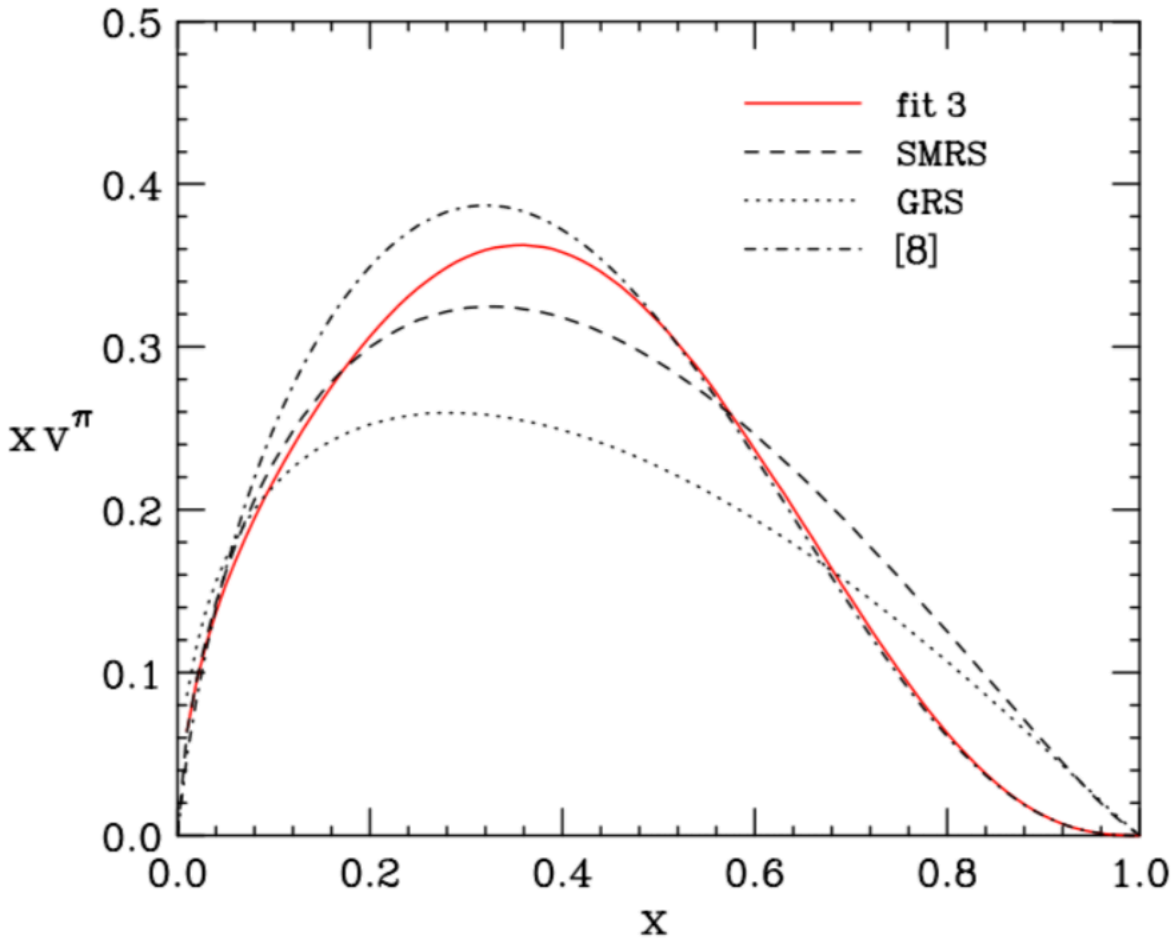
Theoretical Interest

- Behavior of PDF as $x_\pi \rightarrow 1$ ($\mathrm{v}_\pi \sim (1 - x_\pi)^{2\beta}$) can be related to momentum dependence of underlying interaction
- Perturbative QCD predicts that $\beta = 1$

Theoretical Interest

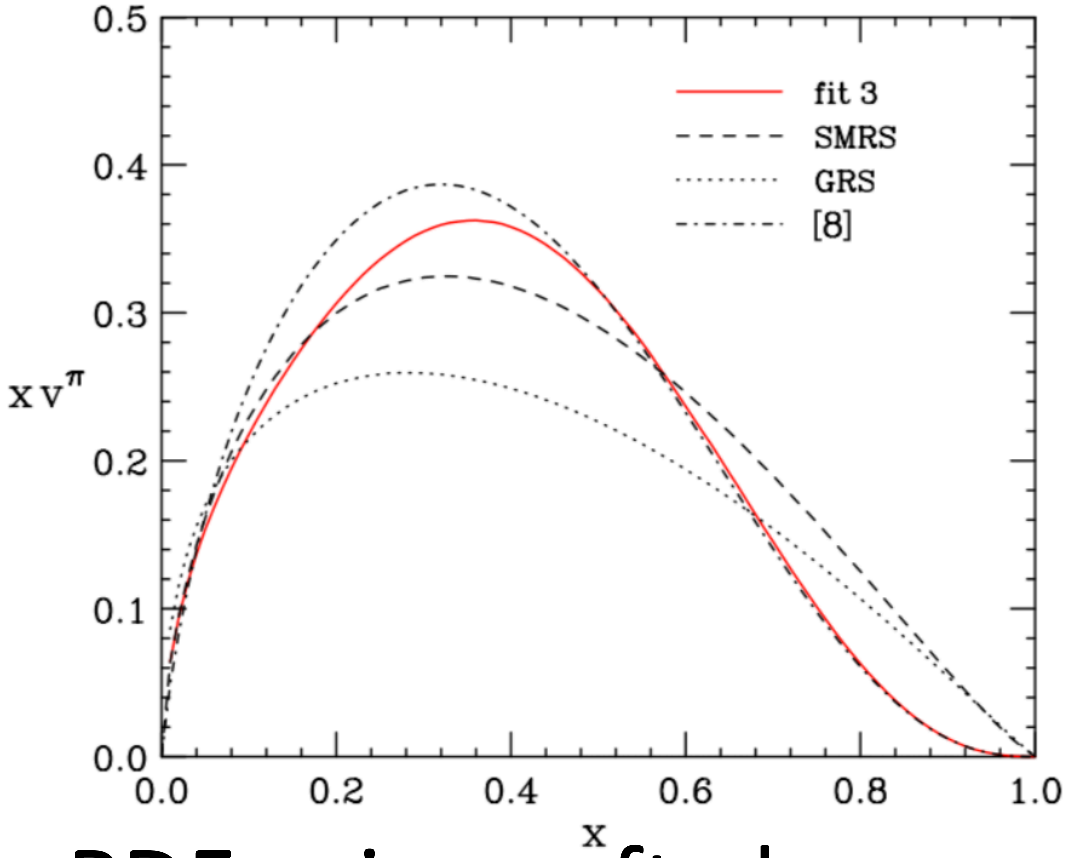
- Recent lattice calculations as well as phenomenologically determined valence quark PDFs using threshold resummation indicate $\beta = 1$ as opposed to fixed order ($\beta = 1/2$)
- Our analysis with threshold resummation will have impact on this question

Previous Pion Fits

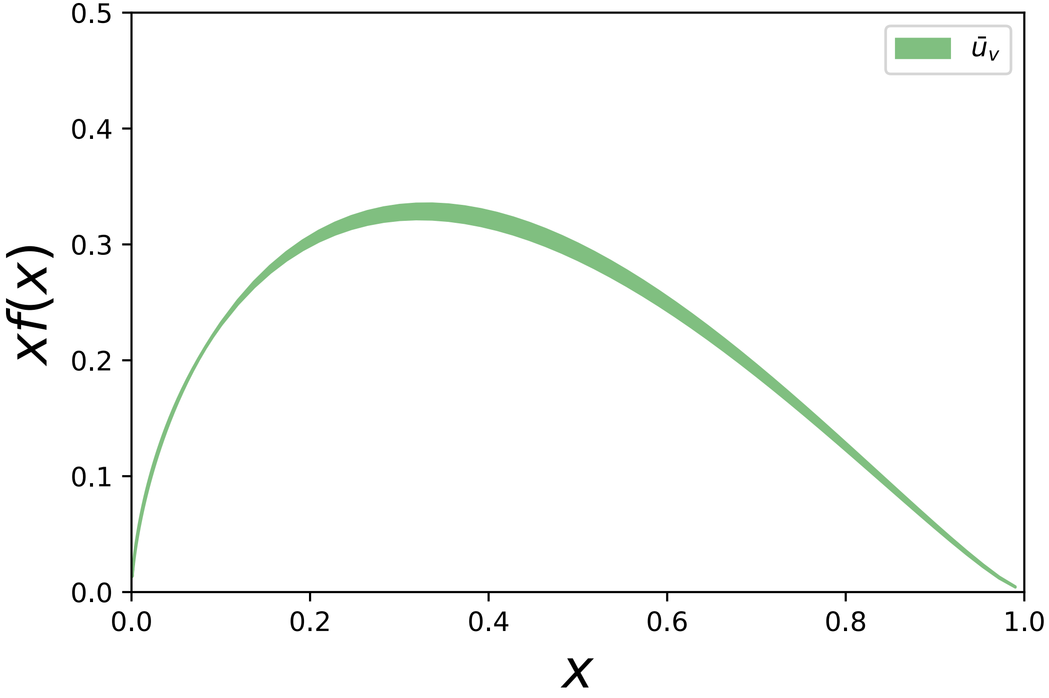


- Most recent (M. Aicher, et al, 2010) pion fit to DY data
- Fit uses soft gluon resummation

