Interpolating sine-Gordon model $(1+1)$

Prof.Ji's group meeting

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Table of Contents

sine-Gordon $(1+1)$

$$\mathcal{L}_{SG} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \frac{\alpha_0}{\beta^2}\left[cos\left(\beta\phi\right) - 1\right] \tag{1}$$

let $\alpha_0 = m^2$ and $\beta = \frac{\sqrt{\lambda}}{m}$. then,

$$\mathcal{L}_{SG} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \frac{m^4}{\lambda}\left[cos\left(\left(\frac{\sqrt{\lambda}}{m}\right)\phi\right) - 1\right] \tag{2}$$

let's rescale the parameters as $x = mx$, $t = mt$ and $\phi = \left(\frac{\sqrt{\lambda}}{m}\right)\phi$.

$$\mathcal{L}_{SG} = \frac{m^4}{\lambda}\left[\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \left[cos\left(\phi\right) - 1\right]\right] \tag{3}$$

then the EoM is

$$\Box\phi \equiv \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = sin\phi \tag{4}$$

zero-mode problem of light-front quantization

Hamiltonian density in IFD,

$$\mathcal{H} = \frac{1}{2}\pi(x)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\phi\right)\right] \qquad (5)$$

Hamiltonian density in IFD,

$$\mathcal{P}_{+} = \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\phi\right)\right] \qquad (6)$$

Hamiltonian density in Interpolation,

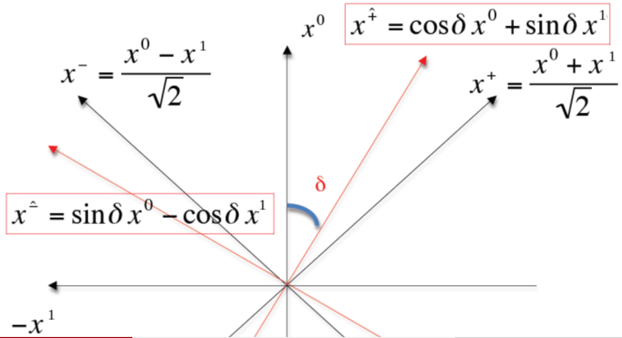
$$\mathcal{P}_{\hat{+}} = \frac{\mathbb{C}}{2}(\partial_{\hat{+}}\phi)^2 + \frac{\mathbb{C}}{2}(\partial_{\hat{-}}\phi)^2 + \frac{\alpha_0}{\beta^2}\left[1 - cos\left(\beta\phi\right)\right] \qquad (7)$$

Interpolation

let $sin2\delta = \mathbb{S}$ and $cos2\delta = \mathbb{C}$,

$$(ds)^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$
$$= g_{\hat{+}\hat{+}} dx^{\hat{+}} dx^{\hat{+}} + g_{\hat{+}\hat{-}} dx^{\hat{+}} dx^{\hat{-}} + g_{\hat{-}\hat{+}} dx^{\hat{-}} dx^{\hat{+}} + g_{\hat{-}\hat{-}} dx^{\hat{-}} dx^{\hat{-}}$$

$$\implies g_{\mu\nu} = \begin{pmatrix} \mathbb{C} & \mathbb{S} \\ \mathbb{S} & -\mathbb{C} \end{pmatrix} = g^{\mu\nu} \tag{8}$$



Free Hamiltonian in Interpolation form

$$\mathcal{L}_{SG} = \frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0 \tag{9}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}\left[a(k_{\hat{-}},m)e^{-ikx} + a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{10}$$

$$\pi(x) = \partial_{\hat{+}}\phi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}k_{\hat{+}}\left[a(k_{\hat{-}},m)e^{-ikx} - a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{11}$$

$$\partial_{\hat{-}}\phi = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{\theta}(k_{\hat{-}})\frac{dk_{\hat{-}}}{\sqrt{2k^{\hat{+}}}}k_{\hat{-}}\left[a(k_{\hat{-}},m)e^{-ikx} - a^{\dagger}(k_{\hat{-}},m)e^{ikx}\right], \tag{12}$$

then the free Hamiltonian density is,

$$\mathcal{P}_{\hat{+}0} = \frac{\mathbb{C}}{2}(\partial_{\hat{+}}\phi)^2 + \frac{\mathbb{C}}{2}(\partial_{\hat{-}}\phi)^2 \tag{13}$$

