Hartree-type Approximation for $(\phi^4)_{1+1}$

Prof. Chueng Ji's group meeting

Hariprashad Ravikumar*

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sine-Gordon $(1+1)$ in IFD

The lagrangian density is given by

$$\mathcal{L}_{SG} = \frac{m^4}{\lambda}\left[\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \left[cos\left(\phi\right) - 1\right]\right], \tag{1}$$

then the static solution is,

$$\phi_{soliton} = 4\,\text{artan}\left(e^x\right), \tag{2}$$

$$\phi_{anti-soliton} = -\,4\,\text{artan}\left(e^x\right), \tag{3}$$

the energy (mass) of this Soliton in IFD,

$$M_{SG} = \frac{8m^3}{\lambda} \tag{4}$$

sine-Gordon $(1+1)$ in the Interpolation

$$\mathcal{L} = \frac{m^4}{\lambda}\left[\frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + [\cos(\phi) - 1]\right] \tag{5}$$

then the wave equation (EoM) is

$$[\mathbb{C}(\partial^2_{\hat{+}} - \partial^2_{\hat{-}}) + 2\mathbb{S}\partial_{\hat{+}}\partial_{\hat{-}}]\phi = -\sin(\phi) \tag{6}$$

then,

$$\phi_{soliton} = 4\,\mathrm{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}) \tag{7}$$

$$\phi_{anti-soliton} = -4\,\mathrm{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}) \tag{8}$$

the energy (mass) of this Soliton in Interpolation,

$$M_{Sol} = \frac{4m^3}{\lambda}\left(\sqrt{\mathbb{C}} + \frac{1}{\sqrt{\mathbb{C}}}\right) \tag{9}$$

rescaling the interpolation spatial coordinate by

$$\tilde{x}^{\hat{-}} = \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \tag{10}$$

$T_{\mu\nu} forsine - Gordon(1+1)$

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha_0}{\beta^2}cos\beta\phi + \gamma_0 \tag{11}$$

$$\Box\phi + \frac{\alpha_0}{\beta^2}\sin\beta\phi = 0 \tag{12}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{13}$$

$$\begin{aligned} T_{00} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi - \frac{\alpha_0}{\beta^2}cos\beta\phi - \gamma_0 \\ T_{01} &= \partial_0\phi\partial_1\phi \\ T_{10} &= \partial_1\phi\partial_0\phi \\ T_{11} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{\alpha_0}{\beta^2}cos\beta\phi + \gamma_0 \end{aligned} \tag{14}$$

4 divergence of $T_{\mu\nu}$

$$
\begin{aligned}
\partial^{\mu} T_{\mu\nu} &= \partial^{\mu}(\partial_{\mu}\phi\partial_{\nu}\phi - g_{\mu\nu}(\frac{1}{2}\partial_{\rho}\phi\partial^{\rho}\phi + \frac{\alpha_0}{\beta^2} cos\beta\phi + \gamma_0)) \\
&= \partial^{\mu}\partial_{\mu}\phi\partial_{\nu}\phi + \partial_{\mu}\phi\partial^{\mu}\partial_{\nu}\phi - \partial_{\nu}(\frac{1}{2}\partial_{\rho}\phi\partial^{\rho}\phi + \frac{\alpha_0}{\beta^2} cos\beta\phi + \gamma_0) \\
&= \Box\phi\partial_{\nu}\phi + \partial_{\mu}\phi\partial^{\mu}\partial_{\nu}\phi - \frac{1}{2}\partial_{\nu}\partial_{\rho}\phi\partial^{\rho}\phi - \frac{1}{2}\partial_{\rho}\phi\partial_{\nu}\partial^{\rho}\phi + \frac{\alpha_0}{\beta^2}\sin(\beta\phi)\partial_{\nu}\phi \\
&= \Box\phi\partial_{\nu}\phi + \frac{\alpha_0}{\beta^2}\sin(\beta\phi)\partial_{\nu}\phi \\
\partial^{\mu} T_{\mu\nu} &= [\Box\phi + \frac{\alpha_0}{\beta^2}\sin(\beta\phi)]\partial_{\nu}\phi = 0
\end{aligned} \tag{15}
$$

Problems in $T_{\mu\nu}$ *commutations*

$$
\begin{aligned}
[T_{00}(x), T_{01}(x')] =& \frac{1}{2}\left(-2i\partial_0\phi(x)\partial_0\phi(x')\partial_{x'}\delta(x-x')\right) \\
&+ \frac{1}{2}\left(2i\partial_x\phi(x)\partial_{x'}\phi(x')\partial_x\delta(x-x')\right) \\
&+ \left(i\partial_x\phi(x)\frac{\alpha}{\beta}sin(\beta\phi)\delta(x-x')\right)
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
[T_{00}(x), T_{10}(x')] =& \frac{1}{2}\left(-2i\partial_0\phi(x)\partial_0\phi(x')\partial_{x'}\delta(x-x')\right) \\
&+ \frac{1}{2}\left(2i\partial_x\phi(x)\partial_{x'}\phi(x')\partial_x\delta(x-x')\right) \\
&+ \left(i\partial_x\phi(x)\frac{\alpha}{\beta}sin(\beta\phi)\delta(x-x')\right)
\end{aligned} \tag{17}
$$

Klein–Gordon (1+1): $T_{\mu\nu}$

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{18}$$

$$\Box\phi + \frac{1}{2}m^2\phi = 0, \tag{19}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{20}$$

