Interpolating the Conformal algebra

Prof. Chueng Ji's group meeting

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Generators of Poincaré group

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}}, \tag{1}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right). \tag{2}$$

In this interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{3}$$

The Poincaré algebra (Contra-variant form) in this interpolating basis is given by

$$\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \tag{4a}$$

$$\left[P^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \tag{4b}$$

$$\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right). \tag{4c}$$

The Poincaré matrix

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{5}$$

transforms as well, so that

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin\delta + K^1 \cos\delta, \\ E^{\hat{2}} &= K^2 \cos\delta - J^1 \sin\delta, \\ F^{\hat{1}} &= K^1 \sin\delta - J^2 \cos\delta, \\ F^{\hat{2}} &= K^2 \sin\delta + J^1 \cos\delta. \end{aligned} \tag{7}$$

The Poincaré matrix

$$M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^1 \sin\delta - J^2 \cos\delta, \\ \mathcal{K}^{\hat{2}} &= J^1 \cos\delta - K^2 \sin\delta, \\ \mathcal{D}^{\hat{1}} &= -K^1 \cos\delta + J^2 \sin\delta, \\ \mathcal{D}^{\hat{2}} &= -J^1 \sin\delta - K^2 \cos\delta. \end{aligned} \tag{9}$$

A comprehensive table of the 45 commutation relations among the co-variant components of the Poincare´ generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).
Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

IFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Instant Form Dynamics (IFD) (when interpolation angle, $\delta = 0$),

	P^0	P^1	P^2	$-K^3$	K^1	K^2	J^3	J^2	$-J^1$	P^3
P^0	0	0	0	iP_3	iP^1	iP^2	0	0	0	0
P^1	0	0	0	0	iP_0	0	$-iP^2$	$-iP_3$	0	0
P^2	0	0	0	0	0	iP_0	iP^1	0	$-iP_3$	0
$-K^3$	$-iP_3$	0	0	0	iJ^2	$-iJ^1$	0	iK^1	iK^2	iP_0
K^1	$-iP^1$	$-iP_0$	0	$-iJ^2$	0	$-iJ^3$	$-iK^2$	iK^3	0	0
K^2	$-iP^2$	0	$-iP_0$	iJ^1	iJ^3	0	iK^1	0	iK^3	0
J^3	0	iP^2	$-iP^1$	0	iK^2	$-iK^1$	0	$-iJ^1$	$-iJ^2$	0
J^2	0	iP_3	0	$-iK^1$	$-iK^3$	0	iJ^1	0	iJ^3	iP^1
$-J^1$	0	0	$+iP_3$	$-iK^2$	0	$-iK^3$	iJ^2	$-iJ^3$	0	iP^2
P^3	0	0	0	$-iP_0$	0	0	0	$-iP^1$	$-iP^2$	0

LFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Light-Front Dynamics (LFD) (when interpolation angle, $\delta = \frac{\pi}{4}$),

	P^+	P^1	P^2	K^3	E^1	E^2	J^3	F^1	F^2	P^-
P^+	0	0	0	iP_-	0	0	0	iP^1	iP^2	0
P^1	0	0	0	0	iP_-	0	$-iP^2$	iP_+	0	
P^2	0	0	0	0	0	iP_-	iP^1	0	iP_+	0
K^3	$-iP_-$	0	0	0	$-iE^1$	$-iE^2$	0	iF^1	iF^2	iP_+
E^1	0	$-iP_-$	0	iE^1	0	0	$-iE^2$	$-iK^3$	$-iJ^3$	$-iP^1$
E^2	0	0	$-iP_-$	iE^2	0	0	iE^1	iJ^3	$-iK^3$	$-iP^2$
J^3	0	iP^2	$-iP^1$	0	iE^2	$-iE^1$	0	iF^2	$-iF^1$	0
F^1	$-iP^1$	$-iP_+$	0	$-iF^1$	iK^3	$-iJ^3$	$-iF^2$	0	0	0
F^2	$-iP^2$	0	$-iP_+$	$-iF^2$	iJ^3	iK^3	iF^1	0	0	0
P^-	0	0	0	$-iP_+$	iP^1	iP^2	0	0	0	0