Free Hamiltonian in Interpolation form

$$\mathcal{P}_{\hat{+}0} = N_m[\mathcal{P}_{\hat{+}0}] + \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{dk_{\hat{-}}}{2k^{\hat{+}}} \left(\mathbb{C}k_{\hat{+}}^2 + \mathbb{C}k_{\hat{-}}^2\right) \tag{14}$$

then,

$$E_0(m) = \frac{1}{8\pi}\int_{-\infty}^{+\infty} \frac{dk_{\hat{-}}}{k^{\hat{+}}} \left(\mathbb{C}k_{\hat{+}}^2 + \mathbb{C}k_{\hat{-}}^2\right) \tag{15}$$

from Hornbostel(1992), we have $k_{\hat{+}} = \frac{\omega_k - \mathbb{S}k_{\hat{-}}}{\mathbb{C}}$, where $\omega_k \equiv \sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2}$.
so, $k^{\hat{+}} = g^{\hat{+}\hat{+}}k_{\hat{+}} + g^{\hat{+}\hat{-}}k_{\hat{-}} = \mathbb{C}k_{\hat{+}} + \mathbb{S}k_{\hat{-}} = \omega_k - \mathbb{S}k_{\hat{-}} + \mathbb{S}k_{\hat{-}} = \omega_k \equiv \sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2}$.
then,

$$E_0(m) = \frac{1}{8\pi}\int_{-\infty}^{+\infty} dk_{\hat{-}} \frac{\mathbb{C}\left(-\frac{\mathbb{S}}{\mathbb{C}}k_{\hat{-}} \pm \frac{\sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2}}{\mathbb{C}}\right)^2 + \mathbb{C}k_{\hat{-}}^2}{\sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2}} \tag{16}$$

$$\lim_{\mathbb{S}\to 1} E_0(m) = \frac{1}{8\pi}\Bigg[\frac{\Lambda}{2\mathbb{C}}\sqrt{\Lambda^2+4\mathbb{C}m^2} - m^2\left(\frac{\Lambda}{\sqrt{\Lambda^2+4\mathbb{C}m^2}}\right) + \\ + \frac{m^2}{2}\log\left(\frac{\sqrt{\Lambda^2+4\mathbb{C}m^2}+\Lambda}{\sqrt{\Lambda^2+4\mathbb{C}m^2}-\Lambda}\right) \\ + \frac{\mathbb{C}\Lambda}{4}\sqrt{\Lambda^2+4\mathbb{C}m^2} - \mathbb{C}^2 m^2\left(\frac{\Lambda}{\sqrt{\Lambda^2+4\mathbb{C}m^2}}\right)\Bigg] \tag{17}$$

when $\lim_{\mathbb{C}\to 0}$, the $\frac{\mathbb{C}\Lambda}{4}\sqrt{\Lambda^2+4\mathbb{C}m^2} \to 0$, so we can ignore this term for now. Other than $\frac{\Lambda}{2\mathbb{C}}\sqrt{\Lambda^2+4\mathbb{C}m^2}$, all the remaining terms are blowing up in either limits $\lim_{\mathbb{C}\to 0}$ and $\lim_{\Lambda\to\infty}$ (or $\Lambda >> m$).
When $\lim_{\Lambda>>m}$

$$\lim_{\Lambda>>m} E_0(m) = \frac{1}{8\pi}\left[\frac{\Lambda^2}{2\mathbb{C}}\left(1+\frac{2\mathbb{C}m^2}{\Lambda^2}+\cdots\right)+\mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)\right] \tag{18}$$

$$\lim_{\Lambda>>m} E_0(m) = \frac{1}{8\pi}\left[\frac{\Lambda^2}{2\mathbb{C}}+m^2+\cdots+\mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)\right] \tag{19}$$

Normal ordering it using different mass μ rather than m,

$$\begin{aligned} N_m[\mathcal{H}_0] &= N_\mu[\mathcal{H}_0] + E_0(\mu) - E_0(m) \\ &= N_\mu[\mathcal{H}_0] + \frac{1}{8\pi}\left[\frac{\Lambda^2}{2\mathbb{C}} + \mu^2 + \cdots + \mathcal{O}\left(\frac{\mu^2}{\Lambda^2}\right)\right] \\ &\quad - \frac{1}{8\pi}\left[\frac{\Lambda^2}{2\mathbb{C}} + m^2 + \cdots + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)\right] \\ N_m[\mathcal{H}_0] &= N_\mu[\mathcal{H}_0] + \frac{1}{8\pi}\left(\mu^2 - m^2\right) \end{aligned} \tag{20}$$

The result is consistent with the IFD.