4 divergence of $T_{\mu\nu}$,

$$\begin{aligned}
\partial^\mu T_{\mu\nu} &= \partial^\mu(\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2)) \\
&= \partial^\mu\partial_\mu\phi\partial_\nu\phi + \partial_\mu\phi\partial^\mu\partial_\nu\phi - \partial_\nu(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2) \\
&= \Box\phi\partial_\nu\phi + \partial_\mu\phi\partial^\mu\partial_\nu\phi - \frac{1}{2}\partial_\nu\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}\partial_\rho\phi\partial_\nu\partial^\rho\phi + \frac{1}{2}m^2\phi\partial_\nu\phi \\
&= \Box\phi\partial_\nu\phi + \frac{1}{2}m^2\phi\partial_\nu\phi \\
\partial^\mu T_{\mu\nu} &= [\Box\phi + \frac{1}{2}m^2\phi]\partial_\nu\phi = 0.
\end{aligned} \tag{21}$$

Klein–Gordon (1+1): $T_{\mu\nu}$, $J_{\mu\nu}$, and Dilatation charge

$$T_{00} = T^{00} = \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{1}{2}m^2\phi^2, \tag{22}$$

$$T_{01} = -T^{01} = \partial_0\phi\partial_1\phi, \tag{23}$$

$$T_{10} = -T^{10} = \partial_1\phi\partial_0\phi, \tag{24}$$

$$T_{11} = T^{11} = \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi - \frac{1}{2}m^2\phi^2 \tag{25}$$

$$J^{01} = \int dx^1 \left(x^0 T^{01} - x^1 T^{00}\right) \tag{26}$$

$$D = \int dx^1 x_\mu T^{0\mu} = \int dx^1 x_0 T^{00} + \int dx^1 x_1 T^{01} \tag{27}$$

$$D = t\ P^0 + \int dx^1 x_1 T^{01} \tag{28}$$

Klein–Gordon (1+1): Poincaré algebra

$$P^0 = \int dk^1\, \omega \left(a^\dagger(k^1, m)a(k^1, m)\right) \tag{29}$$

$$P^1 = \int dk^1\, k^1 \left(a^\dagger(k^1, m)a(k^1, m)\right) \tag{30}$$

Let's fine the Equal-x^0 Commutation,

$$\left[P^0, P^1\right] = \int dk^1 \int dk'^1\, \omega\, k'^1 \Big[\left(a^\dagger(k^1, m)a(k^1, m)\right), \left(a^\dagger(k'^1, m)a(k'^1, m)\right)\Big]$$

$$\left[P^0, P^1\right] = 0 \tag{31}$$

$$[P^\mu, P^\nu] = 0 \ \checkmark \tag{32}$$

$J_{\mu\nu}$ in IFD; Klein–Gordon (1+1)

The Boost operator (K^1) will be,

$$J^{01} = \int dx^1 \left(x^0 T^{01} - x^1 T^{00}\right)$$

$$J^{01} = t\ P^1 - \int dx^1 \left(x^1 T^{00}\right) \tag{33}$$

to evaluate $\int dx^1 \left(x^1 T^{00}\right)$, let's find the space-time evaluation of $a(k^1)$ in Heisenberg's Picture.

$$a(k^1, x^\mu) = e^{iP_\mu x^\mu} a(k^1, 0) e^{-iP_\mu x^\mu} \tag{34}$$

$$a(k^1, x^1) = e^{iP_1 . x^1} a(k^1, 0) e^{-iP_1 . x^1} \tag{35}$$

$$\frac{\partial}{\partial x^1} a(k^1, x^1) = iP_1\ e^{iP_1 . x^1} a(k^1, 0) e^{-iP_1 . x^1} - e^{iP_1 . x^1} a(k^1, 0)\ iP_1\ e^{-iP_1 . x^1} \tag{36}$$

$$\frac{\partial}{\partial x^1} a(k^1, x^1) = ie^{iP_1 . x^1} \left[P_1, a(k^1, 0)\right] e^{-iP_1 . x^1} = -ik_1\ e^{iP_1 . x^1} a(k^1, 0) e^{-iP_1 . x^1}$$

$$\frac{\partial}{\partial x^1} a(k^1, x^1) = -ik\ a(k^1, x^1) \tag{37}$$

(38)

$J_{\mu\nu}$ in IFD; Klein–Gordon (1+1)

$$\implies a(k^1, x^1) = e^{-ik_1.x^1} a(k^1, 0) \tag{39}$$

$$\implies a(k^1, 0) = e^{ik_1.x^1} a(k^1, x^1) = e^{-ik^1.x^1} a(k^1, x^1) \tag{40}$$

$$\implies \frac{\partial}{\partial k_1} a(k^1, 0) = ix^1 \; e^{ik_1.x^1} a(k^1, x^1) = ix^1 \; a(k^1, 0) \tag{41}$$

$$\tag{42}$$

we found that $\int dx^1 T^{00} = \int dk^1 \, \omega \left(a^\dagger(k^1, m) a(k^1, m) \right)$, so

$$\int dx^1 \left(x^1 T^{00} \right) = -i \int dk^1 \, \omega \left(a^\dagger(k^1, m) \frac{\partial}{\partial k_1} a(k^1, m) \right) \tag{43}$$

then,

$$J^{01} = t \; P^1 - \int dx^1 \left(x^1 T^{00} \right)$$

$$J^{01} = t \; P^1 + i \int dk^1 \, \omega \left(a^\dagger(k^1, m) \frac{\partial}{\partial k_1} a(k^1, m) \right) \tag{44}$$

$$J^{10} = -i \int dk^1 \, \omega \left(a^\dagger(k^1, m) \frac{\partial}{\partial k_1} a(k^1, m) \right) - t \; P^1 \tag{45}$$

