# Constraining S-Matrix Bootstrap

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# General Setting(Revision)

Any S-matrix bootstrap problem can be descibed by:-

- **The Setup**: We consider **2-2** scattering of **bosons** in **d+1** dimensions. They can either be **scalars** or have **O(N) symmetry** and are assumed to have the same mass.
- **The Constraints**: The s-matrix should satisfy **Crossing symmetry** and can have analytic structures like **Bound states**, **Branch cuts**, **Poles**, **Resonances** etc.
- **The Ansatz**: We make an ansatz for the S-matrix based on the constraints we have above
- **The S-matrix**: Linear combinations of parameters are **extremized** under **unitarity**(imposed numerically) to get unique S-matrices.

# 1+1 dim Scalars Revision

- The **main** purpose of 1+1 dim bootstrap was to see whether numerical solutions match with known analytic S-matrices under the same extremization scheme.
- This is **VERY** important since analytic maximization cannot be carried out in **3+1 dimensions**. Hence 1+1 dimensions make S-Matrix Bootstrap trustworthy!
- We considered $2 \to 2$ scattering of scalars and made the following ansatz for our S-matrix (through dispersion relation):

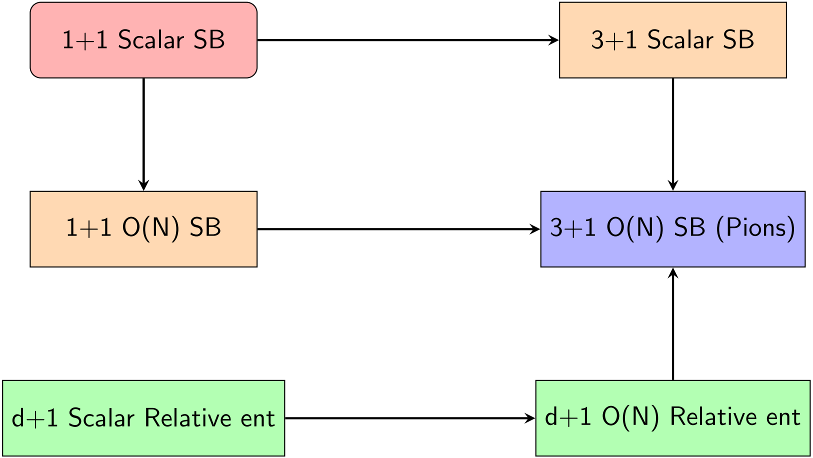
$$S(s) = S_\infty - \sum_j J_j\left(\frac{g_j^2}{s - m_j^2} + \frac{g_j^2}{4m^2 - s - m_j^2}\right) + \sum_{i=0}^{n} \rho_i \, K_i(s)$$

- We then maximized $g^2$ for single bound state case to get the sine gordon model. It is also the result if we maximize $g^2$ analytically. Hence we have agreement

# 1+1 dim scalars doubts

- Meaning of parameters: $g^2$ marks the depth of the potential with which the bound state interacts
- Meaning of parameters: $\rho_i$ is related to the total cross section at that $s_i$ through optical theorem.
- Regge behaviour (J vs $m^2$) is being shown (a recent observation)
- Today we will explore the bootstrap of $2 \rightarrow 2$ scattering of pions in 3+1 d. The S-matrix will have O(3) symmetry.

# General Overview



# Main Objective

Considering $2 \to 2$ pion scattering we get Fig 1. Main components are: The **lake(L)**, The **Peninsula(R)**, The **River(R)**, The **Green regions** and The **Red cross**. Goal: **Understand Fig 1 !**

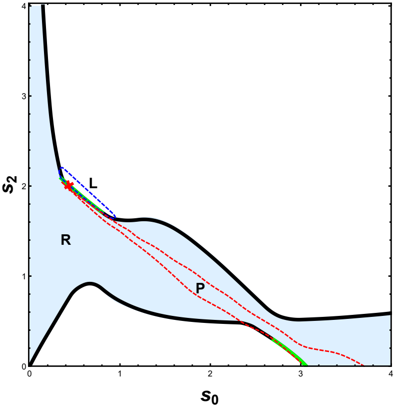



Figure 1: Allowed Space of Adler Zeroes. Red Cross represents the location of adler zeroes of 1-loop $\chi PT$