Kinematic and dynamic generators for different interpolation angles (Phys. Rev. **D 64**, 085013 (2001); Phys. Rev. **D 91**, 065020 (2015))

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

- Among the ten Poincaré generators, the six generators $(\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}})$ are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformations generated by them. The operator $K^3 = M_{\hat{+}\hat{-}}$ is dynamical in the region where $0 \leq \delta < \pi/4$ but becomes kinematic in the light-front limit ($\delta = \pi/4$).
- To understand this, note that $[P^{\hat{+}}, K^{\hat{3}}] = i(\mathbb{S}P^{\hat{+}} - \mathbb{C}P^{\hat{-}}) \rightarrow iP^{\hat{+}}$ as $\delta \rightarrow \pi/4$. Similarly we have $[x^{\hat{+}}, L^{\hat{-}\hat{+}}] = i(\mathbb{S}x^{\hat{+}} - \mathbb{C}x^{\hat{-}}) \rightarrow ix^{\hat{+}}$ as $\delta \rightarrow \pi/4$. Therefore the instant defined by $x^{+} = 0$ becomes invariant under longitudinal boosts as we move to the light front.

Conformal Group

- The set of conformal transformations manifestly forms a group, and it obviously has the Poincaré group as a subgroup.
- We start by introducing conformal transformations and determining the condition for conformal invariance.
- Next, we are going to consider flat space in $d \geq 3$ dimensions and identify the conformal group. And then to $d = 2$ case.

Conformal Transformations

Let us consider a flat space in d dimensions and transformations thereof which locally preserve the angle between any two lines. A conformal transformation is a smooth, invertible map $x \to x'$ such that

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = F(x)g_{\mu\nu}(x), \tag{10}$$

where the positive function $F(x)$ is called the scale factor. We will always consider flat spaces with a constant metric of the form $\eta_{\mu\nu} = diag(1, ..., +1, ...)$. In this case, the condition for a conformal transformation can be written as

$$\boxed{\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = F(x)\eta_{\mu\nu}}. \tag{11}$$

Note furthermore, for flat spaces the scale factor $F(x) = 1$ corresponds to the Poincaré group consisting of translations and rotations, respectively Lorentz transformations.

Let us next study infinitesimal coordinate transformations which up to first order in a small parameter $\epsilon(x) << 1$ read

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2). \tag{12}$$

The left-hand side of $\boxed{\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = F(x)\eta_{\mu\nu}}$ for such a transformation is determined to be of the followingform:

$$\begin{aligned}\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu} + \frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + \mathcal{O}(\epsilon^2)\right)\left(\delta^{\sigma}_{\nu} + \frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \mathcal{O}(\epsilon^2)\right) \\ &= \eta_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \eta_{\rho\nu}\frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + \mathcal{O}(\epsilon^2) \\ &= \eta_{\mu\nu} + \left(\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}\right) + \mathcal{O}(\epsilon^2).\end{aligned}$$

The question we want to ask now is, under what conditions is the transformation ((12)) conformal, i.e. when is Eq. ((11)) satisfied? From the last formula we see that, up to first order in ϵ, we have to demand that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu}\ , \tag{13}$$

where $K(x)$ is some function.

$K(x)$ can be determined by tracing the equation $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K(x)\eta_{\mu\nu}$ with $\eta^{\mu\nu}$

$$\eta^{\mu\nu}\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = K(x)\eta^{\mu\nu}\eta_{\mu\nu}$$
$$2\partial^\mu \epsilon_\mu = K(x)d \tag{14}$$

Using this expression and solving for $K(x)$, we find the following restriction on the transformation $x'^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2)$ to be conformal:

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}}\ , \tag{15}$$

Finally, the scale factor can be read off as $F(x) = 1 + \frac{2}{d}(\partial.\epsilon) + \mathcal{O}(\epsilon^2)$.

