Momentum operator components' transformation under SCT

Prof. Chueng Ji's group meeting

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Generators of Poincaré group

$$\text{(translation)} \quad P^{\hat{\mu}} = -i\partial^{\hat{\mu}}, \tag{1}$$

$$\text{(rotation)} \quad L^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right). \tag{2}$$

In the interpolating basis, the metric becomes

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}, \tag{3}$$

The Poincaré algebra (Contra-variant form) in this interpolating basis is given by

$$\left[P^{\hat{\mu}}, P^{\hat{\nu}}\right] = 0, \tag{4a}$$

$$\left[P^{\hat{\rho}}, L^{\hat{\mu}\hat{\nu}}\right] = i\left(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}}\right), \tag{4b}$$

$$\left[L^{\hat{\alpha}\hat{\beta}}, L^{\hat{\rho}\hat{\sigma}}\right] = -i\left(g^{\hat{\beta}\hat{\sigma}}L^{\hat{\alpha}\hat{\rho}} - g^{\hat{\beta}\hat{\rho}}L^{\hat{\alpha}\hat{\sigma}} + g^{\hat{\alpha}\hat{\rho}}L^{\hat{\beta}\hat{\sigma}} - g^{\hat{\alpha}\hat{\sigma}}L^{\hat{\beta}\hat{\rho}}\right). \tag{4c}$$

The Poincaré matrix

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{5}$$

transforms as well, so that

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin\delta + K^1 \cos\delta, \\ E^{\hat{2}} &= K^2 \cos\delta - J^1 \sin\delta, \\ F^{\hat{1}} &= K^1 \sin\delta - J^2 \cos\delta, \\ F^{\hat{2}} &= K^2 \sin\delta + J^1 \cos\delta. \end{aligned} \tag{7}$$

The Poincaré matrix

$$M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^1 \sin\delta - J^2 \cos\delta, \\ \mathcal{K}^{\hat{2}} &= J^1 \cos\delta - K^2 \sin\delta, \\ \mathcal{D}^{\hat{1}} &= -K^1 \cos\delta + J^2 \sin\delta, \\ \mathcal{D}^{\hat{2}} &= -J^1 \sin\delta - K^2 \cos\delta. \end{aligned} \tag{9}$$

A comprehensive table of the 45 commutation relations among the co-variant components of the Poincare´ generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).
Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

IFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Instant Form Dynamics (IFD) (when interpolation angle, $\delta = 0$),

	P^0	P^1	P^2	$-K^3$	K^1	K^2	J^3	J^2	$-J^1$	P^3
P^0	0	0	0	iP_3	iP^1	iP^2	0	0	0	0
P^1	0	0	0	0	iP_0	0	$-iP^2$	$-iP_3$	0	0
P^2	0	0	0	0	0	iP_0	iP^1	0	$-iP_3$	0
$-K^3$	$-iP_3$	0	0	0	iJ^2	$-iJ^1$	0	iK^1	iK^2	iP_0
K^1	$-iP^1$	$-iP_0$	0	$-iJ^2$	0	$-iJ^3$	$-iK^2$	iK^3	0	0
K^2	$-iP^2$	0	$-iP_0$	iJ^1	iJ^3	0	iK^1	0	iK^3	0
J^3	0	iP^2	$-iP^1$	0	iK^2	$-iK^1$	0	$-iJ^1$	$-iJ^2$	0
J^2	0	iP_3	0	$-iK^1$	$-iK^3$	0	iJ^1	0	iJ^3	iP^1
$-J^1$	0	0	$+iP_3$	$-iK^2$	0	$-iK^3$	iJ^2	$-iJ^3$	0	iP^2
P^3	0	0	0	$-iP_0$	0	0	0	$-iP^1$	$-iP^2$	0

LFD

The following table summarizes the commutation relations (contra-variant form) between the Poincare generators explicitly in Light-Front Dynamics (LFD) (when interpolation angle, $\delta = \frac{\pi}{4}$),

