Interpolating sine-Gordon model $(1+1)$

Prof.Ji's group meeting

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December 18, 2020

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sine-Gordon $(1+1)$

$$L_{SG} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) + \frac{\alpha_0}{\beta^2}\left[\cos\left(\beta\varphi\right) - 1\right] \tag{1}$$

let $\alpha_0 = m^2$ and $\beta = \frac{\sqrt{\lambda}}{m}$. then,

$$L_{SG} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) + \frac{m^4}{\lambda}\left[\cos\left(\left(\frac{\sqrt{\lambda}}{m}\right)\varphi\right) - 1\right] \tag{2}$$

let's rescale the parameters as $x = mx$, $t = mt$ and $\varphi = \left(\frac{\sqrt{\lambda}}{m}\right)\varphi$.

$$L_{SG} = \frac{m^4}{\lambda}\left[\frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) + \left[\cos\left(\varphi\right) - 1\right]\right] \tag{3}$$

then the EoM is

$$\varphi \equiv \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = sin\varphi \tag{4}$$

zero-mode problem of light-front quantization

Hamiltonian density in IFD,

$$H = \frac{1}{2}\pi(x)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\varphi\right)\right] \tag{5}$$

Hamiltonian density in IFD,

$$P_{+} = \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\varphi\right)\right] \tag{6}$$

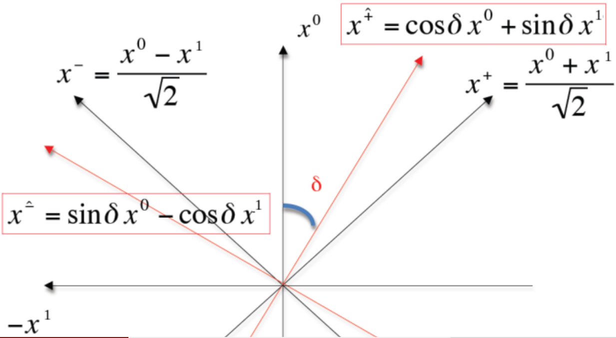
Hamiltonian density in Interpolation,

$$P_{\hat{+}} = \frac{C}{2}(\partial_{\hat{+}}\varphi)^2 + \frac{C}{2}(\partial_{\hat{-}}\varphi)^2 + \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\varphi\right)\right] \tag{7}$$

Interpolation

let $sin2\delta = \mathrm{S}$ and $cos2\delta = \mathrm{C}$,

$$
\begin{aligned}
(ds)^2 &= g_{\mu\nu} dx^{\mu} dx^{\nu} \\
&= g_{\hat{+}\hat{+}} dx^{\hat{+}} dx^{\hat{+}} + g_{\hat{+}\hat{-}} dx^{\hat{+}} dx^{\hat{-}} + g_{\hat{-}\hat{+}} dx^{\hat{-}} dx^{\hat{+}} + g_{\hat{-}\hat{-}} dx^{\hat{-}} dx^{\hat{-}} \\
&= \quad g_{\mu\nu} = \begin{pmatrix} \mathrm{C} & \mathrm{S} \\ \mathrm{S} & -\mathrm{C} \end{pmatrix} = g^{\mu\nu} \qquad (8)
\end{aligned}
$$



Free Hamiltonian in Interpolation form

$$L_{SG} = \frac{1}{2}\left[C(\partial_{+}\varphi)^2 + 2S\partial_{+}\varphi\partial_{-}\varphi - C(\partial_{-}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0 \quad (9)$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{-})\frac{dk_{-}}{\sqrt{2k^{+}}}\left[a(k_{-},m)e^{-ikx} + a^{\dagger}(k_{-},m)e^{ikx}\right], \quad (10)$$

$$\pi(x) = \partial_{+}\varphi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{-})\frac{dk_{-}}{\sqrt{2k^{+}}}k_{+}\left[a(k_{-},m)e^{-ikx} - a^{\dagger}(k_{-},m)e^{ikx}\right], \quad (11)$$

$$\partial_{-}\varphi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{-})\frac{dk_{-}}{\sqrt{2k^{+}}}k_{-}\left[a(k_{-},m)e^{-ikx} - a^{\dagger}(k_{-},m)e^{ikx}\right], \quad (12)$$

then the free Hamiltonian density is,

$$P_{+0} = \frac{C}{2}(\partial_{+}\varphi)^2 + \frac{C}{2}(\partial_{-}\varphi)^2 \quad (13)$$