Let us now derive two useful equations for later purpose. First, we modify $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}$ by taking the derivative ∂^ν and summing over ν. We then obtain

$$\partial^\nu\Big(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\Big) = \frac{2}{d}\partial^\nu(\partial.\epsilon)\eta_{\mu\nu}$$

$$\partial_\mu(\partial.\epsilon) + \Box\epsilon_\mu = \frac{2}{d}\partial_\mu(\partial.\epsilon) \tag{16}$$

Furthermore, we take the derivative ∂_ν to find

$$\partial_\mu\partial_\nu(\partial.\epsilon) + \Box\partial_\nu\epsilon_\mu = \frac{2}{d}\partial_\mu\partial_\nu(\partial.\epsilon) \tag{17}$$

After interchanging $\mu \longleftrightarrow \nu$, adding the resulting expression to Eq.((17)) and using Eq.((15)) we get

$$2\partial_\mu\partial_\nu(\partial.\epsilon) + \Box\Big(\frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}\Big) = \frac{4}{d}\partial_\mu\partial_\nu(\partial.\epsilon)\ ,$$

$$\implies \quad \Big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu\Big)(\partial.\epsilon) = 0. \tag{18}$$

Finally, contracting this equation with $\eta^{\mu\nu}$ gives

$$\boxed{(d-1)\Box(\partial.\epsilon) = 0}. \tag{19}$$

The second expression we want to use later is obtained by taking derivatives ∂_ρ of $\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}}$ and permuting indices

$$\partial_\rho \partial_\mu \epsilon_\nu + \partial_{\rho\partial} \nu \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial.\epsilon),$$
$$\partial_\nu \partial_\rho \epsilon_\mu + \partial_\mu \partial_\rho \epsilon_\nu = \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial.\epsilon),$$
$$\partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\mu \epsilon_\rho = \frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial.\epsilon),$$

Subtracting then the first line from the sum of the last two leads to

$$\boxed{2\partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial.\epsilon)} \qquad (20)$$

After having obtained the condition for an infinitesimal transformations to be conformal, let us now determine the conformal group in the case of dimension $d \geq 3$.

Conformal Generators

We note that $\boxed{(d-1)\Box(\partial.\epsilon) = 0}$ implies that $(\partial.\epsilon)$ is at most linear in x^μ, i.e. $(\partial.\epsilon) = A + B_\mu x^\mu$ with A and B_μ constant. Then it follows that ϵ_μ is at most quadratic in x^ν and so we can make the ansatz:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \ , \tag{21}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} << 1$ are constants and the latter is symmetric in the last two indices, i.e. $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. We now study the various terms in Eq. ((21)) separately. x^μ.

- The constant term a in Eq. ((21)) is not constrained by $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}$. It describes infinitesimal translations $\boxed{x'^\mu = x^\mu + a^\mu}$, for which the generator is the momentum operator $\boxed{P_\mu = -i\partial_\mu}$.

Conformal Generators: $L_{\mu\nu}$ and D

- In order to study the term of $\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$ which is linear in x, we insert ((21)) into the condition $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}$ to find

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}(\eta^{\rho\sigma} b_{\sigma\rho})\eta_{\mu\nu},$$

From this expression, we see that $b_{\mu\nu}$ can be split into a symmetric and an antisymmetric part

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu},$$

where $m_{\mu\nu} = -m_{\nu\mu}$. The symmetric term $\alpha\eta_{\mu\nu}$ describes infinitesimal scale transformations $\boxed{x'^\mu = (1+\alpha)x^\mu}$ with generator $\boxed{D = -ix^\mu \partial_\mu}$. The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $\boxed{x'^\mu = (\delta^\mu_\nu + m^\mu_\nu)x^\nu}$ with generator being the angular momentum operator $\boxed{L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)}$.

Conformal Generators: SCT

- The term of $\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$ at quadratic order in x can be studied by inserting Eq. ((21)) into expression $2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial.\epsilon)$. We then calculate

$$\partial.\epsilon = b^\mu_\mu + 2c^\mu_{\mu\rho}x^\rho \qquad \Longrightarrow \qquad \partial_\nu(\partial.\epsilon) = 2c^\mu_{\mu\nu},$$

from which we find that

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\mu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \qquad \text{with} \qquad b_\mu = \frac{1}{d}c^\rho_{\rho\mu}.$$

