Group Meeting Presentation

Aug. 5, 2020

QCD_{1+1} Mass Gap Equation and Interpretation

Fermion Propagator

Free Propagator ; $S_0(p) = \dfrac{1}{\not{p} - m + i\epsilon}$

Interacting Propagator ;

$$S(p) = \frac{1}{\not{p} - m - \hat{\Sigma}(p) + i\epsilon}$$

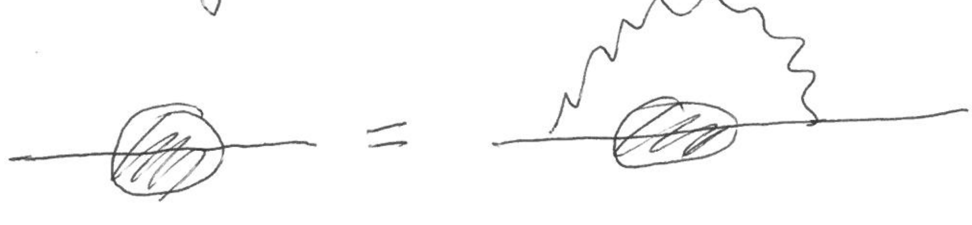
$$= \frac{F(p)}{\not{p} - M(p) + i\epsilon}$$

For $\hat{\Sigma}(p) = \Sigma_v(p)\not{p} + \Sigma_s(p)$,

$$F(p) = (1 - \Sigma_v(p))^{-1}$$ "wave function renormalization"

$$M(p) = \frac{m + \Sigma_s(p)}{1 - \Sigma_v(p)}$$ " physical mass "

Diagramatically,



QCD_{1+1} Interpolating Mass Gap

$$\mathcal{L}_{QCD_{1+1}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\psi,$$

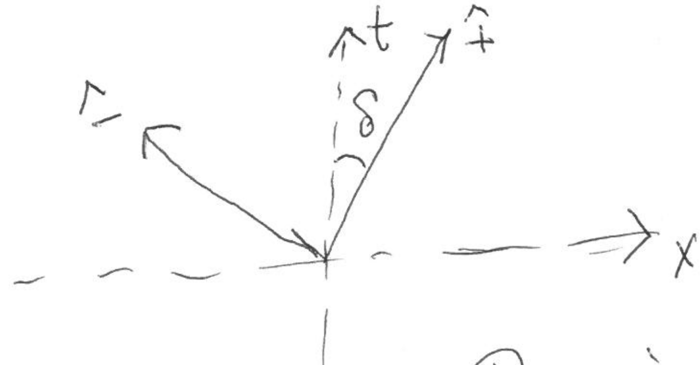
where $D_\mu = \partial_\mu - ig A^a_\mu t^a$

and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$

$$\text{IFD} \longleftrightarrow \text{LFD}$$

IFD		LFD
$\mu = 0, 1$	$\hat{\mu} = \hat{+}, \hat{-}$	$+, -$
t, x	$\cos\delta\, t + \sin\delta\, x$ $\sin\delta\, t - \cos\delta\, x$	$t+x,\ t-x$

$(\delta = 0)$ $\qquad$ $\left(\delta = \frac{\pi}{4}\right)$



Interpolating Dynamics

$$\begin{matrix} \mu, \nu & \longrightarrow & \hat{\mu}, \hat{\nu} & \longrightarrow & \mu, \nu \\ (0, 1) & & (\hat{+}, \hat{-}) & & (+, -) \end{matrix}$$

Interpolating Energy-Momentum Relation

Free particle of mass m in 1+1 Dim.