Comparison - Pion PDFs



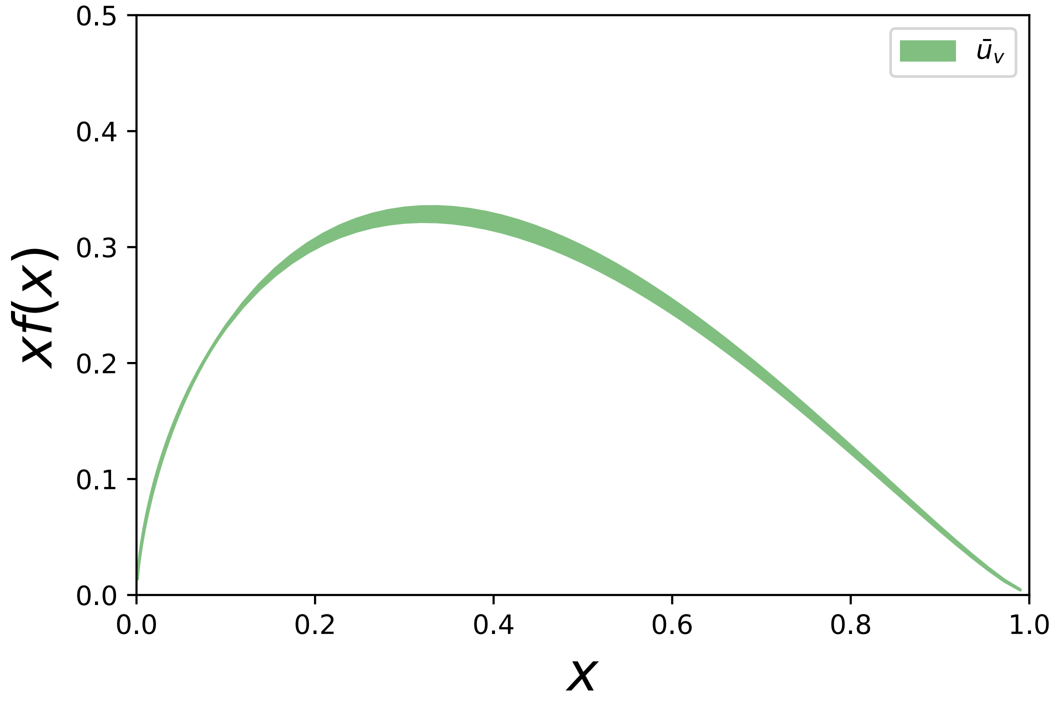
PDF using soft gluon resummation



PDF using only fixed-order pQCD (JAM18)

Uncertainty

- Note uncertainty band on PDFs are strictly from the data errors and parameterization bias
- No theoretical uncertainty shown (more on this later)



Drell-Yan (DY) Definitions

Hadronic variable

$$\tau = \frac{Q^2}{S}$$

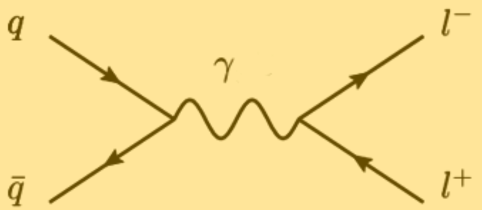
Partonic variable

$\hat{S}$ is the center of mass momentum squared of incoming partons

$$z \equiv \frac{Q^2}{\hat{S}} = \frac{\tau}{x_1 x_2}$$

Fixed Order Up to NLO

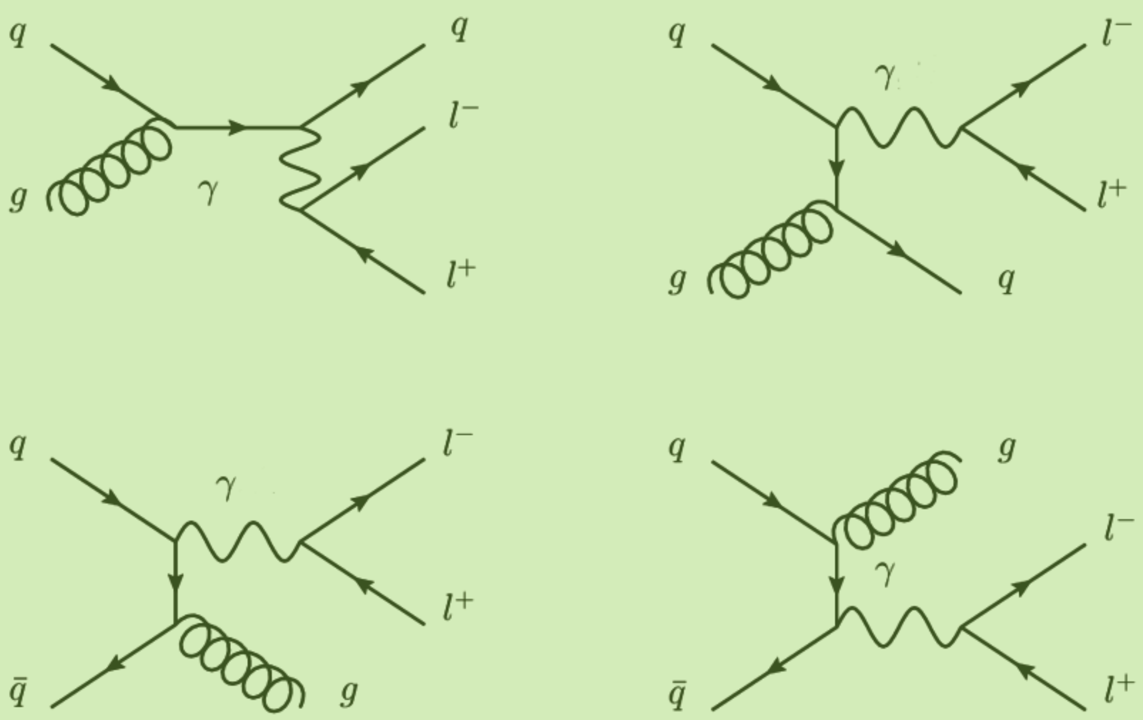
LO: $\mathcal{O}(1)$



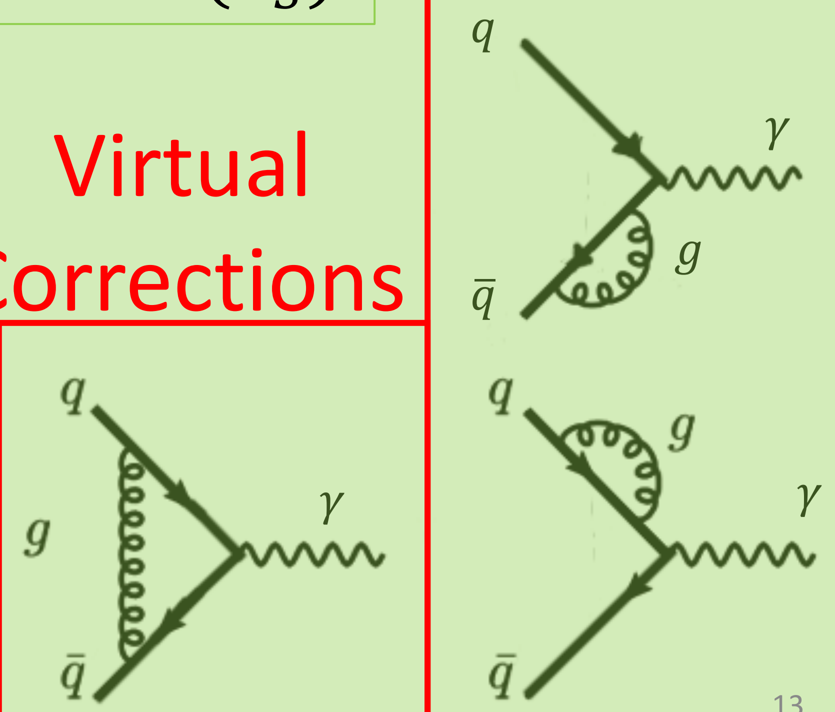
Feynman diagrams for DY amplitudes in collinear factorization

NLO: $\mathcal{O}(\alpha_S)$

Real emissions

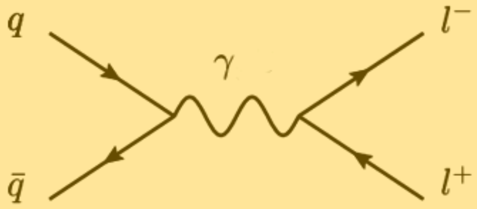


Virtual Corrections



LO

LO: $\mathcal{O}(1)$

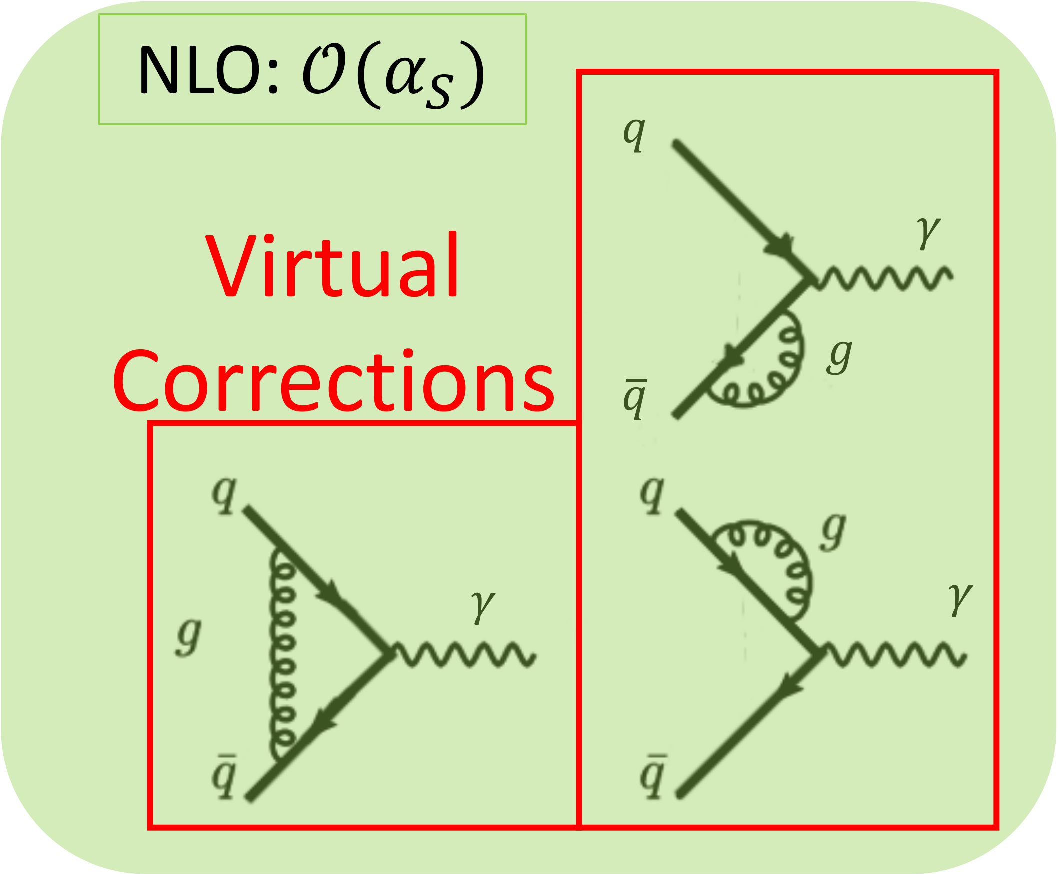


$$C_{q\bar{q}} = \delta(1-z)\frac{\delta(y)+\delta(1-y)}{2}$$

- $z = 1$ corresponds to partonic threshold
- All $\hat{S}$ is equal to Q^2
- All energy of hard partons turns into virtuality of photon