The resulting transformations are called **Special Conformal Transformations (SCT)** and have the following infinitesimal form:

$$x'^\mu = x^\mu + 2(x.b)x^\mu - (x.x)b^\mu \ . \tag{22}$$

Conformal Generators: SCT

- The expression for the full generator G_ν, of a transformation is

$$iG_\nu \Phi = \frac{\delta x^\mu}{\delta \omega_\nu} \partial_\mu \Phi - \frac{\delta F}{\delta \omega_\nu}. \tag{23}$$

For an infinitesimal special conformal transformation (SCT), the coordinates transform like

$$x'^\mu = x^\mu + 2(x \cdot b) x^\mu - b^\mu x^2. \tag{24}$$

If we now suppose the field transforms trivially under a SCT across the entire space, then $\frac{\delta F}{\delta \omega_\nu} = 0$. then,

$$\frac{\delta x^\mu}{\delta b^\nu} = \frac{\delta x^\mu}{\delta (x^\rho b_\rho)} \frac{\delta (x^\gamma b_\gamma)}{\delta b^\nu} = 2 x_\nu x^\mu - x^2 \delta^\mu_\nu. \tag{25}$$

then the Generator for the SCT is,

$$\boxed{\mathfrak{K}_\nu = -i \left(2 x_\nu x^\mu \partial_\mu - x^2 \partial_\nu \right)} \tag{26}$$

We have now identified the infinitesimal conformal transformations.

Finite conformal transformations

$$\begin{aligned} \text{(translation)} \quad & x'^{\mu} = x^{\mu} + a^{\mu} \\ \text{(dilatation)} \quad & x'^{\mu} = \alpha x^{\mu} \\ \text{(rotation)} \quad & x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \\ \text{(SCT)} \quad & x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} \end{aligned}$$

Let us also note that for finite Special Conformal Transformations, we can re-write the expression as follows

$$\begin{aligned} x'^{\mu} &= \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} = \frac{x^2}{|x - bx^2|^2}(x^{\mu} - b^{\mu} x^2), \\ \implies \frac{1}{x'^{\mu}} &= \frac{x^{\mu} - b^{\mu} x^2}{x^2}, \\ \implies \frac{x'^{\mu}}{x'^2} &= \frac{x^{\mu}}{x^2} - b^{\mu} \end{aligned}$$

SCT

From $\boxed{\dfrac{x'^{\mu}}{x'^{2}} = \dfrac{x^{\mu}}{x^{2}} - b^{\mu}}$, we see that the SCT can be understood as an inversion of x^{μ}, followed by a translation b^{μ}, and followed again by an inversion.

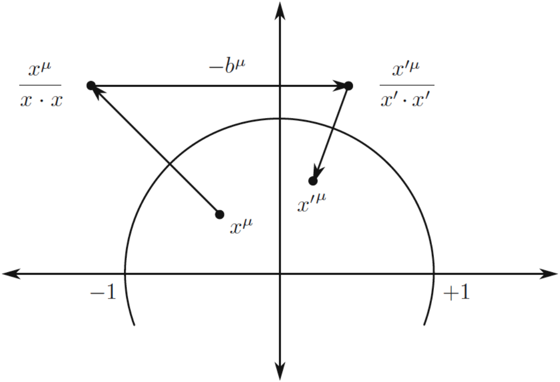

Figure: Illustration of a finite SCT

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}} \tag{27}$$

$$\text{(dilation)} \quad D = -ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{28}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{29}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = -i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{30}$$

Therefore the full Conformal algebra is given by

$$\begin{aligned}
&\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \\
&\left[\mathfrak{K}^{\hat{\mu}}, \mathfrak{K}^{\hat{\nu}}\right] = 0, \\
&\left[D, P^{\hat{\mu}}\right] = iP^{\hat{\mu}}, \\
&\left[D, \mathfrak{K}^{\hat{\mu}}\right] = -i\mathfrak{K}^{\hat{\mu}}, \\
&\left[P^{\hat{\rho}}, L^{\mu\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \\
&\left[\mathfrak{K}^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}\mathfrak{K}^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}\mathfrak{K}^{\hat{\mu}}\right), \\
&\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right), \\
&\left[\mathfrak{K}^{\hat{\mu}}, P^{\hat{\nu}}\right] = 2i\left(g^{\hat{\mu}\hat{\nu}}D - L^{\hat{\mu}\hat{\nu}}\right), \\
&\left[D, L^{\hat{\mu}\hat{\nu}}\right] = 0.
\end{aligned}$$