- $\langle p_3, c, p_4, d|S|p_1, a, p_2, b\rangle = \mathbb{1} + i\,\delta(p_1 + p_2 - p_3 - p_4)\,\mathcal{M}^{c,d}_{a,b}(s,t,u)$
- Due to $O(3)$ symmetry(Invariant decomposition),

$$\mathcal{M}^{c,d}_{a,b}(s,t,u) = A(s|t,u)\delta_{ab}\delta^{cd} + A(t|u,s)\delta^c_a\delta^d_b + A(u|s,t)\delta^d_a\delta^c_b$$

- We also have irrdeducible decomposition,

$$\begin{aligned}\mathcal{M}(s,t,u) = &\,(3A(s|t,u) + A(t|u,s) + A(u|s,t))\mathbb{P}_0 \\ &+ (A(t|u,s) - A(u|s,t))\mathbb{P}_1 + (A(t|u,s) + A(u|s,t))\mathbb{P}_2\end{aligned}$$

- Partial waves 1:

$$S^{(I)}_\ell(s) = 1 + i\,\frac{\sqrt{(s-4)}}{32\,\pi\,\sqrt{s}}\int_{-1}^{1} dx\,P_\ell(x)\,\mathcal{M}^{(I)}(s,t)\big|_{t\to\frac{1}{2}(s-4)(x-1)}$$

- Partial waves 2:

$$\mathbb{M}(s,t,u) = 32\pi \sum_{I=0,1,2} \mathbb{P}_I \sum_{\ell=0} (2l+1)\, T_\ell^{(I)}(s)\, P_\ell\left(x = \frac{u-t}{u+t}\right)$$

- Related to $S_\ell^{(I)}(s)$ by: $T_\ell^{(I)}(s) = \frac{\sqrt{s-4}}{\sqrt{s}} \frac{S_\ell^{(I)}(s)-1}{2i}$
- Scattering lengths($a_\ell^{(I)}$) and Effective energies($b_\ell^{(I)}$) are defined by:

$$\text{Re}(T_\ell^{(I)}) = k^{2l}(a_\ell^{(I)} + k^2 b_\ell^{(I)} + O(k^4))$$

where $k = \frac{\sqrt{s-4}}{2}$

- By crossing symmetry we have $\mathcal{M}_{a,b}^{c,d}(s,t,u) = \mathcal{M}_{a,c}^{b,d}(t,s,u)$, which implies:-

$$A(s|t,u)\delta_{ab}\delta^{cd} + A(t|u,s)\delta_a^c\delta_b^d + A(u|s,t)\delta_a^d\delta_b^c$$
$$= A(t|s,u)\delta_{ac}\delta^{bd} + A(s|u,t)\delta_a^b\delta_c^d + A(u|s,t)\delta_a^d\delta_b^c$$

  Comparing delta functions we realise that $A(s|t,u)$ must be symmetric in their last two entries

- Branch cuts for $(s > 4, t > 4, u > 4)$. Also, no bound states

- We assume maximal analyticity.

- Unitarity : $(\mathbb{S}_{tot}^{(I)})^\dagger \mathbb{S}_{tot}^{(I)} = \mathbb{1} \implies |S_\ell^{(I)}(s)|^2 \leq 1$ for $s \geq 4$

# Pion Bootstrap

## Constraints 2(1/5 Complete)

To specialize for pions we introduce two new constraints:-

- Resonances: Unstable particles

$$S_1^{(1)}(m_\rho^2) = 0$$

where $m_\rho = 5.5 - 0.5i$

- Adler zeroes: Present in the unphysical region of $s \in (0, 4)$.

$$S_0^{(0)}(s_0) = 1 \text{ and } S_0^{(2)}(s_2) = 1$$

Tree Level value is $(s_0, s_2) = (0.5, 2)$ and one loop $\chi PT$ value is $(s_0, s_2) = (0.437, 2.02)$. The one loop value is the **Red Cross**.

# Pion Bootstrap

## Ansatz 1

We first undergo a change of variables :

$$x \to \rho_x = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - x}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - x}}, \qquad x = \frac{s_0(1-\rho_x)^2 + 16m^2\rho_x}{(1+\rho_x)^2}$$