$T_{\mu\nu}$ in LFD

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \tag{21}$$

$$\begin{aligned}
T_{++} &= \partial_+\phi\partial_+\phi \\
T_{+-} &= \partial_+\phi\partial_-\phi - \partial_+\phi\partial_-\phi - \frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 = -\frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 \\
T_{-+} &= \partial_-\phi\partial_+\phi - \partial_+\phi\partial_-\phi - \frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 = -\frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 \\
T_{--} &= \partial_-\phi\partial_-\phi
\end{aligned} \tag{22}$$

$$\implies T_{\mu\nu} = \begin{pmatrix} \partial_+\phi\partial_+\phi & -\frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 \\ -\frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 & \partial_-\phi\partial_-\phi \end{pmatrix} \tag{23}$$

Hamiltonian,

$$\mathcal{P}_- = T_{+-} = -\frac{\alpha_0}{\beta^2} cos\beta\phi - \gamma_0 \tag{24}$$

Momentum,

$$\mathcal{P}_+ = T_{++} = \partial_+\phi\partial_+\phi \tag{25}$$

$T_{\mu\nu}$ in Interpolation Form

$$\mathcal{L}_{SG} = \frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0 \tag{26}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{27}$$

$$\begin{aligned}
T_{\hat{+}\hat{+}} &= \partial_{\hat{+}}\phi\partial_{\hat{+}}\phi - \mathbb{C}\left(\frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0\right)\\
T_{\hat{+}\hat{-}} &= \partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{S}\left(\frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0\right)\\
T_{\hat{-}\hat{+}} &= \partial_{\hat{-}}\phi\partial_{\hat{+}}\phi - \mathbb{S}\left(\frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0\right)\\
T_{\hat{-}\hat{-}} &= \partial_{\hat{-}}\phi\partial_{\hat{-}}\phi + \mathbb{C}\left(\frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0\right)
\end{aligned} \tag{28}$$

Particle spectrum

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Particle spectrum in model field theories from semiclassical functional integral techniques*

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We have used a semiclassical method developed earlier to compute the particle spectrum of a field theory in two-dimensional space-time defined by the (sine-Gordon) Lagrangian $\frac{1}{2}(\partial_\mu\phi)^2 + (m^4/\lambda)\{\cos[(\sqrt{\lambda}/m)\phi]-1\}$. For weak coupling we find a heavy particle, the soliton, corresponding to a peculiar classical field configuration and an antisoliton. Below the soliton-antisoliton threshold there are a large number of further states. They can be viewed either as soliton-antisoliton bound states or as bound states of n of the usual quanta of the theory. The "elementary particle" ϕ is the lowest of these. As the coupling increases, the higher states successively unbind, decaying into soliton-antisoliton pairs. At $\lambda/m^2 = 4\pi$, the "elementary particle" unbinds leaving only solitons and antisolitons for $\lambda/m^2 > 4\pi$. Comparing our semiclassical results with recent exact results of Coleman and with perturbation theory, we find that the semiclassical calculations are *exact*. This field theory seems similar to the hydrogen atom for which the Bohr-Sommerfeld quantization rules give the energy levels exactly. We also treat a ϕ^4 theory in weak coupling and carry out a number of calculations which provide nontrivial illustrations of the semiclassical method.

Soliton solutions

$$\mathcal{L}_{SG} = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) + \frac{m^4}{\lambda} \left[cos \left(\left(\frac{\sqrt{\lambda}}{m} \right) \phi \right) - 1 \right] \tag{29}$$

It is completely solvable at the classical level.
There are two types of solutions. First there is the soliton (and an antisoliton) which is a solution that is time-independent in its rest frame.
The other one the doublet, loosely speaking, a "soliton-antisoliton" bound state. In its rest frame the doublet field oscillates periodically in time.