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$\mathfrak{K}_{\hat{1}}$	$\mathfrak{K}_{\hat{2}}$	$\mathfrak{K}_{\hat{-}}$	D
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$-2i(\mathbb{S}D - K^{\hat{3}})$	$-iP_{\hat{+}}$
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0	$2i\mathcal{D}^{\hat{1}}$	$2iD$	$-2iJ^{\hat{3}}$	$2i\mathcal{K}^{\hat{1}}$	$-iP_{\hat{1}}$
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$2iD$	$2i\mathcal{K}^{\hat{2}}$	$-iP_{\hat{2}}$
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	0	0	$-i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	0
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{+}}$	0	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0	0	$i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{1}}$	0	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{-}}$	0	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{-}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-2i(\mathbb{S}D + K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$2i\mathbb{C}D$	$-iP_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	$2i(\mathbb{S}D + K^{\hat{3}})$	0	0	0	0	$i\mathfrak{K}_{\hat{+}}$
$\mathfrak{K}_{\hat{1}}$	$2i\mathcal{D}^{\hat{1}}$	$-2iD$	$-2iJ^{\hat{3}}$	0	$i\mathfrak{K}_{\hat{+}}$	0	$-i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0	$2i\mathcal{K}^{\hat{1}}$	0	0	0	0	$i\mathfrak{K}_{\hat{1}}$
$\mathfrak{K}_{\hat{2}}$	$2i\mathcal{K}^{\hat{2}}$	$2iJ^{\hat{3}}$	$-2iD$	0	0	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	0	$i\mathfrak{K}_{\hat{-}}$	$2i\mathcal{K}^{\hat{2}}$	0	0	0	0	$i\mathfrak{K}_{\hat{2}}$
$\mathfrak{K}_{\hat{-}}$	$2i(\mathbb{S}D - K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	$-2i\mathbb{C}D$	0	0	0	0	$i\mathfrak{K}_{\hat{-}}$
D	$iP_{\hat{+}}$	$iP_{\hat{1}}$	$iP_{\hat{2}}$	0	0	0	0	0	0	$iP_{\hat{-}}$	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{-}}$	0

Conformal Group in $d = 2$

The condition $\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}}$ for invariance under infinitesimal conformal transformations in two dimensions reads as follows:

$$\partial_0 \epsilon_0 = +\partial_1 \epsilon_1 \ , \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \ , \tag{31}$$

which we recognise as the familiar Cauchy–Riemann equations appearing in complex analysis. A complex function whose real and imaginary parts satisfy Eq. (31) is a holomorphic function (in some open set). We then introduce complex variables in the following way:

$$\begin{aligned} z &= x^0 + ix^1 \ , & \epsilon &= \epsilon^0 + i\epsilon^1 \ , & \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) \ , \\ \bar{z} &= x^0 - ix^1 \ , & \bar{\epsilon} &= \epsilon^0 - i\epsilon^1 \ , & \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1) \ . \end{aligned}$$

Generators

We consider an infinitesimal transformation $z \to z + \epsilon(z)$, $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$. We assume that ϵ admits, the following Laurent series around zero

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \tag{32}$$

where c_n are its Laurent coefficients. Under our infinitesimal conformal transformation we have

$$\begin{aligned} \phi'(z', \bar{z}') &= \phi(z, \bar{z}) \\ &= \phi(z', \bar{z}') - \epsilon(z')\partial_{z'}\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\partial_{\bar{z}'}\phi(z', \bar{z}') \qquad (33) \\ \implies \qquad \delta\phi &= -\epsilon(z)\partial_z\phi - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\phi \\ &= \sum_{n=-\infty}^{\infty} (c_n l_n \phi + \bar{c}_n \bar{l}_n \phi), \qquad (34) \end{aligned}$$

and hence

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{35}$$

It is important to note that since $n \in \mathbb{Z}$, the number of independent infinitesimal conformal transformations is infinite.
It is easy to show that they satisfy the following commutation relations:

$$\begin{aligned} [l_m, l_n] &= z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z \\ &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\ &= -(m-n)z^{m+n+1}\partial_z \\ &= (m-n)l_{m+n} \ , \end{aligned} \tag{36}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \ , \tag{37}$$

$$[l_m, \bar{l}_n] = 0 \ . \tag{38}$$

The first commutation relations define one copy of the so-called Witt algebra, and because of the other two relations, there is a second copy which commutes with the first one. We can then summarise our findings as follows:

- The algebra of infinitesimal conformal transformations in an Euclideantwo-dimensional space is infinite dimensional.