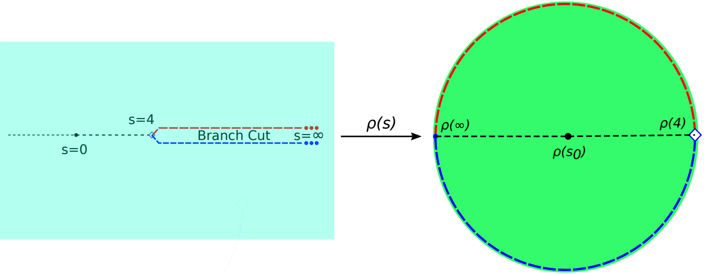



Figure 2: Mapping the complex $s$ plane onto a unit disk

# Pion Bootstrap

## Ansatz 2

- We first relax the constraint of $s + t + u = 4$. Reimposed while calculating partial waves
- Now, inside the poly-disk $\Delta^3$ where $|\rho_s| < 1, |\rho_t| < 1, |\rho_u| < 1$, $A(s|t,u)$ is analytic except for simple poles(There are none here)

$$A(s|t,u) = \sum_{n\leq m}^{N_{max}} a_{nm}(\rho_t^n \rho_u^m + \rho_t^m \rho_u^n) + \sum_{n,m}^{N_{max}} b_{nm}(\rho_t^n + \rho_u^n)\rho_s^m$$

- Crossing symmetry is inbuilt.
- Unitarity , $|S_\ell^{(I)}(s)|^2 \leq 1$ , is imposed using SDPB (Semi-Definite Program for Bootstrap) upto a finite $N_{max}$ and $L_{max}$ for a grid of s-values. Details about SDPB and imposition of unitarity is given in Appendix A.

# Pion Bootstrap

## The Lake 1

Now that we have an ansatz, we require need a valid problem to tackle. This is given by the adler zeroes...

- Recall that the adler zeroes $s_0$ and $s_2$ are located in the unphysical region in $(0, 4)$. Hence we cannot measure them experimentally.
- We will use bootstrap to see which pairs of adler zeroes $s_0$ and $s_2$ are allowed by unitarity and $\rho$ resonance.
- We intend to see whether $T_0^{(0)}(s_0) = 0$ and $T_0^{(2)}(s_2) = 0$ for some $s_0, s_2$ can be imposed together or not.
- We find that not all pairs of adler zeroes $(s_0.s_2)$ are allowed. This gives us a closed figure known as the **Lake**[1].

[1] Andrea L. Guerrieri, Juao Penedones, Pedro Vieira, "Bootstrapping QCD : the Lake, the Peninsula and the Kink, " arXiv:1810.12849v1 [hep-th]

# Pion Bootstrap

## The Lake 2(2/5 complete)

Constraints imposed:

- Unitarity: $|S_\ell^{(I)}(s)|^2 \leq 1$
- Crossing: $A(s|t,u) = A(s|u,t)$
- Rho Resonance : $S_1^{(1)}(m_\rho^2) = 0$
- Checked which pairs of $(s_0, s_2)$ are allowed.

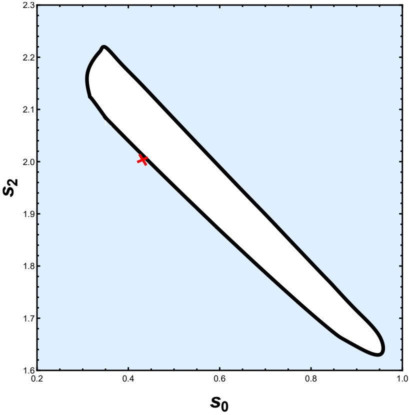



Figure 3: Disallowed region in the space of $s_0$ and $s_2$ where both $s_0$ and $s_2$ cannot be imposed. Cross= $\chi PT$

# Pion Bootstrap

## The Lake 3

Some curious features about the lake boundary:-

- The S-matrices on the lake boundary are unique
- These S-matrices are the "free-est" theories on the adler zero plane. This is because the resonance condition $S_1^{(1)}(m_{\rho^2}) = 0$ is a non-perturbative condition.
- 1-loop $\chi PT$ lies very close to the lower lake boundary.

- Next we constrain $a_0^{(0)}, a_0^{(2)}$ and $a_1^{(1)}$ to be within their experimental errors.
- Re-evaluating the lake, we get the **Peninsula!**

Constraints imposed:

- Unitarity: $|S_\ell^{(I)}(s)|^2 \leq 1$
- Crossing: $A(s|t,u) = A(s|u,t)$
- Rho Resonance : $S_1^{(1)}(m_\rho^2) = 0$
- $\ell = 0$ : $0.2162 \leq a_0^{(0)} \leq 0.2230$ , $-0.0456 \leq a_0^{(2)} \leq -0.0432$
- $\ell = 1$ : $0.036 \leq a_1^{(1)} \leq 0.040$

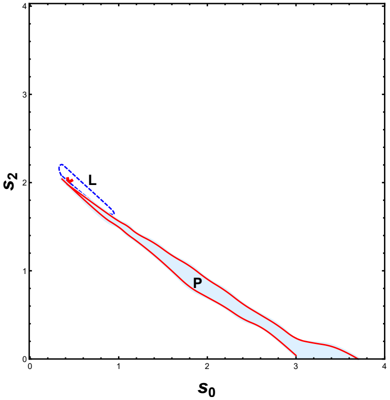



Figure 4: The Peninsula and The Lake

# What Next?

We saw the following:

- With just the resonance condition, the space of S-matrices disallowed (The Lake) was very small.
- To remove more regions they decided to constrain the S and P wave scattering lengths. Though it did have an impact, there was still a huge allowed region left(The Peninsula).