NLO Virtual

- Virtual corrections at NLO are proportional to $\delta(1-z)$
  - Exhibit Born kinematics



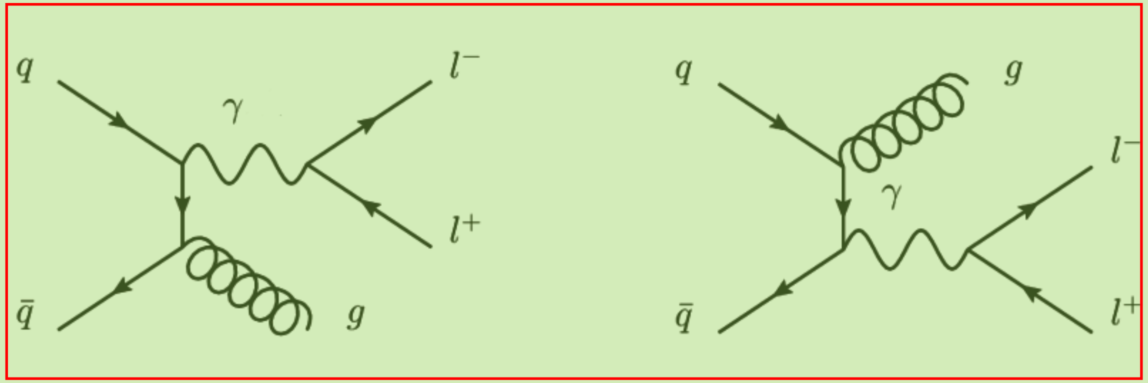
$$C_{q\bar{q}}^{\text{virtual}} = \delta(1-z)\frac{\delta(y)+\delta(1-y)}{2}\left[\frac{C_F\alpha_S}{\pi}\left(\frac{3}{2}\ln\frac{Q^2}{\mu^2}+\frac{2\pi^2}{3}-6\right)\right]$$

NLO Real Emission

- Next to leading order, real gluon emissions

$$C_{q\bar{q}}^{\text{real}} = \frac{C_F \alpha_S}{\pi}\left[\frac{\delta(y)+\delta(1-y)}{2}\left[(1+z^2)\left(\frac{1}{1-z}\ln\frac{Q^2(1-z)^2}{\mu^2 z}\right)_+ + 1 - z\right]\right.$$
$$\left. + \frac{1}{2}\left[\frac{(1-z)^2}{z}y(1-y)\right]\left[\frac{1+z^2}{1-z}\left(\left[\frac{1}{y}\right]_+ + \left[\frac{1}{1-y}\right]_+\right) - 2(1-z)\right]\right]$$

Real emissions

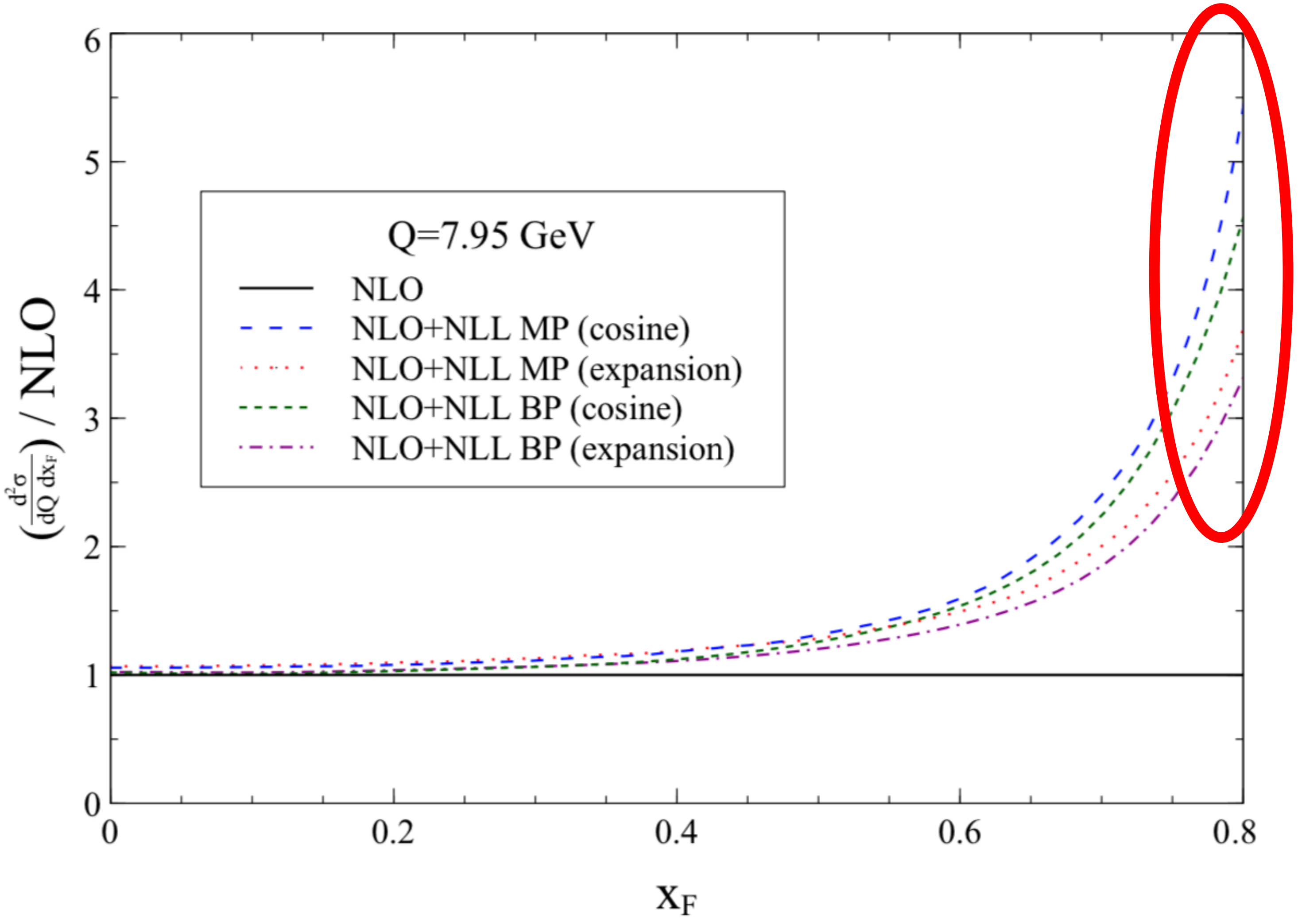


NLO Real Emission

- Plus distributions come from subtraction procedure of collinear singularities
- When $z \to 1, \log(1-z)$ can be large and potentially spoil perturbation
  - Appear in all orders in a predictable manner

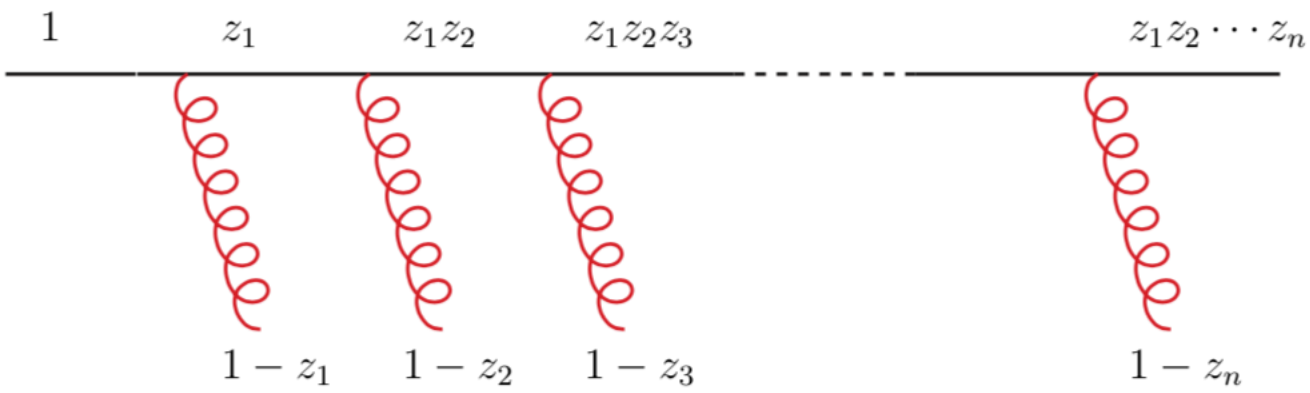
K-factor

- Four different prescriptions and handling of the cosine vs expansion
- See large deviation at large x_F
- Could put a theoretical uncertainty on the pion PDFs



Resummation Formalism

Soft Gluon Resummation



- These are Real Emitted Gluons from a quark line
- Can perturbatively calculate these emissions to all orders of α_S
- Here, z_i near 1

Setting it up

- Because of the Eikonal approximation, in the soft limit, matrix elements of large numbers of emitted gluons can be factorized as such:

$$\mathcal{M}_n(z_1, \ldots, z_n) \overset{\text{soft}}{\simeq} \frac{1}{n!} \prod_{i=1}^{n} \mathcal{M}_1(z_i)$$