$J_{\mu\nu}$ in IFD; Klein–Gordon (1+1)

Now, the commutation relation between J^{01} and P^{μ} will be

$$\begin{aligned}
\left[J^{01}, P^{0}\right] &= \int dk^1\, \omega \left[\left(J^{01}\right), \left(a^{\dagger}(k^1, m) a(k^1, m)\right)\right] \\
&= \int dk^1\, \omega \left(a^{\dagger}(k^1, m)\left[\left(J^{01}\right), a(k^1, m)\right] + \left[\left(J^{01}\right), a^{\dagger}(k^1, m)\right] a(k^1, m)\right) \\
&= -\, i \int dk^1\, \omega\, \omega \left(a^{\dagger}(k^1, m)\left(\frac{\partial}{\partial k_1} a(k^1, m)\right) + \left(\frac{\partial}{\partial k_1} a^{\dagger}(k^1, m)\right) a(k^1, m)\right) \\
&= -\, i \int dk^1\, \omega\, \omega \frac{\partial}{\partial k_1}\left(a^{\dagger}(k^1, m) a(k^1, m)\right) \\
&= i \int dk^1\, \omega \frac{\partial \omega}{\partial k_1}\left(a^{\dagger}(k^1, m) a(k^1, m)\right) \\
&= -\, i \int dk^1\, \omega \frac{\partial \omega}{\partial k^1}\left(a^{\dagger}(k^1, m) a(k^1, m)\right) = -i \int dk^1 \omega \left(\frac{k}{\omega} a^{\dagger}(k^1) a(k^1)\right) \\
&= -\, i \int dk^1 k \left(a^{\dagger}(k^1) a(k^1)\right) = -i\ P^1
\end{aligned}$$

$$\left[J^{01}, P^{0}\right] = -\, i\ P^{1} \ \checkmark$$

Klein–Gordon (1+1): Poincaré algebra

similarly,

$$\left[J^{01}, P^{1}\right] = -\, i\ P^{0}\ \checkmark \tag{48}$$

$$\left[J^{10}, P^{0}\right] = i\ P^{1}\ \checkmark \tag{49}$$

$$\left[J^{10}, P^{1}\right] = i\ P^{0}\ \checkmark \tag{50}$$

$$\left[J^{\lambda\sigma}, P^{\mu}\right] = i\ \left(g^{\sigma\mu} P^{\lambda} - g^{\lambda\mu} P^{\sigma}\right)\ \checkmark \tag{51}$$

$$\left[D, P^{0}\right] = -\, i\ P^{0}\ \checkmark \tag{52}$$

$$\left[D, P^{1}\right] = -\, i\ P^{1}\ \checkmark \tag{53}$$

$$[D, P^{\mu}] = -\, i\ P^{\mu}\ \checkmark \tag{54}$$

$$\left[D, J^{01}\right] = \left[D, J^{10}\right] = 0\ \checkmark \tag{55}$$

$$[P^{\mu}, P^{\nu}] = 0\ \checkmark \tag{56}$$

$$\left[J^{01}, J^{01}\right] = 0\ \checkmark \tag{57}$$

$$\left[J^{\lambda\sigma}, J^{\mu\nu}\right] = ig^{\lambda\mu} J^{\nu\sigma} - ig^{\lambda\nu} J^{\mu\sigma} + ig^{\mu\sigma} J^{\lambda\nu} - ig^{\nu\sigma} J^{\lambda\mu}\ \checkmark \tag{58}$$

Klein–Gordon (1+1): Poincaré algebra in Interpolation

$$\mathcal{L}_{KG} = \frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] - \frac{1}{2}m^2\phi^2 \tag{59}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}\left[a(k_{\hat{-}},m)e^{-ikx} + a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{60}$$

$$\partial_{\hat{+}}\phi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}k_{\hat{+}}\left[a(k_{\hat{-}},m)e^{-ikx} - a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{61}$$

$$\partial_{\hat{-}}\phi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}k_{\hat{-}}\left[a(k_{\hat{-}},m)e^{-ikx} - a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{62}$$

$$\mathcal{P}_{\hat{+}} = \frac{\partial\mathcal{L}}{\partial\partial_{\hat{+}}\phi}\partial_{\hat{+}}\phi - \mathcal{L}$$

$$P_{\hat{+}} = \int dx^{\hat{-}}\left(\frac{\mathbb{C}}{2}(\partial_{\hat{+}}\phi)^2 + \frac{\mathbb{C}}{2}(\partial_{\hat{-}}\phi)^2 + \frac{1}{2}m^2\phi^2\right) \tag{63}$$

Klein–Gordon (1+1): Poincaré algebra in Interpolation

I'm directly writing the results from the Hornbostel(1992),
The Hamiltonian and momentum in terms of creation and annihilation operators is,

$$P_{\hat{+}} = \int_{-\infty}^{+\infty} dk_{\hat{-}} \; \left[k_{\hat{+}}\right] \left(a^{\dagger}(k_{\hat{-}})a(k_{\hat{-}})\right) = \int_{-\infty}^{+\infty} dk_{\hat{-}} \; \left[\frac{\omega_k - \mathbb{S}k_{\hat{-}}}{\mathbb{C}}\right] \left(a^{\dagger}(k_{\hat{-}})a(k_{\hat{-}})\right) \tag{64}$$

$$P_{\hat{-}} = \int_{-\infty}^{+\infty} dk_{\hat{-}} \; \left[k_{\hat{-}}\right] \left(a^{\dagger}(k_{\hat{-}})a(k_{\hat{-}})\right) = \int_{-\infty}^{+\infty} dk_{\hat{-}} \; \left[\frac{\omega_k - \mathbb{C}k_{\hat{+}}}{\mathbb{S}}\right] \left(a^{\dagger}(k_{\hat{-}})a(k_{\hat{-}})\right) \tag{65}$$

we can easily show that,

$$\left[P_{\hat{+}}, P_{\hat{-}}\right] = 0 \;\; \checkmark \tag{66}$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}\phi^4, \tag{67}$$

$$\Box\phi + m^2\phi = \frac{m^2\beta^2}{3!}\phi^3, \tag{68}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{69}$$