We wish to increase the size of the lake using theoretical constraints. This is given by the relative entropy formalism which will be discussed next.

# Introduction to Entropy

## Relative Entropy

The **"Relative Entropy"**[2] gives an idea of the difference between two Probability distributions.

If we observe a random event $X$ for $N$ times and come up with a hypothesis for the distribution as $\mathcal{Q}$, while the underlying theory is $\mathcal{P}$, we get

$$\begin{aligned} \text{likelihood of data } &= P(X^N) \approx 2^{-NS(\mathcal{P}||\mathcal{Q})}\,, \\ S(\mathcal{P}||\mathcal{Q}) &= \sum_i p_i(\ln p_i - \ln q_i)\,, \end{aligned} \tag{2.1}$$

which gives us a concrete method for hypothesis testing and **compare two competing theories.** The quantum equivalent is

$$S(\rho_A||\sigma_A) = \mathrm{Tr}_A\left(\rho_A(\ln(\rho_A) - \ln(\sigma_A))\right)\,. \tag{2.2}$$

[2] E. Witten, "A Mini-Introduction To Information Theory," (2020) [arXiv:1805.11965 [hep-th]].

# Configuration

Consider the reaction
$A(a) + B(b) \to C(c) + D(d)$
of O(3) symmetric particles of same mass, where A,B,C,D denote the particles and a,b,c,d the group indices. The spatial momenta of the outgoing particles are given by

$$\vec{p} \equiv (p, \theta^C, \varphi^C), \quad \vec{q} \equiv (q, \theta^D, \varphi^D) \quad (2.3)$$

In the **CoM frame**, we will necessarily have $\vec{p} + \vec{q} = 0$ and thus

$$p = q, \quad \theta^C + \theta^D = \pi. \quad (2.4)$$

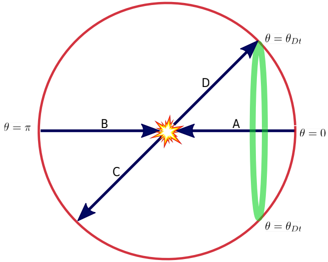



Figure 5: 2-2 scattering configuration in the centre of mass frame, with a Gaussian detector being placed at a point along the ring.

# Density Matrix of joint system $AB$

The initial 2-particle Fock state is chosen to be

$$|i\rangle := |\vec{k}\rangle\!\rangle_{a,b} := \left|\vec{k}, a, -\vec{k}, b\right\rangle \, , \tag{2.5}$$

which is a separable one by construction. The final state is found through a projector as

$$|f_{CD}\rangle = \mathcal{Q}^{(F)}_{CD}\mathcal{S}\,|\vec{k}\rangle\!\rangle = \int d\Pi_{C\vec{p}}d\Pi_{D\vec{q}}\,F(\theta_D)\,|\vec{p}, c, \vec{q}, d\rangle\,\langle\vec{p}, c, \vec{q}, d|\,\mathcal{S}\,|\vec{k}\rangle\!\rangle_{a,b}\,. \tag{2.6}$$

where $F(\theta) = F(\theta;\alpha)$ represents the particle detector at angle $\alpha$ with a **finite angular resolution**, $\Delta\alpha \ll 1$. Thus, the density matrix of the joint system is given by

$$\rho^{(F)}_{CD} := \frac{1}{\mathcal{N}}\,|f_{CD}\rangle\,\langle f_{CD}|\, , \tag{2.7}$$

We then calculate the reduced density matrix of the subsystem C by tracing over momenta with index D thus giving $\rho_C = \mathrm{Tr}_D\left[\rho_{CD}\right]$

# Replica Trick

We then turn to the **Replica Trick**, which is a simple way of calculating the log of a matrix. It arises due to the following identity

$$f(x)\ln g(x) = \lim_{n\to 1} \frac{\partial}{\partial n}\left(f(x)g(x)^{n-1}\right) . \qquad (2.8)$$

This allows us to write

$$D(\rho_C^{(1)} \| \rho_C^{(2)}) = \lim_{n\to 1} \frac{\partial}{\partial n}\left(\mathrm{Tr}_C\left(\rho_C^{(1)}\right)^n - \mathrm{Tr}_C\rho_C^{(1)}\left(\rho_C^{(2)}\right)^{n-1}\right) . \qquad (2.9)$$

where $D(\rho_C^{(1)} \| \rho_C^{(2)})$ is the relative entropy.