The Conformal transformations are globally defined on the whole complex Riemann sphere. The generators l_n are not everywhere defined. In particular, there is an ambiguity at $z = 0$. Even on the Riemann sphere, not all of the generators (2.11) are well defined.
For $z = 0$, we find that

$$l_n = -z^{n+1}\partial_z \ , \quad \text{non-singular at } z = 0 \text{ only for } n \geq -1 \ . \tag{39}$$

The other ambiguous point is $z = \infty$ which is, however, part of the Riemann sphere. To investigate the behaviour of In there, let us perform the change of variable $z = -\frac{1}{w}$ and study $w \to 0$. We then observe that

$$l_n = -(-\frac{1}{w})^{n+1}\partial_w \ , \quad \text{non-singular at } w = 0 \text{ only for } n \leq +1 \ . \tag{40}$$

- Globally defined conformal transformations on the Riemann sphere are generated by (l_{-1}, l_0, l_{+1}).

Therefore, there are only 6 generators that correspond to conformal transformations that are well defined on the complex Riemann sphere:(l_{-1}, l_0, l_{+1}) and ($\bar{l}_{-1}$, $\bar{l}_0$, $\bar{l}_{+1}$). corresponding algebra:

$$[l_0, l_{-1}] = l_{-1}\ ,$$
$$[l_1, l_0] = l_1\ ,$$
$$[l_1, l_{-1}] = 2l_0\ .$$

Here are the infinitesimal and finite transformations corresponding to these generators

$$\begin{aligned} l_{-1}: &\quad z \rightarrow z + \epsilon, &\quad z \rightarrow z + \alpha, \\ l_0: &\quad z \rightarrow z + \epsilon z, &\quad z \rightarrow \lambda z, \\ l_1: &\quad z \rightarrow z + \epsilon z^2, &\quad z \rightarrow \frac{z}{1 - \beta z}. \end{aligned}$$

where ϵ, α, λ and β are complex parameters, ϵ being in addition infinitesimal.

algebra

In $d = 2$: l_{-1} is translation, l_0 is dilatation, and l_1 is SCT.

$$[l_0, l_{-1}] = l_{-1} \ , \quad [l_1, l_0] = l_1 \ , \quad [l_1, l_{-1}] = 2l_0 \ .$$

The full Conformal algebra in $d \geq 3$

$$\checkmark \left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0,$$

$$\checkmark \left[\mathfrak{K}^{\hat{\mu}}, \mathfrak{K}^{\hat{\nu}}\right] = 0,$$

$$\checkmark \left[D, P^{\hat{\mu}}\right] = iP^{\hat{\mu}},$$

$$\checkmark \left[D, \mathfrak{K}^{\hat{\mu}}\right] = -i\mathfrak{K}^{\hat{\mu}},$$

$$\left[P^{\hat{\rho}}, L^{\mu\hat{\hat{\nu}}}\right] = i\left(g^{\hat{\rho}\hat{\mu}} P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}} P^{\hat{\mu}}\right),$$

$$\left[\mathfrak{K}^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}} \mathfrak{K}^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}} \mathfrak{K}^{\hat{\mu}}\right),$$

$$\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}} L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}} L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}} L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}} L^{\hat{\beta}\hat{\rho}}\right),$$

$$\checkmark \left[\mathfrak{K}^{\hat{\mu}}, P^{\hat{\nu}}\right] = 2i\left(g^{\hat{\mu}\hat{\nu}} D - L^{\hat{\mu}\hat{\nu}}\right),$$

$$\left[D, L^{\hat{\mu}\hat{\nu}}\right] = 0.$$

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