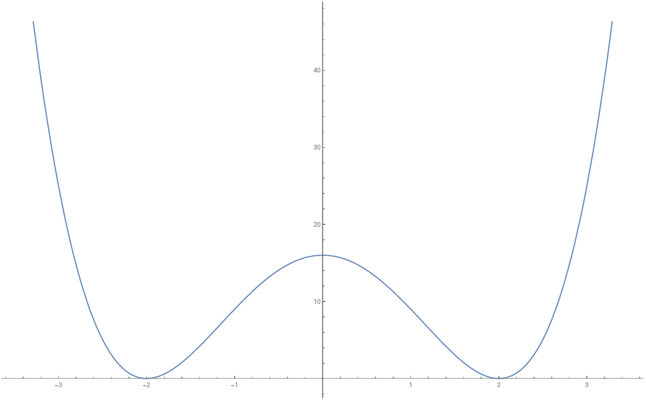
ϕ^4 : Kink soliton

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \frac{\lambda}{2}\left[(\phi^2 - a^2)^2\right] \tag{30}$$

minimum of $U(\phi) = \frac{\lambda}{2}\left[(\phi^2 - a^2)^2\right]$ occure at $\phi = \pm a$

In[∘]:= Plot[{ϕ × ϕ − 4}^2, {ϕ, −3.5, 3.5}]

Out[∘]=



ϕ^4 : Kink soliton

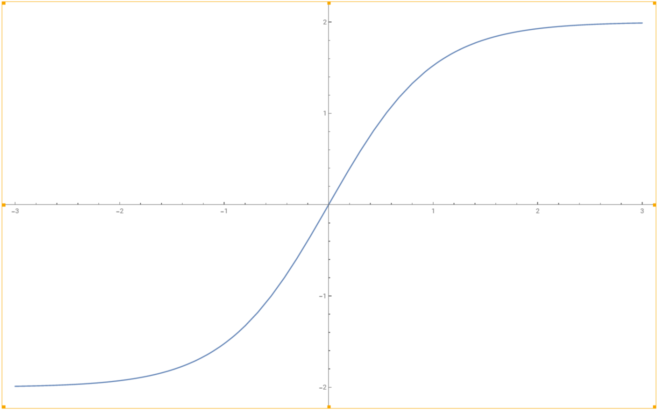
For time-independent($\frac{\partial \phi}{\partial t} = 0$), the solution is,

$$\phi = a\ tanh\ (\mu x) \tag{31}$$

where, $\mu = \lambda a^2$

```
In[]:= Plot[{2 Tanh[x]}, {x, -3, 3}]
```

Out[]=



Solitary wave in (1+1)

$$\mathcal{L}_{SG} = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) + U(\phi) \tag{32}$$

then the wave equation (EoM) is

$$\Box\phi \equiv \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial U(\phi)}{\partial \phi} \tag{33}$$

static solution,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial U(\phi)}{\partial \phi} \tag{34}$$

upon multiplying $\frac{\partial \phi}{\partial x}$,

$$\int \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} dx = \int \frac{\partial U(\phi)}{\partial \phi} \frac{\partial \phi}{\partial x} dx \tag{35}$$

the boundary conditions are as $x \to \pm\infty$, $U(\phi) = (1 - cos\phi) \to 0$ and $\frac{\partial \phi}{\partial x} \to 0$.

$$\implies \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 = (U(\phi)) \tag{36}$$

$$\implies \frac{\partial\phi}{\partial x} = \pm\sqrt{2U(\phi)} \tag{37}$$

on integration,

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2U(\phi)}}, \tag{38}$$

The Soliton for sine-Gordon

$$\mathcal{L}_{SG} = \frac{m^4}{\lambda}\left[\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \left[cos\left(\phi\right) - 1\right]\right] \tag{39}$$

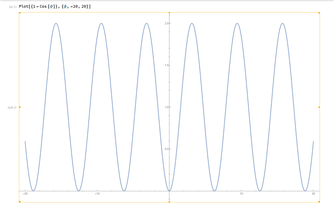
then the EoM is

$$\Box\phi \equiv \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = sin\phi \tag{40}$$

the lagrangian and the fiels equations has the discrete symmetries ,

$$\phi \rightarrow -\phi$$
$$\phi \rightarrow \phi + 2n\pi$$

where, $n \in \mathbb{Z}$. consistent with these symmetries,the Hamiltonian H_{SG} vanishes at the absolute minima of $U(\phi) = 1 - cos\phi$,
which are, $\phi = 2n\pi$