	P^+	P^1	P^2	K^3	E^1	E^2	J^3	F^1	F^2	P^-
P^+	0	0	0	iP_-	0	0	0	iP^1	iP^2	0
P^1	0	0	0	0	iP_-	0	$-iP^2$	iP_+	0	
P^2	0	0	0	0	0	iP_-	iP^1	0	iP_+	0
K^3	$-iP_-$	0	0	0	$-iE^1$	$-iE^2$	0	iF^1	iF^2	iP_+
E^1	0	$-iP_-$	0	iE^1	0	0	$-iE^2$	$-iK^3$	$-iJ^3$	$-iP^1$
E^2	0	0	$-iP_-$	iE^2	0	0	iE^1	iJ^3	$-iK^3$	$-iP^2$
J^3	0	iP^2	$-iP^1$	0	iE^2	$-iE^1$	0	iF^2	$-iF^1$	0
F^1	$-iP^1$	$-iP_+$	0	$-iF^1$	iK^3	$-iJ^3$	$-iF^2$	0	0	0
F^2	$-iP^2$	0	$-iP_+$	$-iF^2$	iJ^3	iK^3	iF^1	0	0	0
P^-	0	0	0	$-iP_+$	iP^1	iP^2	0	0	0	0

Kinematic and dynamic generators for different interpolation angles (Phys. Rev. **D 64**, 085013 (2001); Phys. Rev. **D 91**, 065020 (2015))

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P_-$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P_+$

- Among the ten Poincaré generators, the six generators $(\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}})$ are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformations generated by them. The operator $K^3 = M_{\hat{+}\hat{-}}$ is dynamical in the region where $0 \leq \delta < \pi/4$ but becomes kinematic in the light-front limit ($\delta = \pi/4$).
- To understand this, note that $[P^{\hat{+}}, K^{\hat{3}}] = i(\mathbb{S}P^{\hat{+}} - \mathbb{C}P^{\hat{-}}) \rightarrow iP^{\hat{+}}$ as $\delta \rightarrow \pi/4$. Similarly we have $[x^{\hat{+}}, L^{\hat{-}\hat{+}}] = i(\mathbb{S}x^{\hat{+}} - \mathbb{C}x^{\hat{-}}) \rightarrow ix^{\hat{+}}$ as $\delta \rightarrow \pi/4$. Therefore the instant defined by $x^+ = 0$ becomes invariant under longitudinal boosts as we move to the light front.

Conformal Transformations

The Conformal transformation $x \longmapsto x'$ can be defined by,

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta} = F(x) g_{\mu\nu} \tag{10}$$

Consider an infinitesimal translation,

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \ . \tag{11}$$

The metric changes by,

$$\delta g_{\mu\nu} = \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}} = \partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) \tag{12}$$

Conformality then requires,

$$\boxed{\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) = F(x)\delta_{\mu\nu}} \quad \text{Conformal Killing Equation} \tag{13}$$

contraction with $\delta^{\mu\nu}$ yields

$$2\ \partial^{\mu}\epsilon_{\mu} = F(x)\ d \tag{14}$$

$$\implies \quad F(x) = \frac{2}{d}\partial_{\mu}\epsilon^{\mu} \tag{15}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$

$$\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} \tag{16}$$

$$\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = L^\mu_\nu x^\nu \tag{17}$$

$$\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu \tag{18}$$

$$\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu \tag{19}$$

The generators of conformal transformations are:

$$\begin{aligned}
\text{(translation)} \quad & P^\mu = -i\partial^\mu \ , \\
\text{(dilation)} \quad & D = -ix_\mu \partial^\mu \ , \\
\text{(rotation)} \quad & L^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \ , \\
\text{(SCT)} \quad & \mathfrak{K}^\mu = -i\left(2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu\right) \ .
\end{aligned}$$

Conformal algebra

the full Conformal algebra is given by

$$
\begin{aligned}
&[P^\mu, P^\nu] = 0,\\
&[\mathfrak{K}^\mu, \mathfrak{K}^\nu] = 0,\\
&[D, P^\mu] = iP^\mu,\\
&[D, \mathfrak{K}^\mu] = -i\mathfrak{K}^\mu,\\
&\left[P^\rho, L^{\mu\hat{\nu}}\right] = i\left(g^{\rho\mu}P^\nu - g^{\rho\nu}P^\mu\right),\\
&[\mathfrak{K}^\rho, L^{\mu\nu}] = i\left(g^{\rho\mu}\mathfrak{K}^\nu - g^{\rho\nu}\mathfrak{K}^\mu\right),\\
&\left[L^{\alpha\beta}, L^{\rho\sigma}\right] = -i\left(g^{\beta\sigma}L^{\alpha\rho} - g^{\beta\rho}L^{\alpha\sigma} + g^{\alpha\rho}L^{\beta\sigma} - g^{\alpha\sigma}L^{\beta\rho}\right),\\
&[\mathfrak{K}^\mu, P^\nu] = 2i\left(g^{\mu\nu}D - L^{\mu\nu}\right),\\
&[D, L^{\mu\nu}] = 0.
\end{aligned}
$$