Free Hamiltonian in Interpolation form

$$P_{\hat{+}0} = N_m[P_{\hat{+}0}] + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{dk_{\hat{-}}}{2k^{\hat{+}}}\left(Ck_{\hat{+}}^2 + Ck_{\hat{-}}^2\right) \tag{14}$$

then,

$$E_0(m) = \frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{dk_{\hat{-}}}{k^{\hat{+}}}\left(Ck_{\hat{+}}^2 + Ck_{\hat{-}}^2\right) \tag{15}$$

from Hornbostel(1992), we have $k_{\hat{+}} = \frac{\omega_k - Sk_{\hat{-}}}{C}$, where $\omega_k \equiv \sqrt{k_{\hat{-}}^2 + Cm^2}$.
so, $k^{\hat{+}} = g^{\hat{+}\hat{+}}k_{\hat{+}} + g^{\hat{+}\hat{-}}k_{\hat{-}} = Ck_{\hat{+}} + Sk_{\hat{-}} = \omega_k - Sk_{\hat{-}} + Sk_{\hat{-}} = \omega_k \equiv \sqrt{k_{\hat{-}}^2 + Cm^2}$.
then,

$$E_0(m) = \frac{1}{8\pi}\int_{-\infty}^{+\infty} dk_{\hat{-}}\,\frac{C\left(-\frac{S}{C}k_{\hat{-}} \pm \frac{\sqrt{k_{\hat{-}}^2 + Cm^2}}{C}\right)^2 + Ck_{\hat{-}}^2}{\sqrt{k_{\hat{-}}^2 + Cm^2}} \tag{16}$$

$$\lim_{S\to 1} E_0(m) = \frac{1}{8\pi}\left[\frac{\Lambda}{2C}\left(\sqrt{\Lambda^2+4Cm^2} - m^2\left(\frac{\Lambda}{\sqrt{\Lambda^2+4Cm^2}}\right)\right) + \right.$$
$$+ \frac{m^2}{2}\log\left(\frac{\sqrt{\Lambda^2+4Cm^2}+\Lambda}{\sqrt{\Lambda^2+4Cm^2}-\Lambda}\right)$$
$$\left. + \frac{C\Lambda}{4}\left(\sqrt{\Lambda^2+4Cm^2} - C^2m^2\left(\frac{\Lambda}{\sqrt{\Lambda^2+4Cm^2}}\right)\right)\right] \tag{17}$$

when $\lim_{C\to 0}$, the $\frac{C\Lambda}{4}\sqrt{\Lambda^2+4Cm^2} \to 0$, so we can ignore this term for now. Other than $\frac{\Lambda}{2C}\sqrt{\Lambda^2+4Cm^2}$, all the remaining terms are blowing up in either limits $\lim_{C\to 0}$ and $\lim_{\Lambda\to\infty}$ (or $\Lambda >> m$).

When $\lim_{\Lambda >> m}$

$$\lim_{\Lambda >> m} E_0(m) = \frac{1}{8\pi}\left[\frac{\Lambda^2}{2C}\left(1 + \frac{2Cm^2}{\Lambda^2} + \cdots\right) + O\left(\frac{m^2}{\Lambda^2}\right)\right] \tag{18}$$

$$\lim_{\Lambda >> m} E_0(m) = \frac{1}{8\pi}\left[\frac{\Lambda^2}{2C} + m^2 + \cdots + O\left(\frac{m^2}{\Lambda^2}\right)\right] \tag{19}$$

Normal ordering it using different mass μ rather than m,

$$
\begin{aligned}
N_m[H_0] &= N_\mu[H_0] + E_0(\mu) - E_0(m) \\
&= N_\mu[H_0] + \frac{1}{8\pi}\left(\frac{\Lambda^2}{2C} + \mu^2 + \cdots + O\left(\left(\frac{\mu^2}{\Lambda^2}\right)\right)\right) \\
&\quad - \frac{1}{8\pi}\left(\frac{\Lambda^2}{2C} + m^2 + \cdots + O\left(\left(\frac{m^2}{\Lambda^2}\right)\right)\right) \\
N_m[H_0] &= N_\mu[H_0] + \frac{1}{8\pi}\left(\mu^2 - m^2\right)
\end{aligned}
\tag{20}
$$

The result is consistent with the IFD.