# Probability Density

For this purpose, we calculate the trace of the $n^{th}$ power of the reduced density matrix as

$$\mathrm{Tr}_C \left(\rho_C^{(F)}\right)^n \equiv \mathrm{Tr}_C \left(\rho_C^{(g)}\right)^n = \left[\frac{\delta(0)}{2\pi k^2 \delta^{(3)}(0)}\right]^{n-1} \int_{-1}^{1} dx \left[\mathcal{P}_{(g)}(x)\right]^n \qquad (2.10)$$

with,

$$\mathcal{P}_g(x) = \frac{g(x)\, |_{a,b} \langle\!\langle \vec{p}|\mathbf{s}|\vec{k}\rangle\!\rangle_{c,d}|^2}{\int_{-1}^{1} dx\, g(x)\, |_{a,b} \langle\!\langle \vec{p}|\mathbf{s}|\vec{k}\rangle\!\rangle_{c,d}|^2} \geq 0\,, \qquad (2.11)$$
$$\int_{-1}^{1} dx\, \mathcal{P}_g(x) = 1\,.$$

Hence, $\mathcal{P}_g(x)$ can be interpreted as a **probability density** function which takes into account the probability of the particle scattering at an angle $x = \cos\theta$ along with the probability of the detector detecting it.

# Relative Entropy

Similarly, using the replica trick, we get the quantum relative entropy as

$$\begin{aligned} D\left(\rho_C^{(1)} \| \rho_C^{(2)}\right) &= \int_{-1}^{1} dx\, \mathcal{P}_{g_1}(x)\, \ln\left(\frac{\mathcal{P}_{g_1}(x)}{\mathcal{P}_{g_2}(x)}\right) \\ &= -\int_{-1}^{1} dx\, \mathcal{P}_{g_1}(x)\, \ln\left(\frac{\mathcal{P}_{g_2}(x)}{\mathcal{P}_{g_1}(x)}\right) \\ &\geq \int_{-1}^{1} dx\, \mathcal{P}_{g_1}(x) \left(1 - \frac{\mathcal{P}_{g_2}(x)}{\mathcal{P}_{g_1}(x)}\right) \\ &\geq 0\,, \end{aligned} \tag{2.12}$$

where we have used the concavity of ln function.

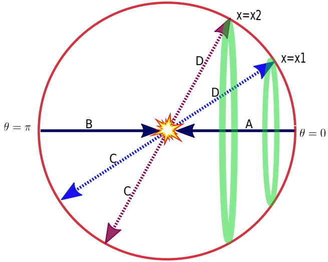



Figure 6: Two different configurations of Gaussian Detectors for 2 to 2 scattering.

# Relative Entropy : Gaussian detectors

We will consider a *Gaussian profile* for our detectors, $g(x)$,

$$g(x) \equiv \delta_\sigma(x-y) = \frac{1}{2\sqrt{2\sigma}}\, e^{-\frac{(x-y)^2}{4\sigma}} \xrightarrow{\sigma\to 0} \delta(x-y)\,. \tag{2.13}$$

With the detectors at two angle, $g_i(x) = \delta_\sigma(x-x_i)$, $x_1$ and $x_2 = x_1 - \Delta x$, we get

$$D\left(\rho_C^{(1)} \| \rho_C^{(2)}\right) \approx \ln\left(\frac{\mathcal{I}_g(s,x_2)}{\mathcal{I}_g(s,x_1)}\right) + (\Delta x)\frac{\partial}{\partial x}\left(\ln\left(\frac{\mathcal{I}_g(s,x)}{\mathcal{I}_g(s,x_1)}\right)\right)\Bigg|_{x_1} + \frac{(\Delta x)^2}{4\sigma}\,, \tag{2.14}$$

where

$$\mathcal{I}_g(s,x_0) = \sum_{i=0}^{\infty} \frac{\partial^{2i}}{\partial x^{2i}}\left(|\mathcal{M}_{a,b}^{c,d}(s,x)|^2\right)\Bigg|_{x_0} \frac{\sigma^i}{i!} = |\mathcal{M}_{a,b}^{c,d}(s,x_0)|^2 + O(\sigma)\,. \tag{2.15}$$