$$p_{\hat\mu}\, p^{\hat\mu} = g_{\hat\mu\hat\nu}\, p^{\hat\mu} p^{\hat\nu} = g^{\hat\mu\hat\nu}\, p_{\hat\mu} p_{\hat\nu} = m^2,$$

where

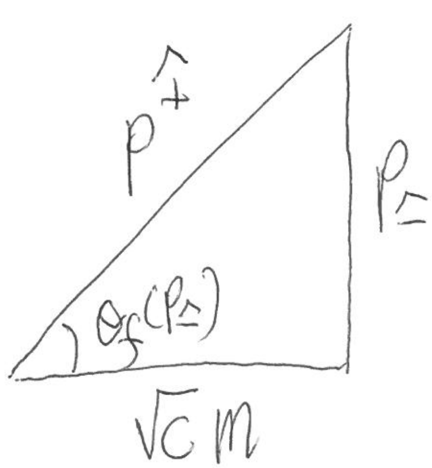
$$g^{\hat\mu\hat\nu} = g_{\hat\mu\hat\nu} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \text{ with } \begin{pmatrix} C = \cos 2\delta \\ S = \sin 2\delta \end{pmatrix}.$$

$$C\left(p_{\hat+}^2 - p_{\hat-}^2\right) + S\left(2 p_{\hat+} p_{\hat-}\right) = \frac{p^{\hat+ 2} - p_{\hat-}^2}{C} = m^2$$

$$p^{\hat+} = \sqrt{p_{\hat-}^2 + C m^2}$$

$C=1$: $p^0 = \sqrt{p_x^2 + m^2}$

$C=0$: $p^+ = p_-$



$$\tan\theta_f(p_{\hat-}) = \frac{p_{\hat-}}{\sqrt{C}\, m} \sim \text{“}\gamma\beta\text{”}$$

$C=1$: $\theta_f(p_x) = \tan^{-1}\frac{p_x}{m}$

$C=0$: $\theta_f(p^+) = \frac{\pi}{2}\,\mathrm{sgn}(p^+)$

In[ ]:= $$\text{Plot}\left[\text{ArcTan}\left[\frac{\text{Tan}[\xi]}{\sqrt{\text{Cos}[2\,\delta]}\ \ \text{m}}\right]\ /.\ \{\delta \to \{0.,\ 0.4,\ 0.6,\ 0.7,\ 0.785398\}\}\ /.\ \text{m} \to 0.18,\ \{\xi,\ -\pi\,/\,2,\ \pi\,/\,2\},\ \text{PlotRange} \to \text{All}\right]$$

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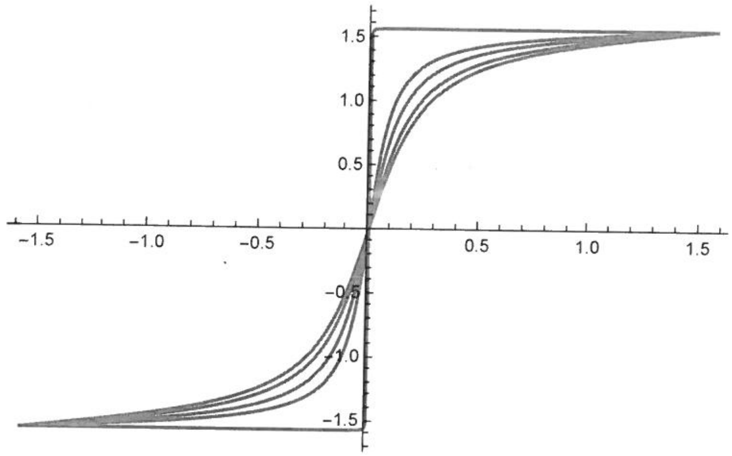

In[ ]:= $$\text{Plot}\left[\text{ArcTan}\left[\frac{\text{Tan}[\xi]}{\sqrt{\text{Cos}[2\,\delta]}\ \ \text{m}}\right]\ /.\ \{\delta \to \{0.,\ 0.4,\ 0.6,\ 0.7,\ 0.75,\ 0.78,\ 0.785,\ 0.785398\}\}\ /.\ \text{m} \to 2.11,\ \{\xi,\ -\pi\,/\,2,\ \pi\,/\,2\},\ \text{PlotRange} \to \text{All}\right]$$