$T_{\mu\nu}$ in LFD

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}L \tag{21}$$

$$\begin{aligned}
T_{++} &= \partial_+\varphi\partial_+\varphi \\
T_{+-} &= \partial_+\varphi\partial_-\varphi - \partial_+\varphi\partial_-\varphi - \frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 = -\frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 \\
T_{-+} &= \partial_-\varphi\partial_+\varphi - \partial_+\varphi\partial_-\varphi - \frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 = -\frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 \\
T_{--} &= \partial_-\varphi\partial_-\varphi
\end{aligned} \tag{22}$$

$$= \quad T_{\mu\nu} = \begin{pmatrix} \partial_+\varphi\partial_+\varphi & -\frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 \\ -\frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 & \partial_-\varphi\partial_-\varphi \end{pmatrix} \tag{23}$$

Hamiltonian,

$$P_- = T_{+-} = -\frac{\alpha_0}{\beta^2}cos\beta\varphi - \gamma_0 \tag{24}$$

Momentum,

$$P_+ = T_{++} = \partial_+\varphi\partial_+\varphi \tag{25}$$

$T_{\mu\nu}$ in Interpolation Form

$$L_{SG} = \frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0 \tag{26}$$

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}L \tag{27}$$

$$\begin{aligned}
T_{\hat{+}\hat{+}} &= \partial_{\hat{+}}\varphi\partial_{\hat{+}}\varphi - C\left[\frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0\right]\\
T_{\hat{+}\hat{-}} &= \partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - S\left[\frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0\right]\\
T_{\hat{-}\hat{+}} &= \partial_{\hat{-}}\varphi\partial_{\hat{+}}\varphi - S\left[\frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0\right]\\
T_{\hat{-}\hat{-}} &= \partial_{\hat{-}}\varphi\partial_{\hat{-}}\varphi + C\left[\frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0\right]
\end{aligned} \tag{28}$$

Particle spectrum

PHYSICAL REVIEW D VOLUME 11, NUMBER 12 15 JUNE 1975

Particle spectrum in model field theories from semiclassical functional integral techniques*

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We have used a semiclassical method developed earlier to compute the particle spectrum of a field theory in two-dimensional space-time defined by the (sine-Gordon) Lagrangian $\frac{1}{2}(\partial_\mu \phi)^2 + (m^4/\lambda)\{\cos[(\sqrt{\lambda}/m)\phi]-1\}$. For weak coupling we find a heavy particle, the soliton, corresponding to a peculiar classical field configuration and an antisoliton. Below the soliton-antisoliton threshold there are a large number of further states. They can be viewed either as soliton-antisoliton bound states or as bound states of n of the usual quanta of the theory. The "elementary particle" ϕ is the lowest of these. As the coupling increases, the higher states successively unbind, decaying into soliton-antisoliton pairs. At $\lambda/m^2 = 4\pi$, the "elementary particle" unbinds leaving only solitons and antisolitons for $\lambda/m^2 > 4\pi$. Comparing our semiclassical results with recent exact results of Coleman and with perturbation theory, we find that the semiclassical calculations are *exact*. This field theory seems similar to the hydrogen atom for which the Bohr-Sommerfeld quantization rules give the energy levels exactly. We also treat a ϕ^4 theory in weak coupling and carry out a number of calculations which provide nontrivial illustrations of the semiclassical method.