# Relative Entropy : Gaussian detectors

In the limit of small $\Delta x := x_1 - x_2$ and $\sigma \to 0$, one obtains

$$D\left(\rho_C^{(1)} \| \rho_D^{(2)}\right) \approx \frac{(\Delta x)^2}{4\sigma} + \frac{(\Delta x)^2}{2}\Bigg( \frac{\partial^2}{\partial x^2}\left(\frac{|\mathcal{M}_{a,b}^{c,d}(s,x)|^2}{|\mathcal{M}_{a,b}^{c,d}(s,x_1)|^2}\right)\Bigg|_{x_1} - \left(\frac{\partial}{\partial x}\left(\frac{|\mathcal{M}_{a,b}^{c,d}(s,x)|^2}{|\mathcal{M}_{a,b}^{c,d}(s,x_1)|^2}\right)\Bigg|$$

$$+ O((\Delta x)^2\sigma)\,. \tag{2.16}$$

The first term is the **hard-sphere scattering part**,since it has no angular dependence. The remaining is what we call the **Quantum Relative Entropy**,

$$D_Q\left(\rho_C^{(1)} \| \rho_C^{(2)}\right) = \frac{(\Delta x)^2}{2}\frac{\partial^2}{\partial x^2}\left(\ln\left(\frac{|\mathcal{M}_{a,b}^{c,d}(s,x)|^2}{|\mathcal{M}_{a,b}^{c,d}(s,x_1)|^2}\right)\right)\Bigg|_{x_1} + O((\Delta x)^2, \sigma)\,. \tag{2.17}$$

# $\chi PT$ amplitudes

We now consider $\chi PT$. When we go beyond the tree level, $\chi PT$ becomes *non-renormalizable.* To incorporate 1-loop effects, one writes down a four-derivative counter-term Lagrangian with several **low energy constants (LEC's)**, 10 at the 1-loop level.

The state of the art is two-loops, although at 1-loop results [3], the $s-$channel amplitude looks like

$$\begin{aligned} A(s,t,u) &= \frac{1}{f^2}(s-1) + \frac{1}{f^4}\left(b_1 + b_2 s + b_3 s^2 + b_4 (t-u)^2\right) \\ &+ \frac{1}{f^4}\left(F^{(1)}(s) + G^{(1)}(s,t) + G^{(1)}(s,u)\right). \end{aligned} \tag{2.18}$$

$$\begin{aligned} \mathcal{M}_{ab\to cd}(s,t(x),u(x)) &= A(s,t,u)\delta_{ab}\delta^{cd} + A(t,u,s)\delta_a^c\delta_b^d + A(u,s,t)\delta_a^d\delta_b^c, \\ &= \sum_I M^{(I)}(s,t,u)P_I \end{aligned} \tag{2.19}$$

where $a\,,b\,,c\,,d$ are $O(3)$ group indices

[3] J. Gasser and H. Leutwyler, "Chiral Perturbation Theory to One Loop," Annals Phys. **158** (1984), 142.

# $\chi PT$ Reactions

We will be studying the following 3 reactions

- $\pi^0 \pi^0 \longrightarrow \pi^0 \pi^0$ : $\mathcal{M}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$
- $\pi^+ \pi^+ \longrightarrow \pi^+ \pi^+$ : $\mathcal{M}(s,t,u) = A(t,u,s) + A(u,s,t)$
- $\pi^+ \pi^- \longrightarrow \pi^0 \pi^0$ : $\mathcal{M}(s,t,u) = A(s,t,u)$

All the rest of the reactions will be related to these 3 through *Crossing Symmetry* which also dictates $A$ to be symmetric in its last two arguments.

# $\chi PT$ Scattering Lengths

For the partial wave expansion

$$\mathcal{M}^{(I)}(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1)\,T_\ell^{(I)}(s)P_\ell(x)\,, \tag{2.20}$$

the **scattering lengths**, $a_\ell^{(I)}$, are defined as

$$T_\ell^{(I)}(s) = \left(\frac{s-4m^2}{4}\right)^{\ell}\left(a_\ell^{(I)}(s) + b_\ell^{(I)}\left(\frac{s-4m^2}{4}\right) + \cdots\right). \tag{2.21}$$

In terms of these, $D_Q$ near the threshold takes the form (in leading order)

$$D_Q(\rho_1||\rho_2) = 15(\Delta x)^2\,\frac{\sum_I c_I\, a_2^{(I)}}{\sum_I c_I\, a_0^{(I)}}\left(s-4m^2\right)^2\,. \tag{2.22}$$

Here $\sum_I c_I\, a_\ell^{(I)}$ is a linear combination of the scattering lengths **depending on the reaction being considered**. To fix the sign of $D_Q(\rho_1||\rho_2)$ we need some scattering length inequalities.