For time-independent, $\frac{\partial \phi}{\partial t} = 0$. Then the Hamiltonian will be,

$$H_{SG} = \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + [1 - cos(\phi)] \right] \tag{41}$$

plug $U = (1 - cos\phi)$ in (62), then

$$\frac{\partial \phi}{\partial x} = \pm \sqrt{2(1 - cos\phi)} = \pm 2sin(\frac{\phi}{2}) \tag{42}$$

on integration,

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{2sin(\frac{\phi}{2})}, \tag{43}$$

then the solution is, (rescale, $x = x - x_0$)

$$\phi_{soliton} = 4\,\text{artan}\left(e^{x}\right) \tag{44}$$

$$\phi_{anti-soliton} = -\,4\,\text{artan}\left(e^{x}\right) \tag{45}$$

solution goed from $\phi = 0$ to $\phi = 2\pi$ or equivalently from $\phi = 2\pi$ to $\phi = 4\pi$ or from $\phi = 4\pi$ to $\phi = 6\pi$,, ,

```
In[=]:= Plot[{4 π + 4 ArcTan[Exp[x]], 2 π + 4 ArcTan[Exp[x]], +4 ArcTan[Exp[x]], -4 ArcTan[Exp[x]], -2 π - 4 ArcTan[Exp[x]], -4 π - 4 ArcTan[Exp[x]]}, {x, -6, 6},
   PlotStyle → Thick, PlotLegends → "Expressions"]
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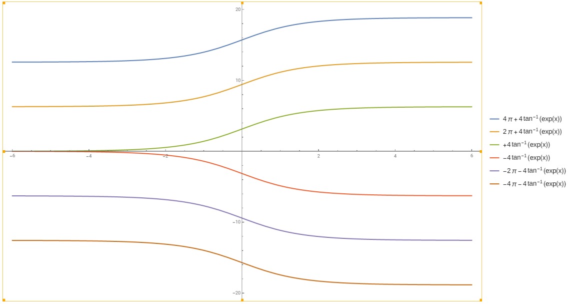
Out[=]=



And by inserting the above solutions into (41), we'll get the energy (mass) of this Soliton,

$$\begin{aligned} M_{SG} =& \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\frac{\partial (4\,\mathrm{artan}\,(e^x))}{\partial x} \right)^2 + [1 - cos\,(4\,\mathrm{artan}\,(e^x))] \right] \\ =& \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{16}{2} \frac{e^{2x}}{(e^{2x}+1)^2} + \frac{8e^{2x}}{(e^{2x}+1)^2} \right] \\ M_{SG} =& \frac{16m^3}{\lambda} \int_{-\infty}^{+\infty} dx \left[\frac{e^{2x}}{(e^{2x}+1)^2} \right] = \frac{16m^3}{\lambda} \left[\frac{-1}{2(e^{2x}+1)} \right]_{-\infty}^{+\infty} = \frac{16m^3}{\lambda} \left[\frac{-1}{\infty} + \frac{1}{2} \right] \\ M_{SG} =& \frac{8m^3}{\lambda} \end{aligned} \tag{46}$$

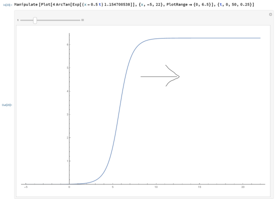
Boosting soliton

A more general solution with time dependence is

$$\phi_{soliton} = 4\,\mathrm{artan}\left(e^{(x-vt)\gamma}\right) \tag{47}$$

$$\phi_{anti-soliton} = -\,4\,\mathrm{artan}\left(e^{(x-vt)\gamma}\right) \tag{48}$$

which shows that the soliton travels with velocity v



Doublet

Our $\mathcal{L}_{SG}$ permits TIME DEPENDENT solution, called Doublet (soliton-antisoliton scattering solution).

$$\phi_{s-a}(x,t) = 4\,\text{artanh}\left(\frac{sinh(ut\gamma)}{u\;cosh(x\gamma)}\right) \tag{49}$$

in asymptotic behaviour in time,