Soliton solutions

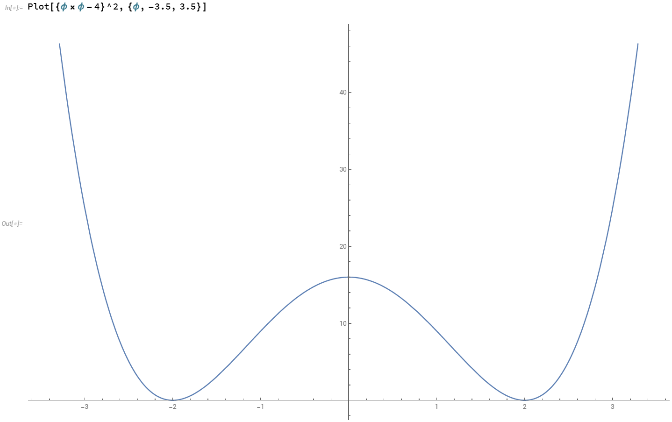
$$L_{SG} = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) + \frac{m^4}{\lambda} \left[\cos \left(\frac{\sqrt{\lambda}}{m} \varphi \right) - 1 \right] \tag{29}$$

It is completely solvable at the classical level.
There are two types of solutions. First there is the soliton (and an antisoliton) which is a solution that is time-independent in its rest frame.
The other one the doublet, loosely speaking, a "soliton-antisoliton" bound state.
In its rest frame the doublet field oscillates periodically in time.

φ^4 : Kink soliton

$$L_{\varphi^4} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + \frac{\lambda}{2}\left((\varphi^2 - a^2)^2\right) \tag{30}$$

minimum of $U(\varphi) = \frac{\lambda}{2}\left((\varphi^2 - a^2)^2\right)$ occure at $\varphi = \pm a$

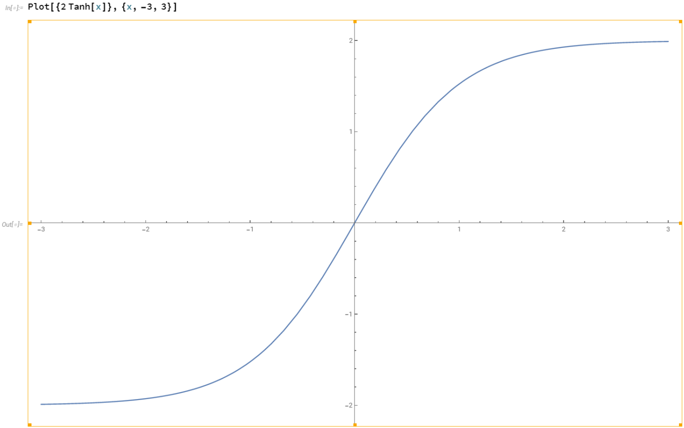


φ^4 : Kink soliton

For time-independent($\frac{\partial \varphi}{\partial t} = 0$), the solution is,

$$\varphi = a \ tanh \ (\mu x) \quad (31)$$

where,$\mu = \lambda a^2$



Solitary wave in (1+1)

$$L_{SG} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) + U(\varphi) \tag{32}$$

then the wave equation (EoM) is

$$\varphi \equiv \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = -\frac{\partial U(\varphi)}{\partial \varphi} \tag{33}$$

static solution,

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial U(\varphi)}{\partial \varphi} \tag{34}$$

upon multiplying $\frac{\partial \varphi}{\partial x}$,

$$\int \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} dx = \int \frac{\partial U(\varphi)}{\partial \varphi}\frac{\partial \varphi}{\partial x} dx \tag{35}$$

the boundary conditions are as $x \rightarrow \pm\infty$, $U(\varphi) = (1 - cos\varphi) \rightarrow 0$ and $\frac{\partial \varphi}{\partial x} \rightarrow 0$.

$$= \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = (U(\varphi)) \tag{36}$$

$$= \frac{\partial\varphi}{\partial x} = \pm\sqrt{2U(\varphi)} \tag{37}$$

on integration,

$$x - x_0 = \pm\int_{\varphi(x_0)}^{\varphi(x)} \frac{d\varphi}{\sqrt{2U(\varphi)}}, \tag{38}$$

The Soliton for sine-Gordon

$$L_{SG} = \frac{m^4}{\lambda} \left[\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) + [\cos(\varphi) - 1] \right] \tag{39}$$