- Even though the amplitudes factorize in z-space in that way, the phase space does not because of the presence of a delta function for conservation of momentum

$$\delta(z - z_1 z_2 \ldots z_n).$$

Setting it up

- In Mellin space, however, we do have factorization of the phase space,

$$\int_0^1 dz z^{N-1} \delta(z - z_1 z_2 ... z_n) = z_1^{N-1} z_2^{N-1} ... z_n^{N-1}$$

- So for hard kernels, we have something as:

$$C^{(n)}(N) \overset{\text{soft}}{\simeq} \frac{1}{n!} \left[C^{(1)}_{\text{soft}}(N) \right]^n$$

- Where $C^1_{soft}(N)$ is the hard kernel for one soft gluon emitted from the quark line

Exponentiation

- Thus, to sum over all the hard kernels is to exponentiate the emission of one soft gluon

$$\sum_{n=1}^{\infty} C^{(n)}(N) = \sum_{n} \frac{1}{n!}[C^{(1)}_{\text{soft}}(N)]^n$$
$$= \exp\left(C^{(1)}_{\text{soft}}(N)\right)$$

- Exponentiation is a key concept in threshold resummation

- A general form for the exponent is

$$C^{(1)}_{\text{soft}}(N) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{\mu_F^2}^{k^2_{T,\max}} \frac{dk_T^2}{k_T^2} \hat{G}(\alpha_S(k_T^2), k_T^2)$$

- Where $\hat{G}$ is different for different processes (DIS, DY, etc.)
- Here, α_S takes the argument of k_T^2 to encompass all threshold effects

Examples

- Each for the different processes, a Δ is included for each initial state parton (in DY there are 2); a J is included for each final state parton (in DIS there is 1)

$$\ln \Delta_p(N) = \int_0^1 \mathrm{d}z \, \frac{z^{N-1} - 1}{1 - z} \int_{\mu_F^2}^{(1-z)^2 Q^2} \frac{\mathrm{d}k_T^2}{k_T^2} \, A_p(\alpha_S(k_T^2)),$$

$$\ln J_p(N) = \int_0^1 \mathrm{d}z \, \frac{z^{N-1} - 1}{1 - z} \left[\int_{(1-z)^2 Q^2}^{(1-z) Q^2} \frac{\mathrm{d}k_T^2}{k_T^2} \, A_p(\alpha_S(k_T^2)) + B_p\Big(\alpha_S((1-z)Q^2)\Big) \right]$$

- A, B are perturbatively calculable in α_S
- Notice the bounds on the k_T^2 integration and the bounds on the z integration: would evaluate $\alpha_S(k_T^2 = 0)$!! Landau pole

Fixed form of α_S

- To perform the calculation, we can use a fixed form for α_S, such as the 2-loop form

$$\alpha_S(k_T^2) = \frac{\alpha_S(\mu^2)}{1 + b_0\alpha_S(\mu^2)\log(\frac{k_T^2}{\mu^2})}\left[1 - \frac{b_1}{b_0}\frac{\alpha_S(\mu^2)\log\left(1 + b_0\alpha_S(\mu^2)\log\left(\frac{k_T^2}{\mu^2}\right)\right)}{1 + b_0\alpha_S(\mu^2)\log\left(\frac{k_T^2}{\mu^2}\right)}\right]$$

- Then we can perform the integrations over k_T^2 then z to get the Mellin transforms using various approximations

Form for exponent

- Performing for the case of DY, we have

$$\log \Delta(N) = 2h^{(1)} \log(\bar{N}) + 2h^{(2)}(\lambda, Q^2/\mu^2)$$

- Where $\bar{N} = Ne^{\gamma_E}$ and $\lambda = b_0 \alpha_s(\mu^2) \ln \bar{N}.$

$$\begin{aligned} h^{(1)}(\lambda) &= \frac{A_q^{(1)}}{2\pi b_0 \lambda} \left[2\lambda + (1-2\lambda)\ln(1-2\lambda)\right], \\ h^{(2)}(\lambda) &= \left(\pi A_q^{(1)} b_1 - b_0 A_q^{(2)}\right) \frac{2\lambda + \ln(1-2\lambda)}{2\pi^2 b_0^3} \\ &+ \frac{A_q^{(1)} b_1}{4\pi b_0^3} \ln^2(1-2\lambda) + \frac{A_q^{(1)}}{2\pi b_0} \ln(1-2\lambda) \ln\frac{Q^2}{\mu^2}, \end{aligned}$$

Hard Kernel to Calculate

$$C^{\mathrm{N}^p\mathrm{LO}}_{\mathrm{N}^k\mathrm{LL}}(N, \alpha_s) = \sum_{j=0}^{p} \alpha_s^j \, C^{(j)}(N) + C^{\mathrm{res}}_{\mathrm{N}^k\mathrm{LL}}(N, \alpha_s) - \sum_{j=0}^{p} \frac{\alpha_s^j}{j!} \left[\frac{d^j C^{\mathrm{res}}_{\mathrm{N}^k\mathrm{LL}}(N, \alpha_s)}{d\alpha_s^j} \right]_{\alpha_s=0}$$

Fixed order Kernel
Already have calculated this at NLO!

New Resummation Kernel
Calculate such as Leading Log,
or Next-to-Leading Log

Matching coefficients
Need to subtract in
order to avoid double
counting

Next-to-Leading + Next-to-Leading Logarithm Order Calculation

Add the rows and columns. Need to make sure only counted once!
- Subtract the matching

	Non-Log terms	LL	NLL	...	N^pLL
LO	--	$constant$	--	...	--
NLO	$\alpha_s(\text{non}-\log)$	$\alpha_s \log(N)^2$	$constant$	...	--
NNLO	$\alpha_s^2(\text{non}-\log)$	$(\alpha_s \log(N)^2)^2$	$(\alpha_s \log(N)^3)$	...	--
...	...	...	...	...	...
N^kLO	$\alpha_s^k(\text{non}-\log)$	$(\alpha_s \log(N)^2)^k$	$(\alpha_s \log(N)^3)^{k-1}$	...	$(\alpha_s \log(N)^{p+2})^{k-p}$

The need for prescriptions

- To compare with data, one must Mellin invert so that the formulas are in momentum-fraction space and not moment space
- The Mellin inversion of the hard kernel appears order-by-order, but it is divergent because of the divergence of α_S
- One can locate the divergences and avoid them (Minimal Prescription)
- Or one can manipulate the summation to make it convergent (Borel prescription)

Minimal Prescription

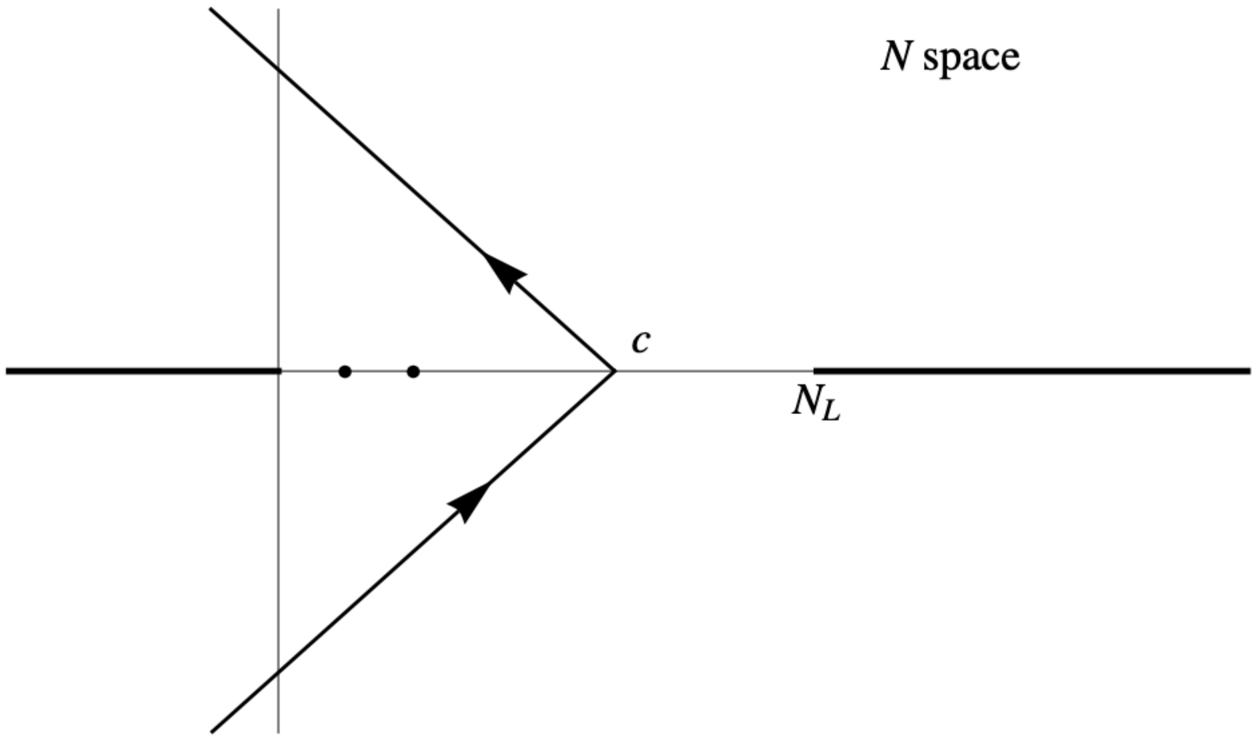
Minimal Prescription

- In principle, one can just do the Mellin inversion exactly
- However, the ambiguity appears in the Landau pole
- We can locate the Landau and avoid it
- By looking at *e.g.* the $h^1(\lambda)$ term, we can see where the arguments of the logarithms go to 0 and become negative
- This location is the Landau pole

$$1 - 2\lambda > 0 \implies \bar{N} < \exp\left(1/2\alpha_S b_0\right)$$

Avoiding the Landau Pole

- The minimal prescription attempts to avoid the Landau pole by making its Mellin inversion contour away the left of the pole and to the right of the other poles