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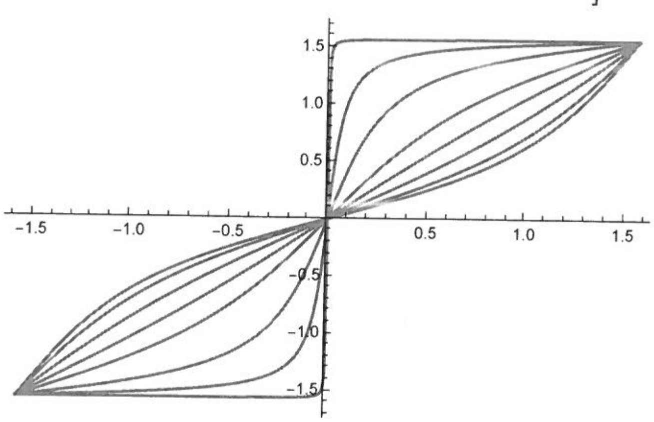


Interpolating Axial Gauge: $A^a_{\hat{-}} = 0$

$c=1$ (IFD) $\quad\quad$ $c=0$ (LFD)

$A^a_x = 0$

Axial Gauge

$A^{+a} = 0$

Light-Front Gauge

Bars & Green, PRD(1978)

Li, PRD(1986)

't Hooft, NPB(1974)

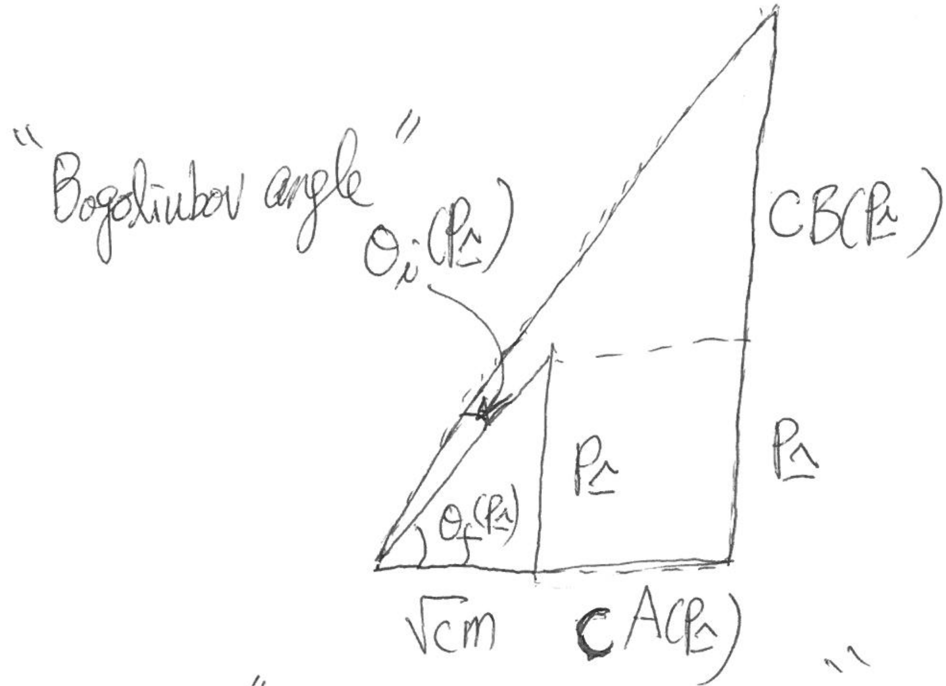
$$\lambda = \frac{g^2(N-\frac{1}{N})}{4\pi} \xrightarrow[g\to 0]{N\to\infty} \lambda = \frac{g^2 N}{4\pi}$$

"finite"

$$\hat{\Sigma}(p_{\hat{-}}) = i\frac{\lambda}{2\pi}\iint \frac{dk_{\hat{-}}\,dk_{\hat{+}}}{(p_{\hat{-}}-k_{\hat{-}})^2}\gamma^{\hat{+}}\frac{1}{\not{k}-m-\hat{\Sigma}(k_{\hat{-}})+i\epsilon}\gamma^{\hat{+}},$$

where $\hat{\Sigma}(p_{\hat{-}}) = \underbrace{\sqrt{C}\,A(p_{\hat{-}})}_{\Sigma_s} + \underbrace{B(p_{\hat{-}})}_{\Sigma_v p^{\hat{-}}}\gamma_{\hat{-}}$

For $\lambda \neq 0$, $A(p_{\hat{-}})$ and $B(p_{\hat{-}})$ exist!