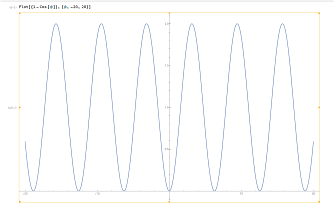
then the EoM is

$$\varphi \equiv \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = \sin\varphi \tag{40}$$

the lagrangian and the fiels equations has the discrete symmetries ,

$$\varphi \to -\varphi$$
$$\varphi \to \varphi + 2n\pi$$

where, $n \in Z$. consistent with these symmetries,the Hamiltonian H_{SG} vanishes at the absolute minima of $U(\varphi) = 1 - \cos\varphi$,
which are, $\varphi = 2n\pi$



For time-independent, $\frac{\partial \varphi}{\partial t} = 0$. Then the Hamiltonian will be,

$$H_{SG} = \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + [1 - cos(\varphi)] \right] \tag{41}$$

plug $U = (1 - cos\varphi)$ in (62), then

$$\frac{\partial \varphi}{\partial x} = \pm \sqrt{2(1 - cos\varphi)} = \pm 2sin(\frac{\varphi}{2}) \tag{42}$$

on integration,

$$x - x_0 = \pm \int_{\varphi(x_0)}^{\varphi(x)} \frac{d\varphi}{2sin(\frac{\varphi}{2})}, \tag{43}$$

then the solution is, (rescale, $x = x - x_0$)