Rapidity Distribution

- In order to compare with data, we need to compare with the rapidity dependent resummed formulas
- Instead of a single Mellin, a Mellin-Fourier transform must be taken

$$\sigma(N,M) \equiv \int_0^1 \mathrm{d}\tau \tau^{N-1} \int_{-\ln\frac{1}{\sqrt{\tau}}}^{\ln\frac{1}{\sqrt{\tau}}} \mathrm{d}\eta e^{iM\eta} \frac{\mathrm{d}\sigma}{\mathrm{d}Q^2 \mathrm{d}\eta},$$

Rapidity Distribution

- Substituting the hadronic rapidity with a partonic rapidity

$$\hat{\eta} = \eta - \frac{1}{2}\log\left(x_1/x_2\right)$$

- We get

$$C^{\text{res}}(N, M) = \int_0^1 dz z^{N-1} \int_{-\log 1/\sqrt{z}}^{\log 1/\sqrt{z}} d\hat{\eta} e^{iM\hat{\eta}} C^{\text{res}}(z)$$

- Since $C^{res}(z)$ is even under $\hat{\eta}$, the exponent can be converted to a cosine

$$C^{\text{res}}(N, M) = \int_0^1 dz z^{N-1} \int_{-\log 1/\sqrt{z}}^{\log 1/\sqrt{z}} d\hat{\eta} \cos(M\hat{\eta}) C^{\text{res}}(z)$$

Rapidity Distribution

- Because in the threshold limit, the hard part has delta functions, the $\hat{\eta}$ integration can be completed, namely

$$C^{\mathrm{res}}(N,M) = \int_0^1 dz z^{N-1} \cos(\frac{M}{2} \log z) C^{\mathrm{res}}(z)$$

Cosine vs Expansion

- Since we focus on the threshold region, that is when $z \to 1$, the log of z will be close to 0, meaning the argument of the cosine will be close to 0
- One can expand to the cosine term such that

$$\cos(\frac{M}{2}\log z) \approx 1$$

- Or, one can take the cosine exactly how it is

$$\cos(\frac{M}{2}\log z) = \frac{1}{2}(e^{i\frac{M}{2}\log z} + e^{-i\frac{M}{2}\log z})$$

Expansion

- If we have the expansion, then

$$C^{\mathrm{res}}(N,M)=\int_0^1 dz z^{N-1}\cos(\frac{M}{2}\log z)C^{\mathrm{res}}(z)$$

- Goes to

$$C^{\mathrm{res}}(N,M)=\int_0^1 dz z^{N-1}C^{\mathrm{res}}(z)=C^{\mathrm{res}}(N)$$

- Note the independence of *C* on *M*

Cosine

- If we have the expansion, then

$$C^{\rm res}(N,M) = \int_0^1 dz z^{N-1} \cos(\frac{M}{2}\log z) C^{\rm res}(z)$$

- Goes to

$$\begin{aligned} C^{\rm res}(N,M) &= \int_0^1 dz z^{N-1} \left[\frac{1}{2}\left(z^{iM/2} + z^{-iM/2}\right)\right] C^{\rm res}(z) \\ &= \int_0^1 dz \frac{1}{2}(z^{(N+iM/2)-1} + z^{(N-iM/2)-1}) C^{\rm res}(z) \end{aligned}$$

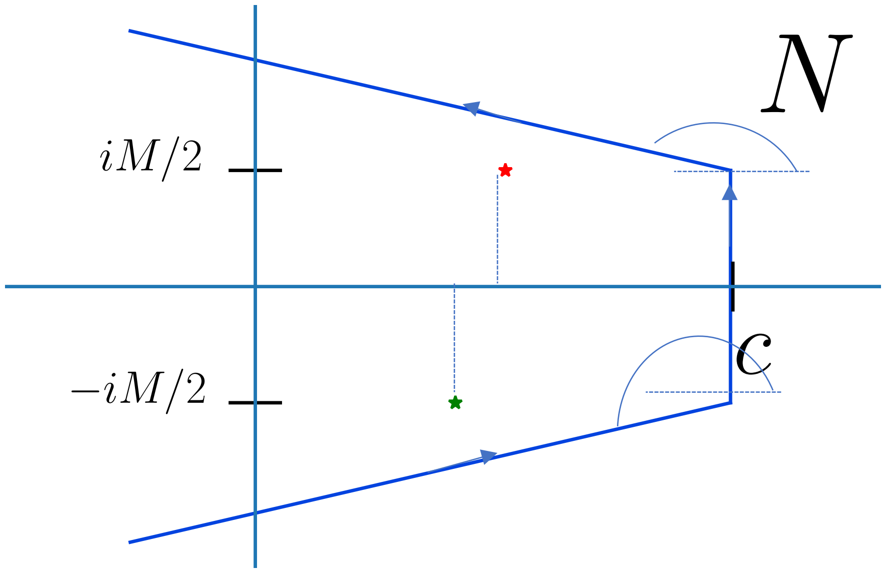
PDFs

- Because of the change of $\eta \to \hat{\eta}$, the PDFs gather a $\pm i\,\frac{M}{2}$ in their Mellin moments

$$\sigma(N,M) = \sigma_0 \sum_{q\bar{q}} f_A^{\pi}(N + iM/2) f_B^{W}(N - iM/2) C^{\mathrm{res}}(N,M)$$

- Whether *C* is dependent on *M* or not

MELLIN CONTOUR



- Here, c is to the right of the PDFs' rightmost poles
- Because the PDF moments are evaluated at $N \pm i\frac{M}{2}$ instead of the usual N, the poles are also located $\pm i\frac{M}{2}$ from the real axis (red and green stars)
- Contour is misshapen to ensure poles are encapsulated

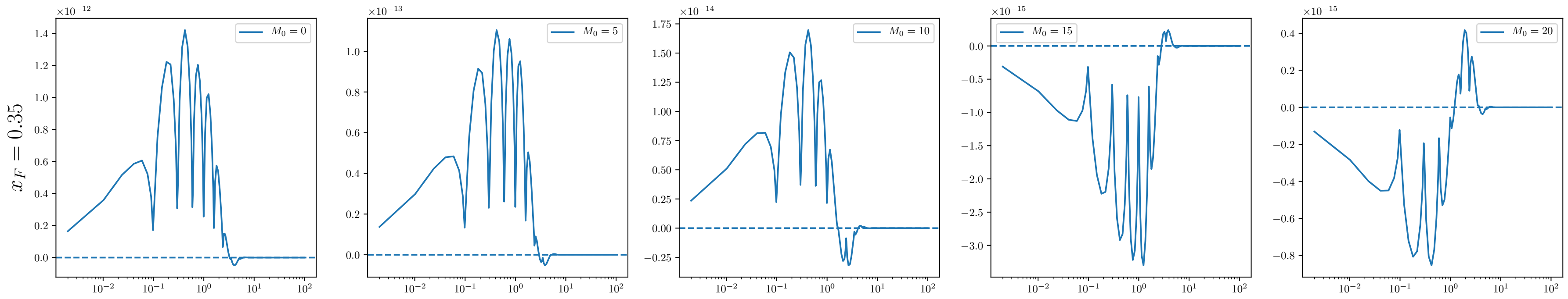
$$N_1 = c - i\frac{M}{2} + z_1\, e^{\phi_1}$$
$$0 < z_1 < \infty$$

$$N_2 = c - i\frac{M}{2} + z_2\, i\, M$$
$$0 < z_2 < 1$$

$$N_3 = c + i\frac{M}{2} + z_3\, e^{\phi_3}$$
$$0 < z_3 < \infty$$

Fast Fourier Transform

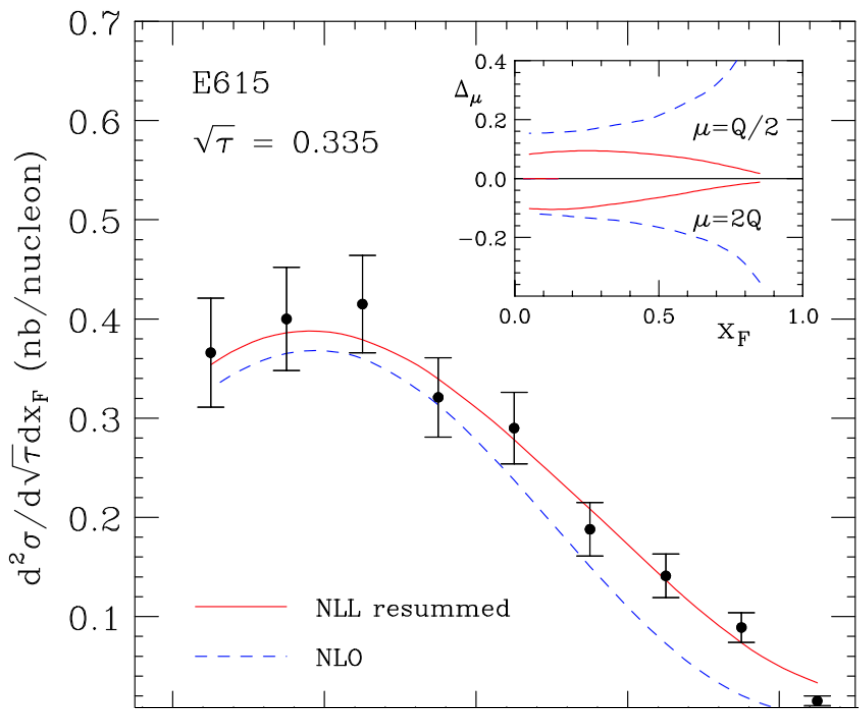
- An integration technique of the Fast Fourier Transform is needed to handle highly oscillatory integrands such as the ones below

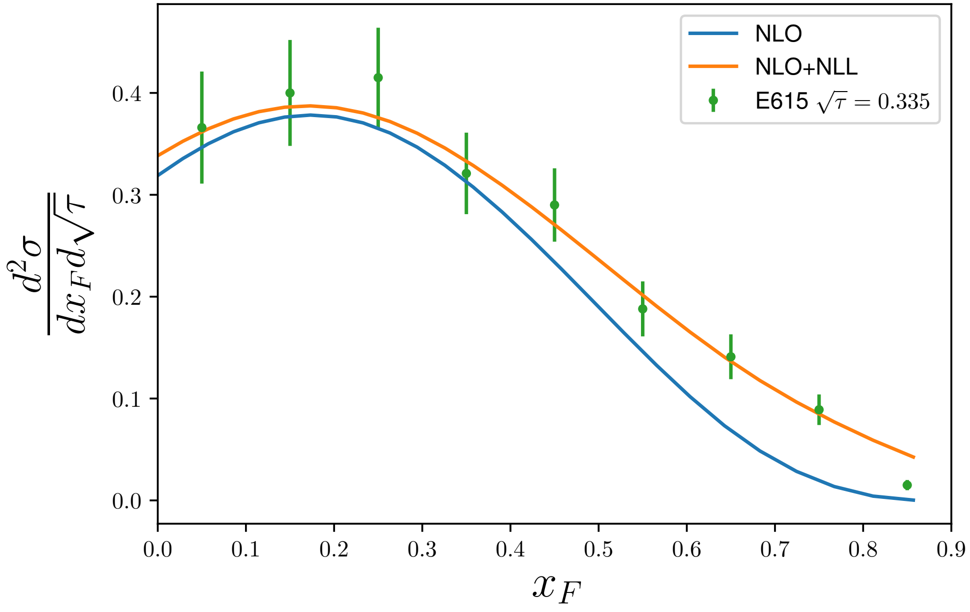


Comparison with Aicher et al.