$$\begin{aligned}\phi_{s-a}(x,t) &= 4\arctan\left(\frac{\left(e^{(ut\gamma)} - e^{-(ut\gamma)}\right)}{u\;\left(e^{(x\gamma)} + e^{-(x\gamma)}\right)}\frac{e^{(x\gamma)}}{e^{(x\gamma)}}\right) \\ &= 4\arctan\left(\frac{\left(e^{(\gamma(x+ut))} - e^{-(\gamma(x-ut))}\right)}{u\;\left(e^{(2x\gamma)} + 1\right)}\right)\end{aligned} \tag{50}$$

let, time delay should be $\Delta \equiv \gamma^{-1}\ln u$,

$$\Longrightarrow e^{-\gamma\Delta} = e^{-\gamma\gamma^{-1}\ln u} = u^{-1} \tag{51}$$

so,

$$\phi_{s-a}(x,t) = 4\arctan\left(\frac{e^{-\gamma\Delta}(e^{(\gamma(x+ut))} - e^{-(\gamma(x-ut))})}{(e^{(2x\gamma)}+1)}\right)$$
$$= 4\arctan\left(\frac{(e^{(\gamma(x+u(t-\Delta)))} - e^{-(\gamma(x-u(t+\Delta)))})}{u\ (e^{(2x\gamma)}+1)}\right) \quad (52)$$

In the $t \to -\infty$ limit the second term in the bracket is large and the first term is exponentially small, so we can make the approximation

$$e^{(u(t-\Delta))} \approx e^{(u(t+\Delta))} \quad (53)$$

Since, $\arctan\left(\frac{x-y}{xy+1}\right) = \arctan(x) - \arctan(y)$. So in the $t \to -\infty$ limit the solution is

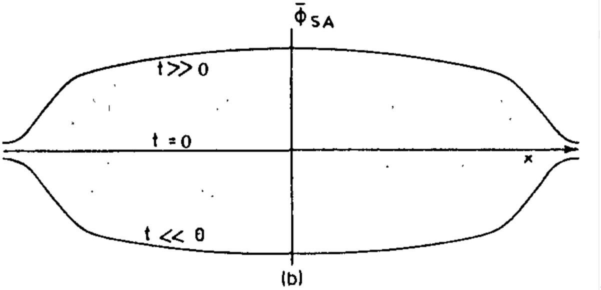
$$\lim_{t\to-\infty} \phi_{s-a}(x,t) \approx 4\arctan\left(\frac{(e^{(\gamma(x+u(t+\Delta)))} - e^{-(\gamma(x-u(t+\Delta)))})}{u\ (e^{(2x\gamma)}+1)}\right)$$
$$= 4\arctan\left(e^{(\gamma(x+u(t+\Delta)))}\right) - 4\arctan\left(e^{-(\gamma(x-u(t+\Delta)))}\right)$$
$$\lim_{t\to-\infty} \phi_{s-a}(x,t) = \phi_{soliton}(\gamma(x+u(t+\Delta))) + \phi_{anti-soliton}(\gamma(x-u(t+\Delta))) \quad (54)$$

Similarly, in the $t \to +\infty$ limit, we can make the approximation

$$e^{(-u(t-\Delta))} \approx e^{(-u(t+\Delta))} \tag{55}$$

then the solution is

$$\begin{aligned}
\lim_{t\to+\infty} \phi_{s-a}(x,t) &\approx 4\arctan\left(\frac{\left(e^{(\gamma(x+u(t-\Delta)))} - e^{-(\gamma(x-u(t-\Delta)))}\right)}{u\left(e^{(2x\gamma)}+1\right)}\right) \\
&= 4\arctan\left(e^{(\gamma(x+u(t-\Delta)))}\right) - 4\arctan\left(e^{-(\gamma(x-u(t-\Delta)))}\right)\right) \\
\lim_{t\to+\infty} \phi_{s-a}(x,t) &= \phi_{soliton}(\gamma(x+u(t-\Delta))) + \phi_{anti-soliton}(\gamma(x-u(t-\Delta)))
\end{aligned} \tag{56}$$



Solitons in Interpolation Form

$$\mathcal{L} = \frac{1}{2}\left[\mathbb{C}(\partial_{\hat{+}}\phi)^2 + 2\mathbb{S}\partial_{\hat{+}}\phi\partial_{\hat{-}}\phi - \mathbb{C}(\partial_{\hat{-}}\phi)^2\right] + U(\phi) \tag{57}$$

then the wave equation (EoM) is

$$[\mathbb{C}(\partial^2_{\hat{+}} - \partial^2_{\hat{-}}) + 2\mathbb{S}\partial_{\hat{+}}\partial_{\hat{-}}]\phi = -\frac{\partial U(\phi)}{\partial\phi} \tag{58}$$

static solution,

$$\mathbb{C}\frac{\partial^2\phi}{\partial x^{\hat{-}2}} = \frac{\partial U(\phi)}{\partial\phi} \tag{59}$$

upon multiplying $\frac{\partial\phi}{\partial x^{\hat{-}}}$,