4 divergence of $T_{\mu\nu}$,

$$\begin{aligned}\partial^\mu T_{\mu\nu} &= \partial^\mu(\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}\phi^4)) \\ &= \Box\phi\partial_\nu\phi + \partial_\mu\phi\partial^\mu\partial_\nu\phi - \frac{1}{2}\partial_\nu\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}\partial_\rho\phi\partial_\nu\partial^\rho\phi + m^2\phi\partial_\nu\phi \\ &\qquad - \frac{m^2\beta^2}{3!}\phi^3\partial_\nu\phi\end{aligned}$$

$$\partial^\mu T_{\mu\nu} = [\Box\phi + \frac{1}{2}m^2\phi - \frac{m^2\beta^2}{3!}\phi^3]\partial_\nu\phi = 0. \tag{70}$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

$$\begin{aligned}
T_{00} = T^{00} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi + \frac{1}{2}m^2\phi^2 - \frac{m^2\beta^2}{4!}\phi^4, \\
T_{01} = -T^{01} &= \partial_0\phi\partial_1\phi, \\
T_{10} = -T^{10} &= \partial_1\phi\partial_0\phi, \\
T_{11} = T^{11} &= \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\partial_1\phi\partial_1\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}\phi^4.
\end{aligned} \tag{71}$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

and our field is,

$$\phi(x) = \int \frac{dk^1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega(k^1, m)}} \left[a(k^1, m) e^{-ikx} + a^\dagger(k^1, m) e^{ikx} \right] \tag{72}$$

There is no change in P^1,

$$H' = -\int dx^1 \frac{m^2\beta^2}{4!}\phi^4 = -\frac{\lambda}{4!}\int dx^1 \phi^4 \tag{73}$$

$$\begin{aligned} H' = &- \frac{\lambda}{4!}\frac{1}{16\pi^2}\int dx dk^1 dk^2 dk^3 dk^4 \frac{1}{\sqrt{\omega^1\omega^2\omega^3\omega^4}} \\ &\times \Big(\left[a(k^1) e^{-ik^1 x} + a^\dagger(k^1) e^{ik^1 x} \right] \times \left[a(k^2) e^{-ik^2 x} + a^\dagger(k^2) e^{ik^2 x} \right] \\ &\times \left[a(k^3) e^{-ik^3 x} + a^\dagger(k^3) e^{ik^3 x} \right] \times \left[a(k^4) e^{-ik^4 x} + a^\dagger(k^4) e^{ik^4 x} \right] \Big) \end{aligned} \tag{74}$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

$$
\begin{aligned}
H' =& N[H'] - \frac{\lambda}{4!}\frac{1}{16\pi^2}\int dx dk^1 dk^2 dk^3 dk^4 \frac{1}{\sqrt{\omega^1\omega^2\omega^3\omega^4}} \\
& \times\Big(a(k^1)a(k^2)a^\dagger(k^3)a^\dagger(k^4)e^{-i(k^1+k^2-k^3-k^4)x} \\
& + a(k^1)a^\dagger(k^2)a(k^3)a^\dagger(k^4)e^{-i(k^1-k^2+k^3-k^4)x}\Big)
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
H' =& N[H'] - \frac{\lambda}{4!}\frac{1}{16\pi^2}\int dx dk^1 dk^2 dk^3 dk^4 \frac{1}{\sqrt{\omega^1\omega^2\omega^3\omega^4}} \\
& \times\Big(a(k^1)a^\dagger(k^4)\delta(k^2-k^3)e^{-i(k^1+k^2-k^3-k^4)x} \\
& + a(k^2)a^\dagger(k^4)\delta(k^1-k^3)e^{-i(k^1+k^2-k^3-k^4)x} \\
& + \delta(k^1-k^2)a(k^3)a^\dagger(k^4)e^{-i(k^1-k^2+k^3-k^4)x}\Big)
\end{aligned} \tag{76}
$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

$$
\begin{aligned}
H' =& N[H'] - \frac{\lambda}{4!}\frac{1}{16\pi^2}\int dx \frac{1}{\sqrt{\omega^1\omega^2\omega^3\omega^4}} \\
&\times\Big(\int dk^1 dk^2 dk^4 a(k^1)a^\dagger(k^4)e^{-i(k^1-k^4)x} \\
&+\int dk^1 dk^2 dk^4 a(k^2)a^\dagger(k^4)e^{-i(k^2-k^4)x} \\
&+\int dk^1 dk^3 dk^4 a(k^3)a^\dagger(k^4)e^{-i(k^3-k^4)x}\Big)
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
H' =& N[H'] - \frac{\lambda}{4!}\frac{1}{16\pi^2}\frac{1}{\sqrt{\omega^1\omega^2\omega^3\omega^4}} \\
&\times\Big(\int dk^1 dk^2 dk^4 a(k^1)a^\dagger(k^4)\delta(k^1-k^4)2\pi \\
&+\int dk^1 dk^2 dk^4 a(k^2)a^\dagger(k^4)\delta(k^2-k^4)2\pi \\
&+\int dk^1 dk^3 dk^4 a(k^3)a^\dagger(k^4)\delta(k^3-k^4)2\pi\Big)
\end{aligned} \tag{78}
$$