- To check the code is working, always a good idea to check against published results
- Aicher et al. fit pion PDFs and studied πW DY
- Take Aicher’s parameters and evaluate the cross sections at the same kinematics

Comparison with Aicher





Borel Prescription

Borel Prescription (BP)

- The Borel Prescription makes use of Borel summation to take care of the divergent series in

$$C^{\mathrm{res}}(z, \alpha_s) = \sum_{k=0}^{\infty} \alpha_s^k \, C_k^{\mathrm{res}}(z)$$

Borel Summation

- Takes a divergent series and gives asymptotic value

$$\sum_k c_k \stackrel{\mathrm{B}}{=} \int_0^\infty dw\, e^{-w} \sum_k \frac{c_k}{k!} w^k$$

- For absolutely convergent series, the integral and the sum can be interchanged,

$$\frac{1}{k!} \int_0^\infty dw\, e^{-w}\, w^k = 1$$

$$= \Gamma(1+k) = k!$$

- So that the sum over c_k is restored
- Start from the divergent series, multiply each term by 1, and write 1 appropriately with k in each term, then exchange the sum and the integral

BP

- It is convenient to write the resumed kernel as

$$C^{\rm res}(N,\alpha_s) = \Sigma\left(\bar\alpha \log\frac{1}{N}, \alpha_s\right)$$

- Which can be re-written as

$$\Sigma(\bar\alpha L, \alpha_s) = \sum_{k=0}^{\infty} h_k(\alpha_s)\,(\bar\alpha L)^k\,.$$

Where $L = \log\left(\frac{1}{N}\right)$

- And we can compute the Mellin inversions term by term

$$C^{\rm res}(z,\alpha_s) = \mathcal{M}^{-1}\left[\Sigma\left(\bar\alpha\log\frac{1}{N},\alpha_s\right)\right](z) = \sum_{k=0}^{\infty} h_k(\alpha_s)\,\bar\alpha^k\,c_k(z)$$

And we know that $c_k(z)$ are the inverse Mellin transforms of the powers of $\log\left(\frac{1}{N}\right)$

BP – c_k

- There are many ways to calculate the Mellin inversion of $\log\left(\frac{1}{N}\right)^k$

$$c_k(z) = \mathcal{M}^{-1}\left[\log^k \frac{1}{N}\right](z)$$

- Skipping ahead for now, we arrive at

$$c_k(z) = \mathcal{M}^{-1}\left[\log^k \frac{1}{N}\right](z) = \frac{k!}{2\pi i}\oint \frac{d\xi}{\xi^{k+1}}\left(\frac{\left[\log^{\xi-1}\frac{1}{z}\right]_+}{\Gamma(\xi)} + \delta(1-z)\right)$$

Certain poles must be encircled in the contour!

- So a new variable to be integrated over is introduced based on

$$\left(\frac{d^k}{d\xi^k}\frac{\log^{\xi-1}\frac{1}{z}}{\Gamma(\xi)}\bigg|_{\xi=0}\right)_+ = \frac{k!}{2\pi i}\left(\oint \frac{d\xi}{\xi^{k+1}}\frac{\log^{\xi-1}\frac{1}{z}}{\Gamma(\xi)}\right)_+$$

BP continuation

- Here, by exchanging the integral and summation, we arrive at

$$C^{\text{res}}(z,\alpha_s) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\left(\frac{\left[\log^{\xi-1}\frac{1}{z}\right]_+}{\Gamma(\xi)} + \delta(1-z)\right)\sum_{k=0}^{\infty} h_k(\alpha_s)\left(\frac{\bar{\alpha}}{\xi}\right)^k k!$$

- We still have divergences here, so now, we can perform the Borel transformation:

$$C^{\text{res}}(z,\alpha_s) = \frac{1}{2\pi i}\oint \frac{d\xi}{\xi}\left(\frac{\left[\log^{\xi-1}\frac{1}{z}\right]_+}{\Gamma(\xi)} + \delta(1-z)\right)\int_0^{\infty} dw\, e^{-w}\sum_{k=0}^{\infty} h_k(\alpha_s)\left(\frac{\bar{\alpha}w}{\xi}\right)^k$$

- We have included the integral over *w*
- We also notice that the summation can be written in the same way as was introduced

$$\Sigma\left(\frac{\bar{\alpha}w}{\xi},\alpha_s\right)$$

BP Comments

- The branch cut (Landau pole) of C^{res} is mapped in terms of ξ to the region $-\bar{\alpha}w \leq \xi \leq 0$, so the contour must enclose this branch cut
- However, the w integral goes up to ∞
- So the $C^{res}(z, \alpha_s)$ is not actually Borel-summable
- We need to introduce an upper cut-off on the w integral, so we get:

$$C^{\mathrm{BP}}(z, \alpha_s, W) = \frac{1}{2\pi i} \oint \frac{d\xi}{\xi} \left(\frac{\left[\log^{\xi-1} \frac{1}{z}\right]_+}{\Gamma(\xi)} + \delta(1-z) \right) \int_0^W \frac{dw}{\bar{\alpha}} \, e^{-\frac{w}{\bar{\alpha}}} \, \Sigma\left(\frac{w}{\xi}, \alpha_s\right)$$

BP Comments

- The divergent series $C^{res}(z)$ is asymptotic to the BP formula

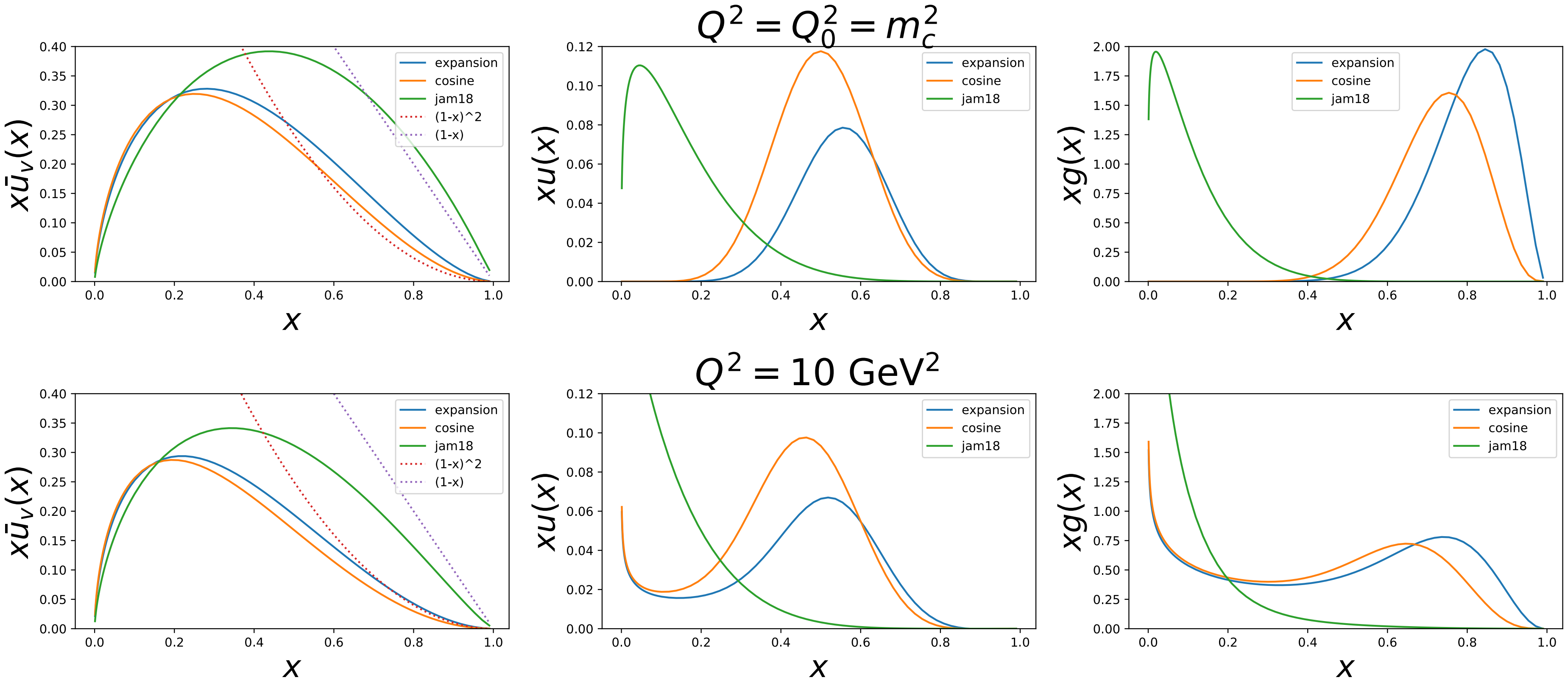
$$C^{\rm BP}(z,\alpha_s,W)=\frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\left(\frac{\left[\log^{\xi-1}\frac{1}{z}\right]_+}{\Gamma(\xi)}+\delta(1-z)\right)\int_0^W\frac{dw}{\bar\alpha}\,e^{-\frac{w}{\bar\alpha}}\,\Sigma\left(\frac{w}{\xi},\alpha_s\right)$$