$$\mathbb{C}\int \frac{\partial\phi}{\partial x^{\hat{-}}}\frac{\partial^2\phi}{\partial x^{\hat{-}2}} d\hat{x} = \int \frac{\partial U(\phi)}{\partial\phi}\frac{\partial\phi}{\partial x^{\hat{-}}} dx^{\hat{-}} \tag{60}$$

the boundary conditions are as $x^{\hat{-}} \to \pm\infty$, $U(\phi) = (1 - cos\phi) \to 0$ and $\frac{\partial\phi}{\partial x^{\hat{-}}} \to 0$.

$$\implies \frac{\mathbb{C}}{2}\left(\frac{\partial\phi}{\partial x^{\hat{-}}}\right)^2 = (U(\phi)) \tag{61}$$

$$\implies \frac{\partial \phi}{\partial x^{\hat{-}}} = \pm\sqrt{\frac{2}{\mathbb{C}}U(\phi)} \tag{62}$$

on integration,

$$x^{\hat{-}} - x_0^{\hat{-}} = \pm \int_{\phi(x_0^{\hat{-}})}^{\phi(x^{\hat{-}})} \frac{d\phi}{\sqrt{\frac{2}{\mathbb{C}}U(\phi)}}, \tag{63}$$

$$x^{\hat{-}} - x_0^{\hat{-}} = \pm \int_{\phi(x_0^{\hat{-}})}^{\phi(x^{\hat{-}})} \frac{d\phi}{\frac{2}{\sqrt{\mathbb{C}}} sin(\frac{\phi}{2})}, \tag{64}$$

then the solution is, (rescale, $x^{\hat{-}} = x^{\hat{-}} - x_0^{\hat{-}}$)

$$\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} = \pm(\ln tan(\frac{\phi}{4})) \tag{65}$$

then,

$$\phi_{soliton} = 4\,\text{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}) \tag{66}$$

$$\phi_{anti-soliton} = -4\,\text{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}) \tag{67}$$

the energy (mass) of this Soliton,

$$M_{Sol} = \frac{m^3}{\lambda}\int_{-\infty}^{+\infty} dx^{\hat{-}} \left[\frac{1}{2}\left(\frac{\partial(4\,\mathrm{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}))}{\partial x^{\hat{-}}}\right)^2 + \left[1 - cos\left(4\,\mathrm{artan}(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}})\right)\right]\right]$$

$$= \frac{m^3}{\lambda}\int_{-\infty}^{+\infty} dx^{\hat{-}} \left[\frac{\frac{1}{\mathbb{C}}8e^{\frac{2x^{\hat{-}}}{\mathbb{C}}}}{(e^{\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}}+1)^2} + \frac{8e^{\frac{2x^{\hat{-}}}{\mathbb{C}}}}{(e^{\frac{2x^{\hat{-}}}{\sqrt{\mathbb{C}}}}+1)^2}\right]$$

$$M_{Sol} = \frac{8(1+\frac{1}{\mathbb{C}})m^3}{\lambda}\int_{-\infty}^{+\infty} dx^{\hat{-}} \left[\frac{e^{\frac{2x^{\hat{-}}}{\sqrt{\mathbb{C}}}}}{(e^{\frac{2x^{\hat{-}}}{\sqrt{\mathbb{C}}}}+1)^2}\right] = \frac{8(1+\frac{1}{\mathbb{C}})m^3}{\lambda}\left[\frac{-\sqrt{\mathbb{C}}}{2(e^{2x^{\hat{-}}}+1)}\right]_{-\infty}^{+\infty}$$

$$= \frac{8(1+\frac{1}{\mathbb{C}})m^3}{\lambda}\left[\frac{-\sqrt{\mathbb{C}}}{\infty} + \frac{\sqrt{\mathbb{C}}}{2}\right]$$

$$M_{Sol} = \frac{4m^3}{\lambda}(\sqrt{\mathbb{C}} + \frac{1}{\sqrt{\mathbb{C}}}) \quad (68)$$

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