$T_{\mu\nu}$: for $(\phi^4)_{1+1}$ scalar field

$$H' = N[H'] - \frac{\lambda}{4!}\frac{1}{8\pi}\Big[\frac{1}{\omega^1\omega^2}\int dk^1 dk^2 \left(a(k^1)a^\dagger(k^1) + a(k^2)a^\dagger(k^2)\right) + \frac{1}{\omega^1\omega^3}\int dk^1 dk^3 a(k^3)a^\dagger(k^3)\Big] \tag{79}$$

$$H' = N[H'] - \frac{\lambda}{4!}\frac{1}{8\pi}\Big[\frac{1}{\omega\omega'}\int dk dk' \left(a(k)a^\dagger(k) + a(k')a^\dagger(k')\right) + \frac{1}{\omega\omega''}\int dk dk'' a(k'')a^\dagger(k'')\Big] \tag{80}$$

Using the Hartree Approximation and Bogoliubov transformation.

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}\phi^4 \tag{81}$$

$$\phi^4 \longrightarrow 6\phi^2\langle\phi^2\rangle - 3\langle\phi^2\rangle^2 \tag{82}$$

$$\mathcal{L}_{\phi^4} \longrightarrow \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}(6\phi^2\langle\phi^2\rangle - 3\langle\phi^2\rangle^2) \tag{83}$$

$$\mathcal{L}_{\phi^4} \longrightarrow \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2\beta^2}{4!}6\phi^2\langle\phi^2\rangle \tag{84}$$

$$\mathcal{L}_{\phi^4} \longrightarrow \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M\phi^2 \tag{85}$$

where, $M = m^2 - \frac{m^2\beta^2}{2}\langle\phi^2\rangle$

Bogoliubov transformation

For the free field operators, the change of mass $m \to M$ of particles is described by the Bogoliubov-Valatin transformation:

$$a_M(k^1) = \frac{1}{2}\left(\sqrt{\frac{\omega_m}{\omega_M}} + \sqrt{\frac{\omega_M}{\omega_m}}\right) a_m(k^1) - \frac{1}{2}\left(\sqrt{\frac{\omega_m}{\omega_M}} - \sqrt{\frac{\omega_M}{\omega_m}}\right) a_m^\dagger(-k^1) \quad (86)$$

$$a_M^\dagger(k^1) = \frac{1}{2}\left(\sqrt{\frac{\omega_m}{\omega_M}} + \sqrt{\frac{\omega_M}{\omega_m}}\right) a_m^\dagger(k^1) - \frac{1}{2}\left(\sqrt{\frac{\omega_m}{\omega_M}} - \sqrt{\frac{\omega_M}{\omega_m}}\right) a_m(-k^1) \quad (87)$$

we can show that the operators of creation and annihilation of the particle of mass M and momentum k^1 also satisfy the commutation relations,

$$[a_M(k^1), a_M^\dagger(k^{1\prime})] = \delta(k^1 - k^{1\prime}) \quad (88)$$

$$\left[a_M(k^1), a_M(k^{1\prime})\right] = 0 \; ; \; \left[a_M^\dagger(k^1), a_M^\dagger(k^{1\prime})\right] = 0 \quad (89)$$

Bogoliubov transformation

let $\lambda(k^1) = \frac{1}{2}\ln\frac{\omega_m(k^1)}{\omega_M(k^1)} \overset{\text{in general}}{=} \frac{1}{2}\ln\frac{\omega(k^1)}{\omega(k^1,t)}$, where $\omega(k^1) = \sqrt{k^2 + t^2 m^2}$. then,

$$\frac{1}{2}\left(\sqrt{\frac{\omega_m(k^1)}{\omega_M(k^1)}} + \sqrt{\frac{\omega_M(k^1)}{\omega_m(k^1)}}\right) = \frac{e^{\frac{1}{2}\ln\frac{\omega(k^1)}{\omega(k^1,t)}} + e^{-\frac{1}{2}\ln\frac{\omega(k^1)}{\omega(k^1,t)}}}{2} = \cosh(\lambda), \tag{90}$$

$$\frac{1}{2}\left(\sqrt{\frac{\omega_m(k^1)}{\omega_M(k^1)}} - \sqrt{\frac{\omega_M(k^1)}{\omega_m(k^1)}}\right) = \frac{e^{\frac{1}{2}\ln\frac{\omega(k^1)}{\omega(k^1,t)}} - e^{-\frac{1}{2}\ln\frac{\omega(k^1)}{\omega(k^1,t)}}}{2} = \sinh(\lambda), \tag{91}$$

then,

$$a_m(k^1) \longrightarrow a_M(k^1) = a_m(k^1)\cosh(\lambda) - a_m^\dagger(-k^1)\sinh(\lambda), \tag{92}$$

$$a_m^\dagger(k^1) \longrightarrow a_M^\dagger(k^1) = a_m^\dagger(k^1)\cosh(\lambda) - a_m(-k^1)\sinh(\lambda). \tag{93}$$