"Free" $\longrightarrow$ "Interacting"

$$\theta_f(p_\triangle) \longrightarrow \theta_f(p_\triangle) + \theta_i(p_\triangle) \equiv \theta(p_\triangle)$$

$$\tan\theta_f(p_\triangle) = \frac{p_\triangle}{\sqrt{c}\,m} \longrightarrow \tan\theta(p_\triangle) = \frac{p_\triangle + CB(p_\triangle)}{\sqrt{c}\,m + CA(p_\triangle)}$$

Interpolating Mass Gap Equation for $\hat{\Sigma}(p_\triangle) = \sqrt{c}A(p_\triangle) + \gamma_\perp B(p_\triangle)$ can be summarized as the equation for $\theta(p_\triangle)$.

$$\boxed{\frac{p_\triangle}{\sqrt{c}}\cos\theta(p_\triangle) - m\sin\theta(p_\triangle) = \frac{\sqrt{c}\,\lambda}{2}\int_{-\infty}^{\infty}\frac{dk_\triangle}{(p_\triangle - k_\triangle)^2}\sin(\theta(p_\triangle) - \theta(k_\triangle))}$$

If $\lambda = 0$, then $\tan\theta(p_\triangle) = \dfrac{p_\triangle}{\sqrt{c}\,m}$ or $\theta(p_\triangle) = \theta_f(p_\triangle)$ as expected.

As $\sin(\theta(p_\Delta) - \theta(k_\Delta)) = \sin\theta(p_\Delta)\cos\theta(k_\Delta) - \cos\theta(p_\Delta)\sin\theta(k_\Delta)$,

we get

$$\left(\frac{p_\Delta}{\sqrt{c}} + \sqrt{c}\, B(p_\Delta)\right)\cos\theta(p_\Delta) = \left(m + \sqrt{c}\, A(p_\Delta)\right)\sin\theta(p_\Delta),$$

where

$$A(p_\Delta) = \frac{\lambda}{2}\int_{-\infty}^{\infty} \frac{dk_\Delta}{(p_\Delta - k_\Delta)^2}\cos\theta(k_\Delta)$$

and

$$B(p_\Delta) = \frac{\lambda}{2}\int_{-\infty}^{\infty} \frac{dk_\Delta}{(p_\Delta - k_\Delta)^2}\sin\theta(k_\Delta).$$

Let's scale to use dimensionless variables.

$$\left(\frac{p_\Delta}{\sqrt{c}\,m} + \frac{\sqrt{c}}{m} B(p_\Delta)\right)\cos\theta(p_\Delta) = \left(1 + \frac{\sqrt{c}}{m} A(p_\Delta)\right)\sin\theta(p_\Delta)$$

Using $\frac{p_\Delta}{\sqrt{c}\,m} = \tan\theta_f(p_\Delta)$, let's write $A(p_\Delta)$ and $B(p_\Delta)$.

$$\frac{\sqrt{c}}{m} A(p_\Delta) = \frac{\lambda}{2m^2} \int_{-\infty}^{\infty} \frac{d\left(\frac{k_\Delta}{\sqrt{c}m}\right)}{\left(\frac{p_\Delta}{\sqrt{c}m} - \frac{k_\Delta}{\sqrt{c}m}\right)^2} \cos\theta(k_\Delta)$$

$$\frac{\sqrt{c}}{m} B(p_\Delta) = \frac{\lambda}{2m^2} \int_{-\infty}^{\infty} \frac{d\left(\frac{k_\Delta}{\sqrt{c}m}\right)}{\left(\frac{p_\Delta}{\sqrt{c}m} - \frac{k_\Delta}{\sqrt{c}m}\right)^2} \sin\theta(k_\Delta)$$