- Does not spoil convolution since formulated at the partonic level
- Based on subleading term arguments, more BP kernels can be generated (not in this talk)
  - Try to reproduce correct instances of $\frac{\log(1-z)}{1-z}$ and constant terms
  - Also has to do with taking the large-N approximation when N is not always large

Preliminary Results

Minimal Prescription

- From fitting E615 experimental data

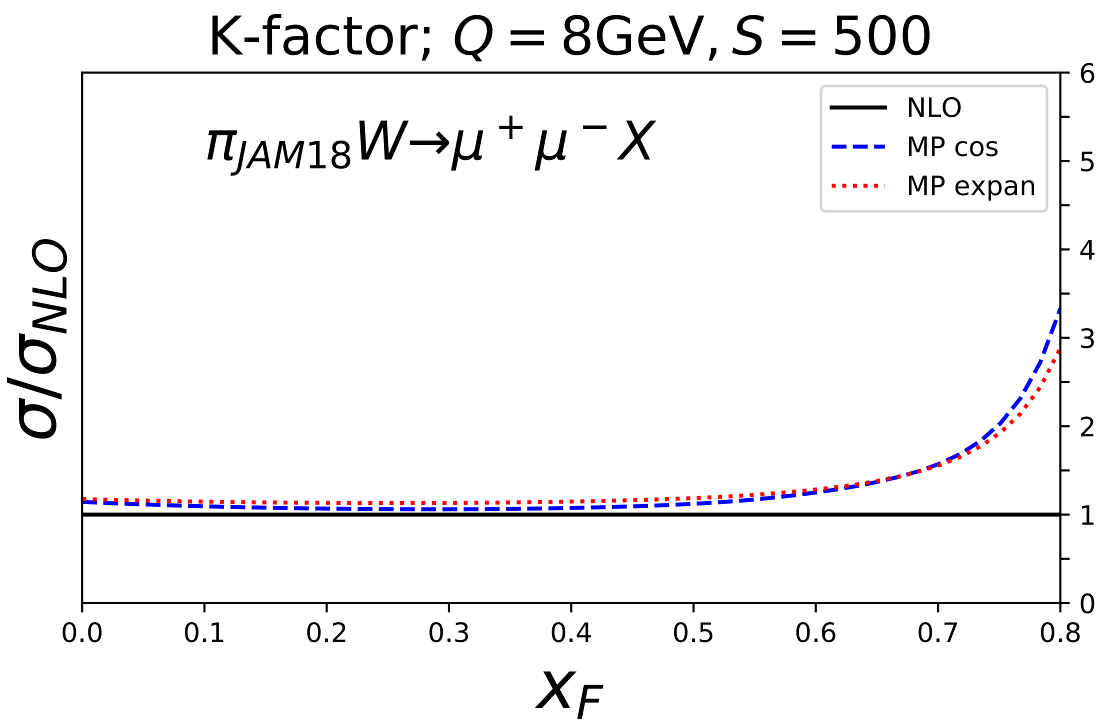


$\chi^2 \sim 2$/npts, not very good

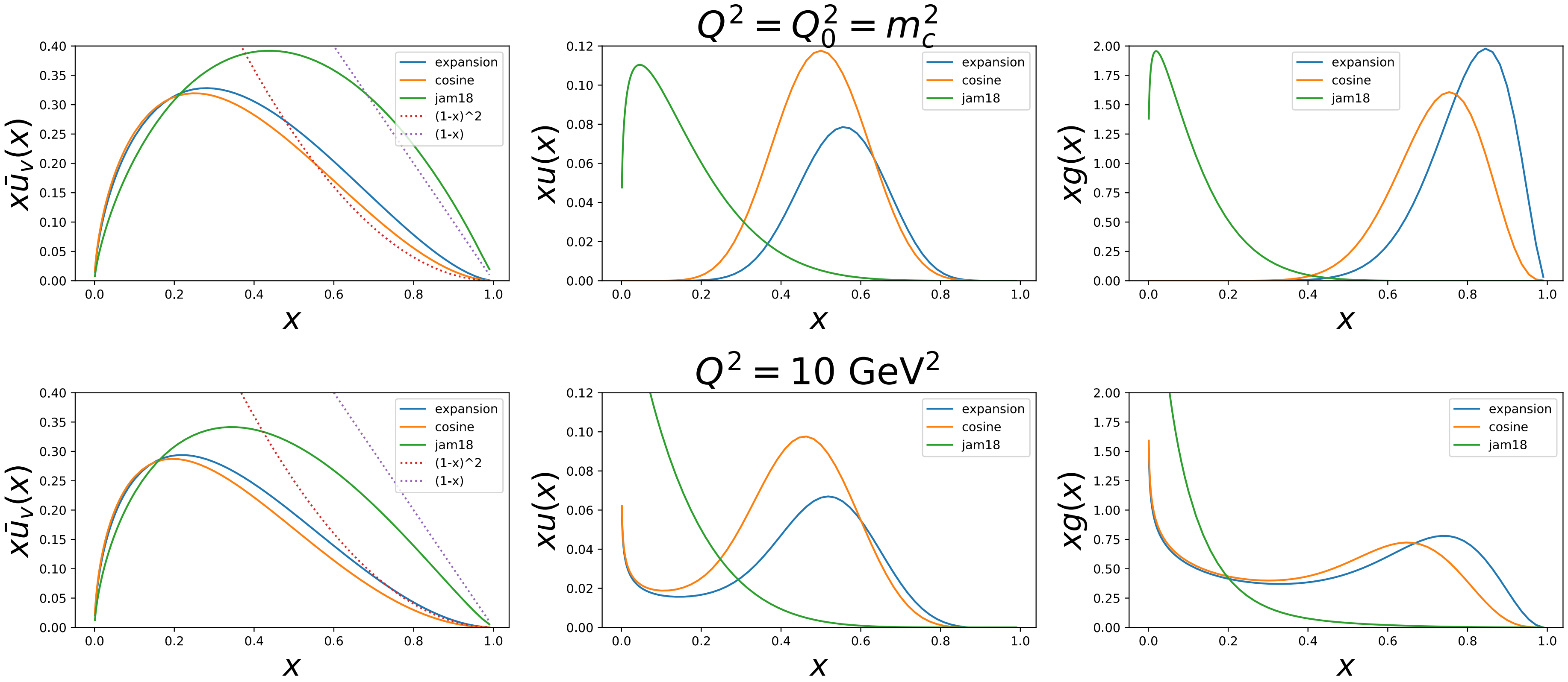
This large of a gluon distribution at high-x is unphysical

K-factor Pions

- For JAM18 Pion PDFs plugged into the threshold resummed DY cross section
- Not much difference at large x_F, especially compared with the proton

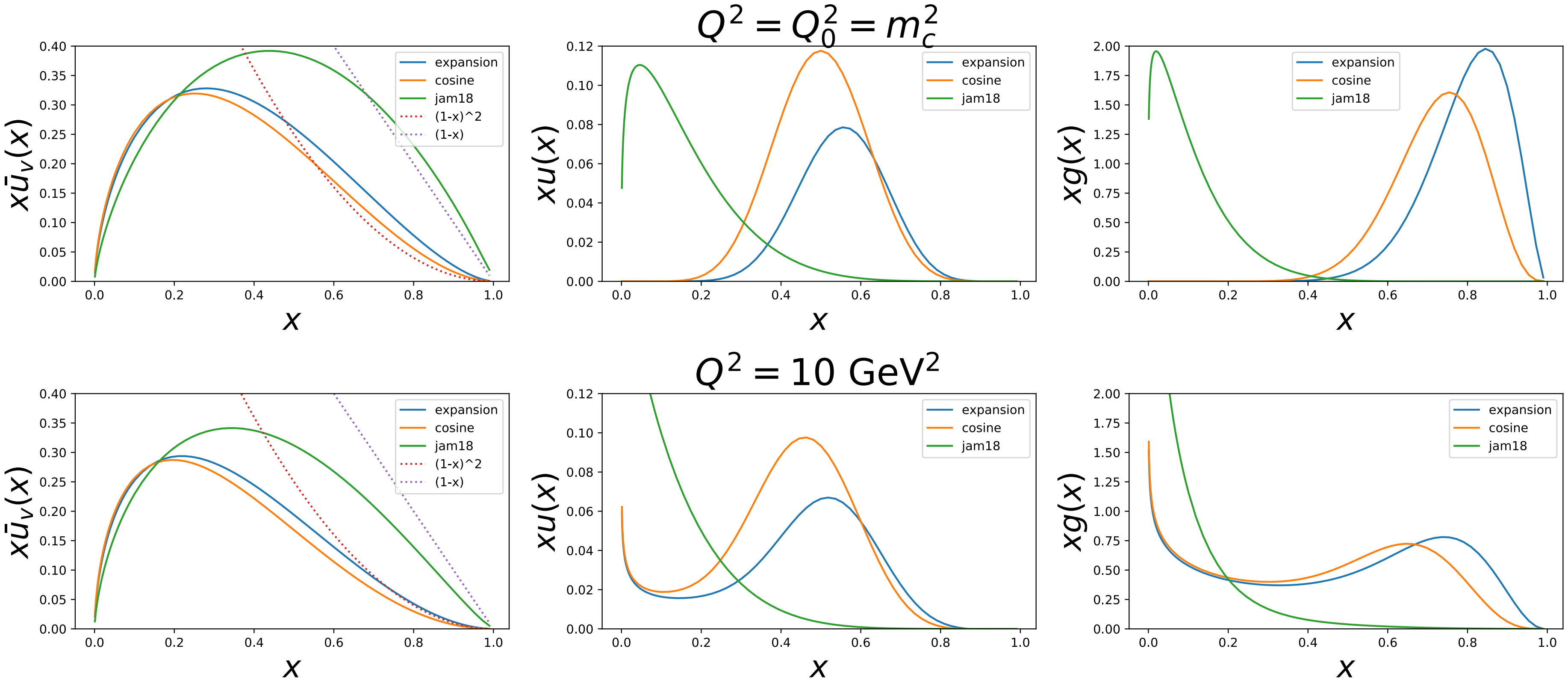


Minimal Prescription



- Momentum fractions for cosine:
  - $\langle x \rangle_{val} = 0.359, \langle x \rangle_{sea} = 0.205, \langle x \rangle = 0.436$