- The scattering lengths were given by:

$$a_\ell^{(I)} = \frac{4^\ell \ell!}{(2\ell+1)} \frac{\partial^\ell}{\partial t^\ell} \left(\mathcal{M}^{(I)}(s,t)\right)\Big|_{s=4,t=0}$$

- We use the froissart gribov formula to get:

$$\frac{\partial^2}{\partial s^2} \left(\mathcal{M}^{(I)}(s,t)\right)\Big|_{s=0,t=4} = \frac{2}{\pi} \sum_J \left( \int_{4m^2}^{\infty} \frac{ds'}{(s')^3} \left(\delta^{IJ} + C_{su}^{IJ}\right) \mathrm{Im}[\mathcal{M}^{(J)}(s',4)] \right)$$

- Using positivity of Absorptive part(Optical Theorem) in the physical region and positivity of legendre polynomials(in $x \geq 1$) we get(in [5][4]):

$$a_2^{(0)} + 2a_2^{(2)} \geq 0\,, \quad a_2^{(0)} - a_2^{(2)} \geq 0\,, \quad 2a_2^{(0)} + a_2^{(2)} \geq 0\,, \quad a_2^{(0)} \geq 0$$

[4] A. V. Manohar and V. Mateu, "Dispersion Relation Bounds for pi pi Scattering," [arXiv:0801.3222 [hep-ph]].

# Scattering Length Constraints

## From Chiral Perturbation Theory

- To get $I = 0$ scattering length we use results from 1-loop $\chi PT$.
- The scattering lengths have the following expressions in terms of LEC's:

$$
\begin{aligned}
a_0^{(0)} &= \frac{7}{f^2} + \frac{1}{f^4}\left(5\, b_1 + 12\, b_2 + 48\, b_3 + 32\, b_4 + \frac{49}{16\,\pi^2}\right), \\
a_0^{(2)} &= -\frac{2}{f^2} + \frac{2}{f^4}\left( b_1 + 16\, b_4 + \frac{1}{8\,\pi^2}\right),
\end{aligned}
$$

- Using the sign of the leading order result the following inequalties hold:

$$
a_0^{(0)} + 2a_0^{(2)} \geq 0\,, \quad 2a_0^{(0)} + a_0^{(2)} \geq 0\,, \quad a_0^{(0)} - a_0^{(2)} \geq 0\,, \quad a_0^{(2)} \leq 0\,.
$$

- We assume any physical theory to respect these two sets of inequalities

- Using the S and D wave inequalities we constrain any physical theory to respect the following signs of $D_Q$:

$$D_Q(\pi^0+\pi^0\to\pi^0+\pi^0)\geq 0\,,\quad D_Q(\pi^++\pi^-\to\pi^0+\pi^0)\geq 0\,,$$
$$D_Q(\pi^++\pi^+\to\pi^++\pi^+)\leq 0\,,\quad D_Q(\pi^++\pi^0\to\pi^++\pi^0)\leq 0\,.$$

- Next we turn to impose these signs.

# S and D Constraints

## The River(4/5 complete)

We re-evaluate the lake with the S and D scattering length inequalities. This gives us the **River!** (4/5 complete).

Constraints imposed:

- Unitarity: $|S_\ell^{(I)}(s)|^2 \leq 1$
- Crossing: $A(s|t,u) = A(s|u,t)$
- Rho Resonance : $S_1^{(1)}(m_\rho^2) = 0$
- S-wave Inequalities
- D-wave Inequalities

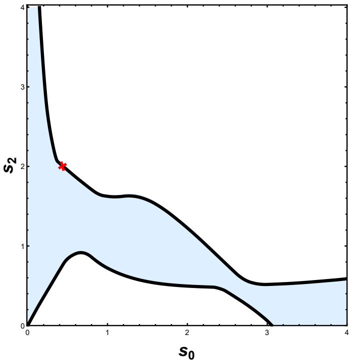



Figure 7: New allowed region after imposing the constraints. Note that the 1-loop $\chi PT$ lives close to the "kink" in the upper bank.