$$\varphi_{soliton} = 4\,\text{artan}\left(e^{x}\right) \tag{44}$$

$$\varphi_{anti-soliton} = -\,4\,\text{artan}\left(e^{x}\right) \tag{45}$$

solution goed from $\varphi = 0$ to $\varphi = 2\pi$ or equivalently from $\varphi = 2\pi$ to $\varphi = 4\pi$ or from $\varphi = 4\pi$ to $\varphi = 6\pi$,, ,

```
In[=]:= Plot[{4 π + 4 ArcTan[Exp[x]], 2 π + 4 ArcTan[Exp[x]], +4 ArcTan[Exp[x]], -4 ArcTan[Exp[x]], -2 π - 4 ArcTan[Exp[x]], -4 π - 4 ArcTan[Exp[x]]}, {x, -6, 6},
  PlotStyle → Thick, PlotLegends → "Expressions"]
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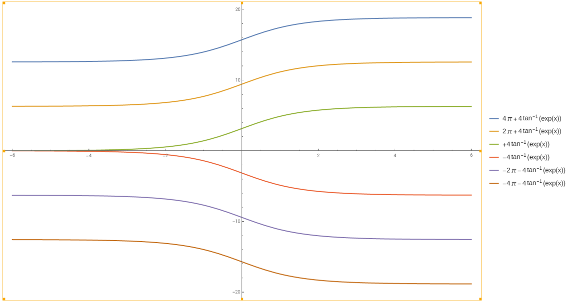
Out[=]=



And by inserting the above solutions into (41), we'll get the energy (mass) of this Soliton,

$$M_{SG} = \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\frac{\partial (4\,\text{artan}\,(e^x))}{\partial x} \right)^2 + [1 - cos\,(4\,\text{artan}\,(e^x))] \right]$$

$$= \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{16}{2} \frac{e^{2x}}{(e^{2x}+1)^2} + \frac{8e^{2x}}{(e^{2x}+1)^2} \right]$$

$$M_{SG} = \frac{16m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left(\frac{e^{2x}}{(e^{2x}+1)^2} \right) = \frac{16m^3}{\lambda} \left[\frac{-1}{2(e^{2x}+1)} \right]_{-\infty}^{+\infty} = \frac{16m^3}{\lambda} \left[\frac{-1}{\infty} + \frac{1}{2} \right]$$

$$M_{SG} = \frac{8m^3}{\lambda} \tag{46}$$

Boosting soliton

A more general solution with time dependence is

$$_{\text{soliton}} = 4\,\text{artan}\quad e^{(x \quad vt)} \tag{47}$$

$$_{\text{anti} \quad \text{soliton}} = \quad 4\,\text{artan}\quad e^{(x \quad vt)} \tag{48}$$

which shows that the soliton travels with velocity v

Doublet

Our L_{SG} permits TIME DEPENDENT solution, called Doublet (soliton-antisoliton scattering solution).

$$_{s\ a}(x;t) = 4\,\mathrm{artanh}\ \frac{\sinh(ut\ \ )}{u\ \cosh(x\ \ )} \tag{49}$$

in asymptotic behaviour in time,

$$\begin{aligned} _{s\ a}(x;t) &= 4\,\mathrm{arctan}\ \frac{(e^{(ut\ \ )}\ \ e^{\ (ut\ \ )})}{u\ (e^{(x\ \ )} + e^{\ (x\ \ )})}\,\frac{e^{(x\ \ )}}{e^{(x\ \ )}} \\ &= 4\,\mathrm{arctan}\ \frac{(e^{(\ \ (x+ut))}\ \ e^{\ (\ \ (x\ \ ut))})}{u\ (e^{(2x\ \ )} + 1)} \end{aligned} \tag{50}$$

let, time delay should be $\quad ^{1}\ln u,$

$$=)\ \ e\ \ \ = e\ \ \ ^{1}\ln u = u\ \ ^{1} \tag{51}$$

so,

$$ {}_{s\ \ a}(x;t) = 4\arctan \frac{e \quad (e^{(\ (x+ut))} \quad e^{\ (\ (x \quad ut))})}{(e^{(2x \ )}+1)} $$

$$ = 4\arctan \frac{(e^{(\ (x+u(t \quad)))} \quad e^{\ (\ (x \quad u(t+)))} \ )}{u \ (e^{(2x \ )}+1)} \tag{52} $$

In the t ! 1 limit the second term in the bracket is large and the rst term is exponentially small, so we can make the approximation

$$ e^{(u(t \quad))} \qquad e^{(u(t+))} \tag{53} $$

Since, $\arctan \frac{x \quad y}{xy+1} = \arctan(x) \quad \arctan(y)$. So in the t ! 1 limit the solution is

$$ \lim_{t!1} \ {}_{s\ \ a}(x;t) \quad 4\arctan \frac{(e^{(\ (x+u(t+)))} \quad e^{\ (\ (x \quad u(t+)))} \ )}{u \ (e^{(2x \ )}+1)} $$

$$ = 4\arctan \ e^{(\ (x+u(t+)))} \qquad 4\arctan \ e^{\ (\ (x \quad u(t+)))} \ ) $$

$$ \lim_{t!1} \ {}_{s\ \ a}(x;t) = {}_{soliton}(\ (x+u(t+))) + {}_{anti \quad soliton}(\ (x \quad u(t+))) \tag{54} $$

Similarly, in the $t \to +\infty$ limit, we can make the approximation

$$e^{(\ \ u(t\ \ ))} \qquad e^{(\ \ u(t+))} \tag{55}$$

then the solution is

$$\lim_{t \to +\infty} \ _s\ _a(x;t) \quad 4\arctan \frac{(e^{(\ (x+u(t\ \ )))} \quad e^{\ (\ (x\ \ u(t\ \ )))}\ )}{u\ (e^{(2x\ \ )}+1)}$$