Minimal Prescription



- Momentum fractions for expansion:
  - $\langle x \rangle_{val} = 0.390, \langle x \rangle_{sea} = 0.130, \langle x \rangle = 0.480$

Momentum Fractions

- JAM18 vs resummation fits

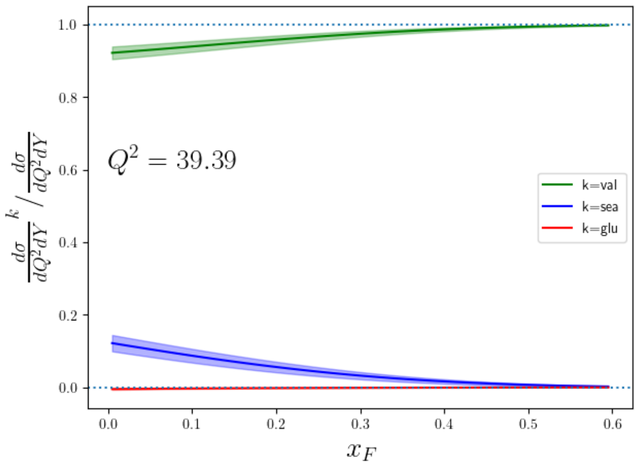
Flavor / Determination	JAM18 Pion	Cosine	Expansion
$\langle x_\pi \rangle_{val}$	0.54	0.36	0.39
$\langle x_\pi \rangle_{sea}$	0.16	0.21	0.13
$\langle x_\pi \rangle_{g}$	0.30	0.44	0.48

Stark contrast in valence and gluon momentum fractions!

Future Work and Summary

Next Steps for Minimal Prescription

- Gluon and sea could be frozen from most recent analysis, allowing fit to be just for the valence (as was done in Aicher)
- Recall the valence is the main contributor to the DY cross section



- Include LN data as well to constrain the gluon

Future for Minimal Prescription

- Calculation takes a long time, Monte Carlo needs improvement
- Memory taken for the Mellin contours is large
- Sophisticated parallelization of tasks is needed
- Possible AI applications

Next Steps for Borel Prescription

- Need to make sure the NLO+NLL is in line with Bonvini calculations
  - Check the plus distribution function
- Construct Mellin tables to speed up Borel calculation
- Integrate the Borel code into the JAM fitting framework
- Perform fits on DY data, perhaps in a similar way to the Minimal Prescription

Borel Prescription Tables (Cosine)

- The Borel prescription is best described in x_π-space

$$\frac{1}{\sigma_0}\frac{d\sigma^{\rm BP}}{dQ^2dY} = \frac{1}{2}\sum_{q\bar{q}} e_q^2\Big[\int_{x_1^0}^1 \frac{dx_1}{x_1} f_A^\pi(x_1,Q^2) f_B^W(x_2^0,Q^2) C_{\rm BP}^{\rm res}\big(\frac{x_1^0}{x_1},Q^2\big) + \int_{x_2^0}^1 \frac{dx_2}{x_2} f_A^\pi(x_1^0,Q^2) f_B^W(x_2,Q^2) C_{\rm BP}^{\rm res}\big(\frac{x_2^0}{x_2},Q^2\big)\Big].$$

- Each line has a convolution form, so we can write it as

$$\frac{1}{\sigma_0}\frac{d\sigma^{\rm BP}}{dQ^2dY} = \frac{1}{2}\sum_{q\bar{q}} e_q^2\Big[\int_0^1 d\xi \int_0^1 dx_1 f_A^\pi(x_1,Q^2) f_B^W(x_2^0,Q^2) C_{\rm BP}^{\rm res}(\xi,Q^2)\delta(x_1^0 - x_1\xi) + \int_0^1 d\xi \int_0^1 dx_2 f_A^\pi(x_1^0,Q^2) f_B^W(x_2,Q^2) C_{\rm BP}^{\rm res}(\xi,Q^2)\delta(x_2^0 - x_2\xi)\Big].$$

Borel Prescription Tables (Cosine)

- Recall that Mellin transforms of convolutions are products of individual Mellin transforms
- Line by line, I show

$$\sigma^{\mathrm{BP}}_{\mathrm{line}\ 1} = \frac{1}{2}\sum_{q\bar{q}} e_q^2 f_B^W(x_2^0, Q^2) \int_0^1 dx_1^0 (x_1^0)^{N-1} \int_0^1 d\xi \int_0^1 dx_1 f_A^\pi(x_1, Q^2) C^{\mathrm{res}}_{\mathrm{BP}}(\xi, Q^2)\delta(x_1^0 - x_1\xi).$$

$$\sigma^{\mathrm{BP}}_{\mathrm{line}\ 1} = \frac{1}{2}\sum_{q\bar{q}} e_q^2 f_B^W(x_2^0, Q^2) \int_0^1 \xi^{N-1} C^{\mathrm{res}}_{\mathrm{BP}}(\xi, Q^2) \int_0^1 dx_1 x_1^{N-1} f_A^\pi(x_1, Q^2)$$

$$= \frac{1}{2}\sum_{q\bar{q}} e_q^2 f_B^W(x_2^0, Q^2) C^{\mathrm{res}}_{\mathrm{BP}}(N, Q^2) f_A^\pi(N, Q^2).$$

- Line 2 follows similarly

Borel Prescription Tables (Cosine)

- Combining, we write

$$\frac{1}{\sigma_0}\frac{d\sigma^{\rm BP}}{dQ^2 dY} = \frac{1}{2}\sum_{q\bar{q}} e_q^2\Big[f_B^W(x_2^0,Q^2)\frac{1}{2\pi i}\int_{C_N} dN (x_1^0)^{-N} C_{\rm BP}^{\rm res}(N,Q^2) f_A^{\pi}(N,Q^2)$$

$$+ f_A^{\pi}(x_1^0,Q^2)\frac{1}{2\pi i}\int_{C_M} dM (x_2^0)^{-M} C_{\rm BP}^{\rm res}(M,Q^2) f_B^W(M,Q^2)\Big].$$

- For each line, we can construct Mellin tables to be multiplied with the pion PDF

$$T_1(N) = \frac{1}{2} e_q^2 (x_1^0)^{-N} f_B^W(x_2^0,Q^2) C_{\rm BP}^{\rm res}(N,Q^2)$$

$$\frac{1}{\sigma_0}\frac{d\sigma^{\rm BP1}}{dQ^2 dY} = \sum_{q\bar{q}} \frac{1}{2\pi i}\int_{C_N} dN f_A^{\pi}(N,Q^2) T_1(N)$$

Borel Prescription Tables (Cosine)

- The 2nd line table can perform the Mellin inversion

$$T_2 = \frac{1}{2}\int_{C_M} dM (x_2^0)^{-M} C_{\mathrm{BP}}^{\mathrm{res}}(M, Q^2) f_B^W(M, Q^2)$$

$$\frac{1}{\sigma_0}\frac{d\sigma^{\mathrm{BP2}}}{dQ^2 dY} = \sum_{q\bar{q}} f_A^\pi(x_1^0, Q^2) T_2$$

- Doing so, we can calculate the cross section much faster for Monte Carlo global fits

Borel Prescription Tables (Expansion)

- For the expansion, we take the cosine to be 1. In doing so, we have the following cross section

$$\frac{1}{\sigma_0}\frac{d\sigma^{BP}}{dQ^2dY} = \sum_{q\bar{q}} \int_{\tau e^{2|Y|}}^{1} \frac{dz}{z} f_A^{\pi}(\frac{x_1^0}{\sqrt{z}}, Q^2) f_B^{W}(\frac{x_2^0}{\sqrt{z}}, Q^2) C_{\rm BP}^{\rm res}(z, Q^2).$$

- We can replace the pion PDF by the Mellin inversion of its Mellin transform,

$$f_A^{\pi}(x_1, Q^2) = \frac{1}{2\pi i}\int_{C_N} dN x_1^{-N} f_A^{\pi}(N, Q^2),$$

Borel Prescription Tables (Expansion)

- By plugging that back into the cross section, we get

$$\frac{1}{\sigma_0}\frac{d\sigma^{BP}}{dQ^2dY} = \sum_{q\bar{q}} e_q^2 \int_{\tau e^{2|Y|}}^{1} \frac{dz}{z}\Big[\frac{1}{2\pi i}\int_{C_N} dN(\frac{x_1^0}{\sqrt{z}})^{-N} f_A^{\pi}(N,Q^2)\Big] f_B^{W}(\frac{x_2^0}{\sqrt{z}},Q^2) C_{\rm BP}^{\rm res}(z,Q^2).$$

- And we can write our tables as

$$T(N,Q^2) = e_q^2 \int_{\tau e^{2|Y|}}^{1} \frac{dz}{z}(\frac{x_1^0}{\sqrt{z}})^{-N} f_B^{W}(\frac{x_2^0}{\sqrt{z}},Q^2) C_{\rm BP}^{\rm res}(z,Q^2)$$

- Such that the cross section is

$$\frac{1}{\sigma_0}\frac{d\sigma^{BP}}{dQ^2dY} = \sum_{q\bar{q}} \frac{1}{2\pi i}\int_{C_N} dN f_A^{\pi}(N,Q^2) T(N,Q^2).$$

Conclusions

- Resummation is important for describing the high-x behavior of PDFs
- We can have input on the debate on whether the pion PDF's high x_π behavior goes as $(1 - x_\pi)$ or $(1 - x_\pi)^2$
- Different prescriptions will give some theoretical uncertainties to the PDFs
- Single fits for the minimal prescription have been done, but need to be improved
- Codes for the Borel prescription are vastly improved, and close to being able to perform fits