# Hypothesis Testing

## Green Regions(5/5 complete)

- Now, the river boundary satisfies all the monotonicity constraints by definition. But which of these S-matrices are close to $\chi PT$?
- This is answered by Hypothesis testing. We calculate the relative entropy $D(\rho_{Boot}||\rho_{\chi PT})$ for different $\chi PT$ reactions around the river and see where it is minimized overall.
- We find two regions, one in the upper bank and another in the lower bank. These are the **Green Regions**.
- Quite remarkably the scattering lengths in these regions are close to the experimental values in comparison to other regions.

# Fig 1 revisited

We have now understood this figure! **Mission Accomplished**...

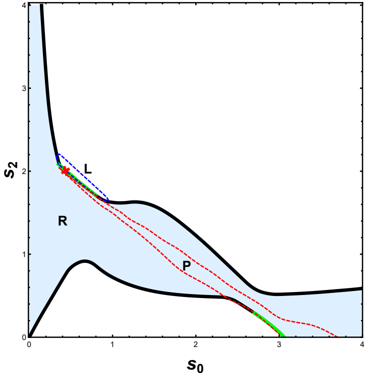



Figure 8: Allowed Space of Adler Zeroes

# Discussion

## Stuff we are trying in Bootstrap

- We want to extract the resonances of points on the River boundary and see whether they fall on the Regge Trajectory(Plot $\ell$ *vs* $m^2$).
- We are also looking at Positivity constraints and the Martin inequalities as a substitute of the unitarity conditions.
- We are searching for more theoretical/perturbative motivations to further decrease the allowed region.
- Looking for a way to include subadditivity constraints in bootstrap.

To impose unitarity, we shall use SDPB as defined in [4][5]. SDPB solves the following problem: It maximizes,

$$b_0 + \vec{b}.\vec{y} \quad \text{over } \vec{y} \in \mathbb{R}^N$$

where $y$ is the set of $N$ parameters under the constraint that,

$$M_j^0(x) + \sum_{n=1}^{N} y_n M_j^n(x) \geq 0 \quad \forall\, x \geq 0 \text{ and } 1 \leq j \leq J$$

where $M_j^i(x)$ are symmetric matrices with polynomial entries.

We will show that unitarity constraints can be written in the form of the above constaint with $2 \times 2$ matrices.

[5]D. Simmons-Duffin, "A Semi-definite Program Solver for the Conformal Bootstrap," JHEP 06 (2015) 174, arXiv:1502.02033 [hep-th]

- We can write

$$\mathcal{M}(s,t,4-s-t) = \vec{y}.\vec{\mathcal{M}}(s,t) \tag{5.1}$$

which leads to $S_\ell(s) = 1 + i\vec{y}.\vec{f}_\ell(s)$

- The unitarity constraints :

$$\begin{aligned} 0 \leq \left(1-\vec{y}.\vec{I}_\ell\right)^2 + \left(\vec{y}.\vec{R}_\ell\right)^2 \leq 1 \\ \implies U_\ell \equiv 2\vec{y}.\vec{I}_\ell - (\vec{y}.\vec{I}_\ell)^2 - (\vec{y}.\vec{R}_\ell)^2 \geq 0 \\ \text{and also} \quad U_\ell \leq 1 \end{aligned} \tag{5.2}$$

- This is equivalent to a PMP with

$$M_\ell \equiv \begin{pmatrix} 1+\vec{y}.\vec{R}_\ell & 1-\vec{y}.\vec{I}_\ell \\ 1-\vec{y}.\vec{I}_\ell & 1-\vec{y}.\vec{R}_\ell \end{pmatrix} \tag{5.3}$$

- So our input consists of these unitarity matrices,conditions like adler zeroes and resonances and what to maximize.

We turn to $\chi PT$ [6] which is a famous effective field theory to understand the *low energy phenomenology* of QCD, while being approximately consistent with the underlying symmetry (which exactly match in the chiral limit *i.e.,* quark mass going to 0).

The lowest order effective chiral lagrangian in the 8 pseudo-Goldstone bosons, $\phi_a$ (3 **Pions**, 4 **Kaons** and 1 $\eta$-**meson**), is

$$\mathcal{L}_2 = \frac{1}{4} f^2 \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{2} f^2 B\, \mathrm{Tr}\left(M^\dagger U + M U^\dagger\right) , \tag{5.4}$$

where the $M$ is the QCD mass matrix, $U = \exp(i\phi_a T^a / f)$ with $T^a$ being the generators of the $SU(3)$, and $f$ turns out to be the pion decay constant, $f_\pi \approx 93 MeV$.

[6] B. Ananthanarayan, "Chiral perturbation theory for nuclear physicists," [arXiv:hep-ph/9712525 [hep-ph]].