$$= 4\arctan\ e^{(\ (x+u(t\ \ )))} \quad 4\arctan\ e^{\ (\ (x\ \ u(t\ \ )))}\ )$$

$$\lim_{t \to +\infty} \ _s\ _a(x;t) = \ _{soliton}(\ (x+u(t\ \ \ ))) + \ _{anti\ \ soliton}(\ (x\ \ \ u(t\ \ \ ))) \tag{56}$$

Solitons in Interpolation Form

$$L = \frac{1}{2}\left[C(\partial_{\hat{+}}\varphi)^2 + 2S\partial_{\hat{+}}\varphi\partial_{\hat{-}}\varphi - C(\partial_{\hat{-}}\varphi)^2\right] + U(\varphi) \quad (57)$$

then the wave equation (EoM) is

$$[C(\partial_{\hat{+}}^2 - \partial_{\hat{-}}^2) + 2S\partial_{\hat{+}}\partial_{\hat{-}}]\varphi = -\frac{\partial U(\varphi)}{\partial \varphi} \quad (58)$$

static solution,

$$C\frac{\partial^2\varphi}{\partial x^{\hat{-}2}} = \frac{\partial U(\varphi)}{\partial \varphi} \quad (59)$$

upon multiplying $\frac{\partial \varphi}{\partial x^{\hat{-}}}$,

$$C\int \frac{\partial \varphi}{\partial x^{\hat{-}}}\frac{\partial^2\varphi}{\partial x^{\hat{-}2}} d\hat{x} = \int \frac{\partial U(\varphi)}{\partial \varphi}\frac{\partial \varphi}{\partial x^{\hat{-}}} dx^{\hat{-}} \quad (60)$$

the boundary conditions are as $x^{\hat{-}} \to \pm\infty$, $U(\varphi) = (1 - cos\varphi) \to 0$ and $\frac{\partial \varphi}{\partial x^{\hat{-}}} \to 0$.

$$= \frac{C}{2}\left(\frac{\partial \varphi}{\partial x^{\hat{-}}}\right)^2 = (U(\varphi)) \quad (61)$$

$$= \frac{\partial \varphi}{\partial \hat{x}} = \pm \sqrt{\frac{2}{C} U(\varphi)} \tag{62}$$

on integration,

$$\hat{x} - \hat{x}_0 = \pm \int_{\varphi(\hat{x}_0)}^{\varphi(\hat{x})} \frac{d\varphi}{\sqrt{\frac{2}{C} U(\varphi)}}, \tag{63}$$

$$\hat{x} - \hat{x}_0 = \pm \int_{\varphi(\hat{x}_0)}^{\varphi(\hat{x})} \frac{d\varphi}{\sqrt{\frac{2}{C}} sin(\frac{\varphi}{2})}, \tag{64}$$

then the solution is, (rescale, $\hat{x} = \hat{x} - \hat{x}_0$)

$$\frac{\hat{x}}{\sqrt{C}} = \pm(\ln tan(\frac{\varphi}{4})) \tag{65}$$

then,

$$\varphi_{soliton} = 4 \,\text{artan}(e^{\frac{\hat{x}}{\sqrt{C}}}) \tag{66}$$

$$\varphi_{anti-soliton} = -4 \,\text{artan}(e^{\frac{\hat{x}}{\sqrt{C}}}) \tag{67}$$

the energy (mass) of this Soliton,

$$M_{Sol} = \frac{m^3}{\lambda}\int_{-\infty}^{+\infty} d\hat{x} \left[\frac{1}{2}\left(\frac{\partial(4\,\mathrm{artan}(e^{\hat{x}/\sqrt{C}}))}{\partial \hat{x}}\right)^2 + \left(1 - cos\left(4\,\mathrm{artan}(e^{\hat{x}/\sqrt{C}})\right)\right)\right]$$

$$= \frac{m^3}{\lambda}\int_{-\infty}^{+\infty} d\hat{x} \left[\frac{\frac{1}{C}8e^{\frac{2\hat{x}}{C}}}{(e^{\hat{x}/\sqrt{C}}+1)^2} + \frac{8e^{\frac{2\hat{x}}{C}}}{(e^{2\hat{x}/\sqrt{C}}+1)^2}\right]$$

$$M_{Sol} = \frac{8(1+\frac{1}{C})m^3}{\lambda}\int_{-\infty}^{+\infty} d\hat{x} \left[\frac{e^{2\hat{x}/\sqrt{C}}}{(e^{2\hat{x}/\sqrt{C}}+1)^2}\right] = \frac{8(1+\frac{1}{C})m^3}{\lambda}\left[\frac{-\sqrt{C}}{2(e^{2\hat{x}}+1)}\right]_{-\infty}^{+\infty}$$

$$= \frac{8(1+\frac{1}{C})m^3}{\lambda}\left[\frac{-\sqrt{C}}{\infty} + \frac{\sqrt{C}}{2}\right]$$

$$M_{Sol} = \frac{4m^3}{\lambda}\left(\sqrt{C} + \frac{1}{\sqrt{C}}\right) \tag{68}$$

Acknowledgment

I like to thank Prof.Ji, Dr.Harleen Dahiya, and Bailing for clarifying numerous points, encouraging the interest that made this progress possible, and for their continuous guidance.

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[1] Solitons and Instantons: An Introduction by Ramamurti Rajaraman

[2] M. Burkardt, Phys. Rev. D, 47, 4628

[3] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 11, 3424 (1975)

[4] P. Griffin, Phys. Rev. D, 46, 3538

[5] S. Coleman, Phys. Rev. D, 11, 2088

[6] Hornbostel(1995)