Defining the dimensionless coupling $\alpha \equiv \frac{\lambda}{2m^2}$,

we get

$$\frac{\sqrt{c}\, A(p_\Delta)}{m} = \alpha \int_{-\infty}^{\infty} \frac{d \tan\theta_f(k_\Delta) \cos\theta(k_\Delta)}{\left(\tan\theta_f(p_\Delta) - \tan\theta_f(k_\Delta)\right)^2}$$

$$= \alpha \cos^2\theta_f(p_\Delta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_f(k_\Delta) \frac{\cos\theta(k_\Delta)}{\sin^2\left(\theta_f(p_\Delta) - \theta_f(k_\Delta)\right)}$$

$$\frac{\sqrt{c}\, B(p_\Delta)}{m} = \alpha \cos^2\theta_f(p_\Delta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_f(k_\Delta) \frac{\sin\theta(k_\Delta)}{\sin^2\left(\theta_f(p_\Delta) - \theta_f(k_\Delta)\right)}$$

$$\left(\tan\theta_f(p_\Delta) + \frac{\sqrt{c}\,B(p_\Delta)}{m}\right)\cos\theta(p_\Delta) = \left(1+\frac{\sqrt{c}\,A(k_\Delta)}{m}\right)\sin\theta(k_\Delta)$$

$$\tan\theta_f(p_\Delta)\cos\theta(p_\Delta) - \sin\theta(p_\Delta) = \frac{\sqrt{c}\,A(k_\Delta)}{m}\sin\theta(p_\Delta)$$

$$-\frac{\sqrt{c}\,B(k_\Delta)}{m}\cos\theta(p_\Delta)$$

$$= \alpha\cos^2\theta_f(p_\Delta)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_f(k_\Delta)\,\frac{\sin(\theta(p_\Delta)-\theta(k_\Delta))}{\sin^2(\theta_f(p_\Delta)-\theta_f(k_\Delta))}$$

Thus, we get

$$\sin(\theta_f(p_\Delta)-\theta(p_\Delta)) = \alpha\cos^3\theta_f(p_\Delta)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta_f(k_\Delta)\,\frac{\sin(\theta(p_\Delta)-\theta(k_\Delta))}{\sin^2(\theta_f(p_\Delta)-\theta_f(k_\Delta))}$$

$$\boxed{\sin\theta_i(p_\Delta) = \alpha\cos^3\theta_f(p_\Delta)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_f(k_\Delta)\,\frac{\sin(\theta(k_\Delta)-\theta(p_\Delta))}{\sin^2(\theta_f(k_\Delta)-\theta_f(p_\Delta))}}$$

Note that, as $\theta_f(k_\triangle) \to \pm\frac{\pi}{2}$, $\theta_i(k_\triangle) \to 0$

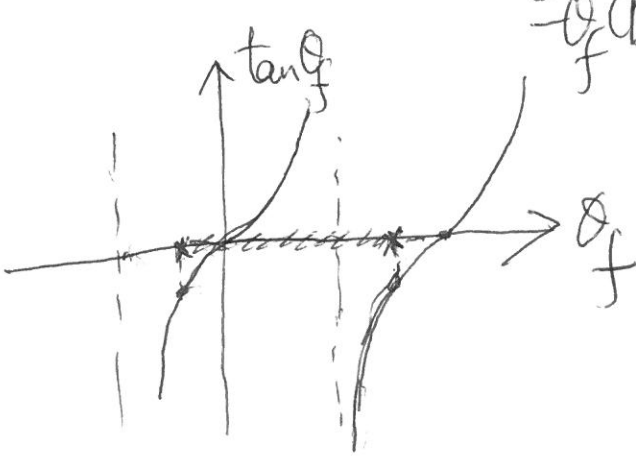
Shift the boundary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ by subtracting $\theta_f(P_\triangle)$, i.e. $\displaystyle\int_{-\frac{\pi}{2}-\theta_f(P_\triangle)}^{\frac{\pi}{2}-\theta_f(P_\triangle)} d\theta_f(k_\triangle) \cdots$,