Bogoliubov transformation

This transformation is unitary if and only if there exists a unitary operator U such that

$$a_M(k^1) = U a_m(k^1) U^{-1}, \quad (94)$$

$$a_M^\dagger(k^1) = U a_m^\dagger(k^1) U^{-1}. \quad (95)$$

it is fulfilled with the following choice,(QFT Lecture Notes of M. Blasone)

$$U = \exp\left(\frac{-1}{2}\int dk^1 \lambda(k^1)\left[a_m(-k^1)a_m(k^1) - a_m^\dagger(k^1)a_m^\dagger(-k^1)\right]\right) \checkmark \quad (96)$$

Bogoliubov transformation

From the QFT Lecture Notes of M. Blasone and Hiroomi Umezawa's Thermo Field Book,

$$a_M(k^1)\,|\Omega\rangle = U a_m(k^1) U^{-1}\,|\Omega\rangle = 0 \tag{97}$$

$$\implies a_m(k^1) U^{-1}\,|\Omega\rangle = 0 \tag{98}$$

$$\text{also,} \qquad a_m(k^1)\,|0\rangle = 0 \tag{99}$$

$$\implies |0\rangle = U^{-1}\,|\Omega\rangle \tag{100}$$

$$\implies |\Omega\rangle = U\,|0\rangle \tag{101}$$

$$|\Omega\rangle = \left[\exp\left(-\frac{1}{2}\delta(0)\int dk^1 \log\cosh\lambda\right)\exp\left(\frac{1}{2}\delta(0)\int dk^1 \tanh\lambda a_m^\dagger(k^1) a_m^\dagger(-k^1)\right)\right]|0\rangle \tag{102}$$

Hartree-type Approximation

$$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger) \tag{103}$$

$$\langle x \rangle = \frac{x_0}{\sqrt{2}}(\langle 0| a |0\rangle + \langle 0| a^\dagger |0\rangle) = 0 \tag{104}$$

$$\langle x^2 \rangle = \frac{x_0^2}{2} \langle 0| (aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) |0\rangle = \frac{x_0^2}{2} \langle 0| (1 + N) |0\rangle = \frac{x_0^2}{2} \tag{105}$$

$$\begin{aligned} \langle x^3 \rangle &= \frac{x_0^3}{2\sqrt{2}} \langle 0| (\cdots + aa^\dagger a^\dagger + aaa^\dagger + \dots) |0\rangle \\ &= \frac{x_0^3}{2\sqrt{2}} \langle 0| (\cdots + (1+N)a^\dagger + a(1+N) + \dots) |0\rangle = 0 \end{aligned} \tag{106}$$

$$\begin{aligned} \langle x^4 \rangle &= \frac{x_0^4}{4} \langle 0| (\cdots + aaaa^\dagger + aa^\dagger a^\dagger a^\dagger + aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + \dots) |0\rangle \\ &= \frac{x_0^4}{4} \langle 0| (\cdots + (aa(1+N)) + ((1+N)a^\dagger a^\dagger) + (a(1+N)a^\dagger) + ((1+N)aa^\dagger) + \\ \langle x^4 \rangle &= \frac{x_0^4}{4} \langle 0| (\cdots + 0 + 0 + 2 + 1 + \dots) |0\rangle = \frac{3x_0^4}{4} \end{aligned} \tag{107}$$

Hartree-type Approximation

$$x^3 \longrightarrow 3\langle x^2\rangle x - 2\langle x\rangle^3 \tag{108}$$

$$\langle x^3\rangle = 0 \tag{109}$$

$$\langle\left(3\langle x^2\rangle x - 2\langle x\rangle^3\right)\rangle = 3\langle x^2\rangle\langle x\rangle - 2\langle x\rangle^3 = 0 \tag{110}$$

also,

$$x^4 \longrightarrow 6\langle x^2\rangle x^2 - 3\langle x^2\rangle^2 \tag{111}$$

$$\langle x^4\rangle = \frac{3x_0^4}{4} \tag{112}$$

$$\langle\left(6\langle x^2\rangle x^2 - 3\langle x^2\rangle^2\right)\rangle = 6\langle x^2\rangle\langle x^2\rangle - 3\langle x^2\rangle^2 = \frac{6x_0^4}{4} - \frac{3x_0^4}{4} = \frac{3x_0^4}{4} \tag{113}$$

Hartree-type Approximation for $\mathcal{L}_{\phi^4}$

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4} g \left(\phi^2 - c^2\right)^2 \tag{114}$$

$$\Box\phi + g\left(\phi^2 - c^2\right)\phi = 0 \tag{115}$$

we can separate ϕ into a c-numaber part ϕ_c and an operator part ϕ_q through

$$\phi = \phi_c + \phi_q \tag{116}$$

with

$$\phi_c = \langle \phi \rangle \tag{117}$$

and

$$\langle \phi_q \rangle = 0 \tag{118}$$

then the EOM,

$$\Box\phi + g\phi^3 - gc^2\phi = 0 \tag{119}$$

Hartree-type Approximation for $\mathcal{L}_{\phi^4}$

we make the Hartree-type approximation (Phys. Rev. D12, 1071 (1975))

$$\phi^3 \longrightarrow 3\langle\phi^2\rangle\phi - 2\langle\phi\rangle^3 \tag{120}$$

then,

$$\Box\phi + g\left(3\langle\phi^2\rangle\phi - 2\langle\phi\rangle^3\right) - gc^2\phi = 0 \tag{121}$$

$$\Box\phi + g\left(3\langle\phi^2\rangle - c^2\right)\phi - 2g\langle\phi\rangle^3 = 0 \tag{122}$$

plug, $\phi = \phi_c + \phi_q$ in the above relation. Then,

$$\Box\phi_c + \Box\phi_q + g\left(3\left(\phi_c^2 + \langle\phi_q^2\rangle\right) - c^2\right)\phi_c + g\left(3\left(\phi_c^2 + \langle\phi_q^2\rangle\right) - c^2\right)\phi_q - 2g\phi_c^3 = 0 \tag{123}$$

$$\Box\phi_c + \Box\phi_q + g\left(\phi_c^2 - c^2\right)\phi_c + 3g\langle\phi_q^2\rangle\phi_c + g\left(3\phi_c^2 - c^2\right)\phi_q + 3g\langle\phi_q^2\rangle\phi_q = 0 \tag{124}$$