Then,

$$\sin\theta_i(P_\triangle) = \alpha\cos^3\theta_f(P_\triangle)\int_{-\frac{\pi}{2}-\theta_f(P_\triangle)}^{\frac{\pi}{2}-\theta_f(P_\triangle)} d\theta_f(k_\triangle)\,\frac{\sin\theta(k_\triangle)}{\sin^2\theta_f(k_\triangle)}$$

$$= -\alpha\cos^3\theta_f(P_\triangle)\int_{-\theta_f(P_\triangle)}^{\pi-\theta_f(P_\triangle)} d\theta_f(k_\triangle)\,\frac{\cos\theta(k_\triangle)}{\cos^2\theta_f(k_\triangle)}$$

$$= -\alpha\cos^3\theta_f(P_\triangle)\int_{-\theta_f(P_\triangle)}^{\pi-\theta_f(P_\triangle)} d\tan\theta_f(k_\triangle)\,\cos\left(\theta_f(k_\triangle)+\theta_i(k_\triangle)\right)$$



Note that

$$\int_{-\tan\theta_f(p_\Delta)}^{\tan(\pi-\theta_f(p_\Delta))} dt \cos\left(\tan^{-1}t + \theta_i(t)\right)$$

$$= \int_{-\tan\theta(p_\Delta)}^{\tan(\pi-\theta(p_\Delta))} dt \cos\left(\tan^{-1}t\right)$$

$$= 2\sinh^{-1}\left(\tan\theta(p_\Delta)\right)$$

$$\sin\theta_i(p_\Delta) = -2\alpha\cos^3\theta_f(p_\Delta)\sinh^{-1}\left(\tan\theta(p_\Delta)\right)$$

$$\boxed{\sinh\left[\frac{\sin\theta_i(p_\Delta)}{-2\alpha\cos^3\theta_f(p_\Delta)}\right] = \tan\left(\theta_f(p_\Delta)+\theta_i(p_\Delta)\right)}$$

If $\theta_i(p_\Delta)$ in the RHS is taken to be zero, then the result agrees with the first order approximation, i.e.

$$\sin\theta_i(p_\Delta) = -2\alpha\cos^3\theta_f(p_\Delta)\sinh^{-1}\left(\tan\theta_f(p_\Delta)\right)$$

$$= \alpha\cos^3\theta_f(p_\Delta)\log\left(\frac{\tan\frac{\frac{\pi}{2}-\theta_f(p_\Delta)}{2}}{\tan\frac{\pi+\theta_f(p_\Delta)}{2}}\right)^2$$

$$= \alpha\left(\frac{cm^2}{p_\Delta^2+cm^2}\right)^{3/2}\log\left(\frac{\sqrt{cm^2+p_\Delta^2}-p_\Delta}{\sqrt{cm^2+p_\Delta^2}+p_\Delta}\right)$$