Hartree-type Approximation for $\mathcal{L}_{\phi^4}$

then the above equation can be seperated into two equations

$$\Box\phi_c + g\left(\phi_c^2 - c^2\right)\phi_c + 3g\langle\phi_q^2\rangle\phi_c = 0 \tag{125}$$

$$\Box\phi_q + g\left(3\phi_c^2 - c^2\right)\phi_q + 3g\langle\phi_q^2\rangle\phi_q = 0 \tag{126}$$

Equation (125) is ac- number equation, and (126) is linear in the field operator ϕ_q. It is now possible to expand ϕ_q as combination of creation and annihilation operators

$$\phi_q = \sum_n [\psi_n^*(x)e^{i\omega_n t}a_n^\dagger + \psi_n(x)e^{-i\omega_n t}a_n] \tag{127}$$

then

$$\left(-\omega_m^2 - \nabla^2\right)\psi_n + g\left(3\phi_c^2 - c^2\right)\psi_n + 3g\left(\sum_n C_n{\psi_n}^2\right)\psi_n \tag{128}$$

Hartree-type Approximation for $\mathcal{L}_{\phi^4}$

The wave function are normalized by

$$2\omega_n \int dx \psi^*_{n'}(x)\psi_n(x) = \delta_{n'n} \tag{129}$$

for the vacuum reference state,

$$\langle \phi_q^2(x) \rangle = \sum_n \psi^*_n(x)\psi_n(x) \tag{130}$$

For an arbitary n-particle reference state with occupation number $(N_1, N_2, ...)$, we have

$$\langle \phi_q^2(x) \rangle = 2\sum_n (N_n + \frac{1}{2})\psi^*_n(x)\psi_n(x) \tag{131}$$

Hamiltonian

The Hamiltonian associated with the $\mathcal{L}_{\phi^4}$ is given by

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{4}g(\phi^2 - c^2)^2 \tag{132}$$

under the splitting $\phi = \phi_c + \phi_q$, we can decompose the Hamiltonian into three parts

$$\begin{aligned} H = \int dx \Bigg[& \left[\frac{1}{2}\dot{\phi_c}^2 + \frac{1}{2}(\nabla\phi_c)^2 + \frac{1}{4}g(\phi_c^2 - c^2)^2\right] \\ & + \left[\dot{\phi_c}\dot{\phi_q} + \nabla\phi_c\nabla\phi_q + g(\phi_c^2 - c^2)\phi_c\phi_q + g\phi_c\phi_q^3\right] \\ & + \left[\frac{1}{2}\dot{\phi_q}^2 + \frac{1}{2}(\nabla\phi_q)^2 + \frac{1}{2}g(3\phi_c^2 - c^2)\phi_q^2 + \frac{1}{4}g\phi_q^4\right]\Bigg] \end{aligned} \tag{133}$$

$$H = H_c + H_{cq} + H_q \tag{134}$$

Hamiltonian

make the Hartree-type approximation (Phys. Rev. D12, 1071 (1975))

$$\phi^3 \longrightarrow 3\langle\phi^2\rangle\phi - 2\langle\phi\rangle^3 \tag{135}$$

$$\phi_q^4 \longrightarrow 6\langle\phi_q^2\rangle\phi_q^2 - 3\langle\phi_q^2\rangle^2 \tag{136}$$

under these approximation, we have

$$H_{cq} = \int dx \left[\dot{\phi}_c\dot{\phi}_q + \nabla\phi_c\nabla\phi_q + g(\phi_c^2 - c^2)\phi_c\phi_q + g\phi_c\phi_q^3\right] \tag{137}$$

$$= \int dx \left[\dot{\phi}_c\dot{\phi}_q + \nabla(\phi_q\nabla\phi_c) - (\nabla^2\phi_c)\phi_q + g(\phi_c^2 - c^2)\phi_c\phi_q + g\phi_c 3\langle\phi_q^2\rangle\phi_q\right] \tag{138}$$

$$= \int dx \left[\dot{\phi}_c\dot{\phi}_q + \nabla(\phi_q\nabla\phi_c) + \left(-(\nabla^2\phi_c) + g(\phi_c^2 - c^2)\phi_c + g\phi_c 3\langle\phi_q^2\rangle\right)\phi_q\right] \tag{139}$$

$$H_{cq} = \int dx \left[\dot{\phi}_c\dot{\phi}_q + \nabla(\phi_q\nabla\phi_c) - \ddot{\phi}_c\phi_q\right] \tag{140}$$

Hamiltonian

also,

$$H_q = \int dx \left[\frac{1}{2}\dot{\phi_q}^2 + \frac{1}{2}(\nabla \phi_q)^2 + \frac{1}{2} g(3\phi_c^2 - c^2)\phi_q^2 + \frac{1}{4} g(6\langle \phi_q^2 \rangle \phi_q^2 - 3\langle \phi_q^2 \rangle^2) \right] \tag{141}$$

$$= \int dx \left[\frac{1}{2}\dot{\phi_q}^2 + \frac{1}{2}(\nabla \phi_q)^2 + \frac{1}{2} g(3\phi_c^2 - c^2)\phi_q^2 + \frac{1}{2} 3g\langle \phi_q^2 \rangle \phi_q^2 - \frac{3}{4} g\langle \phi_q^2 \rangle^2 \right] \tag{142}$$

$$= \int dx \left[\frac{1}{2}\dot{\phi_q}^2 + \frac{1}{2}(\nabla \phi_q)^2 + \frac{1}{2} \left(g(3\phi_c^2 - c^2)\phi_q + 3g\langle \phi_q^2 \rangle \phi_q \right) \phi_q - \frac{3}{4} g\langle \phi_q^2 \rangle^2 \right] \tag{143}$$

$$H_q = \int dx \left[\frac{1}{2}\dot{\phi_q}^2 + \frac{1}{2}(\nabla \phi_q)^2 - \frac{1}{2} \left(\Box \phi_q \right) \phi_q - \frac{3}{4} g\langle \phi_q^2 \rangle^2 \right] \tag{144}$$

Hamiltonian

The energy associated with the reference state is given by

$$E = \int dx \left[\frac{1}{2}\dot{\phi_c}^2 + \frac{1}{2}(\nabla\phi_c)^2 + \frac{1}{4}g(\phi_c^2 - c^2)^2 \right] + \sum_n (N_n + \frac{1}{2})\omega_n - \frac{3}{4}g \int dx \langle \phi_q^2 \rangle^2 \tag{145}$$

Variational Principle

A time independent solution $\phi_c(x)$ for a given referance state can be obtained bt the Variational Principle,
In other words, the requirement

$$\frac{\delta E}{\delta \phi_c} = 0 \tag{146}$$

The energy associated with the reference state is given by

$$\begin{aligned} E = \sum_n \frac{C_n \omega_n}{2} + \int dx \Bigg[\\ &+ \frac{1}{2}(\nabla\phi_c)^2 + \frac{1}{4}g(\phi_c^2 - c^2)^2 + \frac{1}{2}\sum_n C_n \psi_n^* \left(-\omega_m^2 - \nabla^2\right)\psi_n \\ &+ \frac{1}{2}g\left(3\phi_c^2 - c^2\right)\sum_n C_n {\psi_n}^2 + \frac{3}{4}g\left(\sum_n C_n {\psi_n}^2\right)^2 \Bigg] \end{aligned} \tag{147}$$

with $C_n = 2(N_n + \frac{1}{2})$

Variational Principle

Now,

$$\frac{\delta E}{\delta \phi_c(x)} = \left(\frac{\delta E}{\delta \phi_c(x)}\right)_{explicit} + \sum_n \frac{\delta E}{\delta \omega}\frac{\delta \omega}{\delta \phi_c(x)} + \sum_n \int dy \frac{\delta E}{\delta \psi_n(y)}\frac{\delta \psi_n(y)}{\delta \phi_c(x)} + \sum_n \int dy \frac{\delta E}{\delta \psi_n^*(y)}\frac{\delta \psi_n^*(y)}{\delta \phi_c(x)} \tag{148}$$

$$\frac{\delta E}{\delta \omega} = \frac{C_n}{2}\left(1 - 2\omega_n \int dx {\psi_n}^2\right) = 0, \tag{149}$$

$$\frac{\delta E}{\delta \psi_n(y)} = \frac{C_n}{2}\left(\left(-\omega_m^2 - \nabla^2\right)\psi_n + g\left(3\phi_c^2 - c^2\right)\psi_n + 3g\left(\sum_n C_n {\psi_n}^2\right)\psi_n\right) = 0, \tag{150}$$

$$\frac{\delta E}{\delta \psi_n^*(y)} = \frac{C_n}{2}\left(\left(-\omega_m^2 - \nabla^2\right)\psi_n^* + g\left(3\phi_c^2 - c^2\right)\psi_n^* + 3g\left(\sum_n C_n {\psi_n}^2\right)\psi_n^*\right) = 0, \tag{151}$$

Variational Principle

Hence,

$$\frac{\delta E}{\delta \phi_c(x)} = \left(\frac{\delta E}{\delta \phi_c(x)}\right)_{explicit} \tag{152}$$

$$= -\nabla^2 \phi_c + g\left(\phi_c^2 - c^2\right)\phi_c + 3g \sum_n C_n {\psi_n}^2 \phi_c \tag{153}$$

$$= -\nabla^2 \phi_c + g\left(\phi_c^2 - c^2\right)\phi_c + 3g\langle \phi_q^2 \rangle \phi_c = 0 \tag{154}$$

Renormalization

$$
\begin{aligned}
\Delta E(unnormalized) \equiv & E(\phi_c) - E(c) && (155)\\
= & \int dx \left[\frac{1}{2}\dot{\phi_c}^2 + \frac{1}{2}(\nabla\phi_c)^2 + \frac{1}{4}g(\phi_c^2 - c^2)^2\right] \\
& + \sum_n \frac{1}{2}\omega_n(\phi_c) - \frac{3}{4}g \int dx \left(\sum_n C_n\psi_n(\phi_c)^2\right)^2 \\
& - \sum_n \frac{1}{2}\omega_n(c) + \frac{3}{4}g \int dx \left(\sum_n C_n\psi_n(c)^2\right)^2 && (156)
\end{aligned}
$$

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