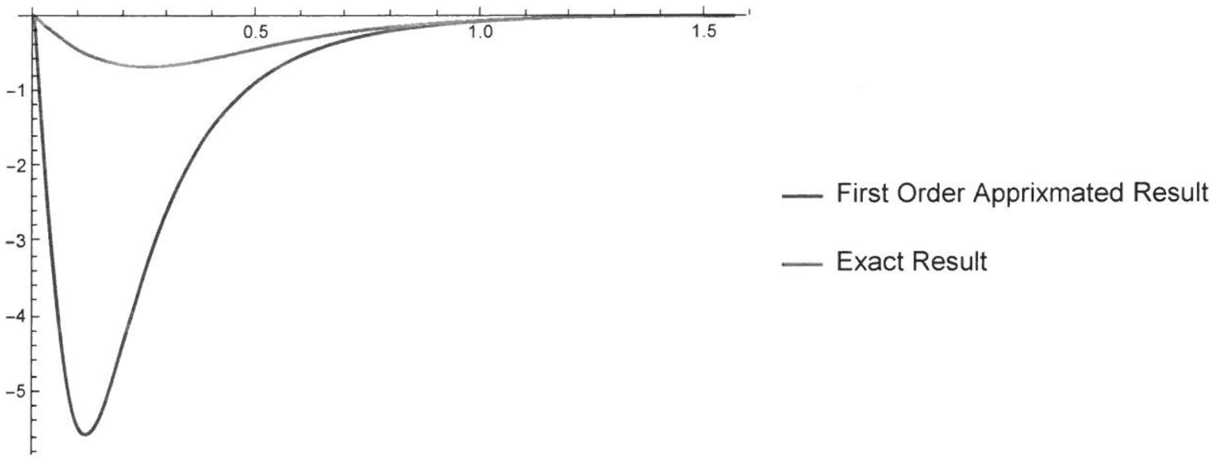
$$\exp\left[\frac{\sin\theta_i(P_\triangle)}{-2\alpha\cos^3\theta_f(P_\triangle)}\right] - \exp\left[\frac{\sin\theta_i(P_\triangle)}{2\alpha\cos^3\theta_f(P_\triangle)}\right]$$

$$= 2\tan\left(\theta_f(P_\triangle) + \theta_i(P_\triangle)\right)$$

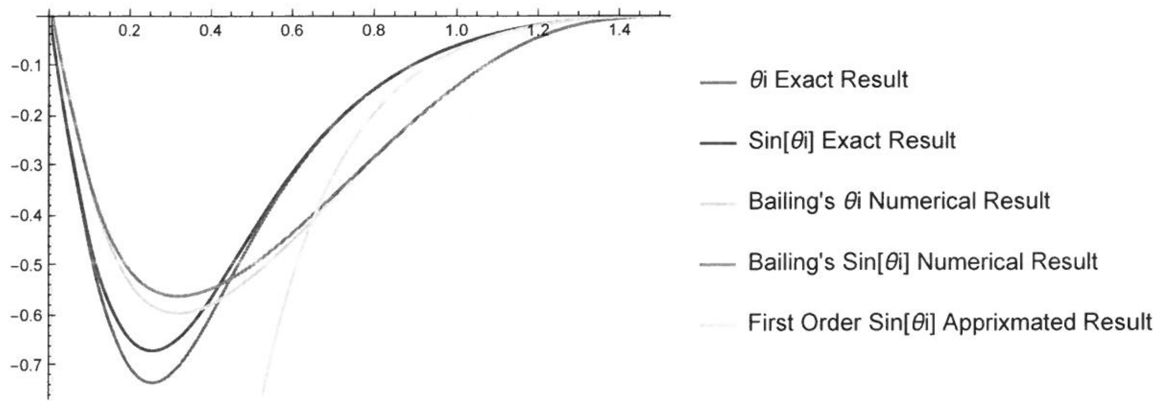
in comparison with the first-order approximated result

$$\exp\left[\frac{\sin\theta_i(P_\triangle)}{\cancel{2}\alpha\cos^3\theta_f(P_\triangle)}\right] = \left[\frac{1-\tan\frac{\theta_f(P_\triangle)}{2}}{1+\tan\frac{\theta_f(P_\triangle)}{2}}\right]^2$$

Out[ ]=


0.5
1.0
1.5
−1
−2
−3
−4
−5
First Order Apprixmated Result
Exact Result

Out[ ]=


0.2
0.4
0.6
0.8
1.0
1.2
1.4
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
θi Exact Result
Sin[θi] Exact Result
Bailing's θi Numerical Result
Bailing's Sin[θi] Numerical Result
First Order Sin[θi] Apprixmated Result