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$\mathfrak{K}_{\hat{1}}$	$\mathfrak{K}_{\hat{2}}$	$\mathfrak{K}_{\hat{-}}$	D
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$-2i(\mathbb{S}D - K^{\hat{3}})$	$-iP_{\hat{+}}$
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0	$2i\mathcal{D}^{\hat{1}}$	$2iD$	$-2iJ^{\hat{3}}$	$2i\mathcal{K}^{\hat{1}}$	$-iP_{\hat{1}}$
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$2iD$	$2i\mathcal{K}^{\hat{2}}$	$-iP_{\hat{2}}$
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	0	0	$-i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	0
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{+}}$	0	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0	0	$i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{1}}$	0	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{-}}$	0	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{-}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-2i(\mathbb{S}D + K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$2i\mathbb{C}D$	$-iP_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	$2i(\mathbb{S}D + K^{\hat{3}})$	0	0	0	0	$i\mathfrak{K}_{\hat{+}}$
$\mathfrak{K}_{\hat{1}}$	$2i\mathcal{D}^{\hat{1}}$	$-2iD$	$-2iJ^{\hat{3}}$	0	$i\mathfrak{K}_{\hat{+}}$	0	$-i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0	$2i\mathcal{K}^{\hat{1}}$	0	0	0	0	$i\mathfrak{K}_{\hat{1}}$
$\mathfrak{K}_{\hat{2}}$	$2i\mathcal{K}^{\hat{2}}$	$2iJ^{\hat{3}}$	$-2iD$	0	0	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	0	$i\mathfrak{K}_{\hat{-}}$	$2i\mathcal{K}^{\hat{2}}$	0	0	0	0	$i\mathfrak{K}_{\hat{2}}$
$\mathfrak{K}_{\hat{-}}$	$2i(\mathbb{S}D - K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	$-2i\mathbb{C}D$	0	0	0	0	$i\mathfrak{K}_{\hat{-}}$
D	$iP_{\hat{+}}$	$iP_{\hat{1}}$	$iP_{\hat{2}}$	0	0	0	0	0	0	$iP_{\hat{-}}$	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{-}}$	0

Applying $R_3 = e^{-i\alpha_3 J^3}$ over the momentum operator components

We apply $R_3 = e^{-i\alpha_3 J^3}$ to each of the momentum operator components ($\widehat{\mu} = \widehat{+}, \widehat{-}, \widehat{1}, \widehat{2}$):

$$
\begin{aligned}
R_3^\dagger \mathcal{P}_{\widehat{\mu}} R_3 &= e^{i\alpha_3 J^3} \mathcal{P}_{\widehat{\mu}} e^{-i\alpha_3 J^3} \\
&= \mathcal{P}_{\widehat{\mu}} + i\left[\alpha_3 J^3, \mathcal{P}_{\widehat{\mu}}\right] + \frac{i^2}{2!}\left[\alpha_3 J^3, \left[\alpha_3 J^3, \mathcal{P}_{\widehat{\mu}}\right]\right] + \cdots
\end{aligned}
\tag{20}
$$

This yields

$$
\begin{aligned}
R_3^\dagger \mathcal{P}_{\widehat{+}} R_3 &= \mathcal{P}_{\widehat{+}} \\
R_3^\dagger \mathcal{P}_{\widehat{-}} R_3 &= \mathcal{P}_{\widehat{-}} \\
R_3^\dagger \mathcal{P}^{\widehat{1}} R_3 &= \mathcal{P}^{\widehat{1}} \cos\alpha_3 - \mathcal{P}^{\widehat{2}} \sin\alpha_3 \\
R_3^\dagger \mathcal{P}^{\widehat{2}} R_3 &= \mathcal{P}^{\widehat{2}} \cos\alpha_3 + \mathcal{P}^{\widehat{1}} \sin\alpha_3
\end{aligned}
\tag{21}
$$

Applying $R_3 = e^{-i\alpha_3 J^3}$ over the momentum operator components

Taking the limit $\delta \to 0$ in Eq.(**??**), we get

$$\begin{aligned} P_0' &= P_0 \\ P_3' &= P_3 \\ P'^1 &= P^1 \cos\alpha_3 - P^2 \sin\alpha_3 \\ P'^2 &= P^2 \cos\alpha_3 + P^1 \sin\alpha_3 \end{aligned}$$

Taking the limit $\delta \to \frac{\pi}{4}$, on the other hand, we get

$$\begin{aligned} P_+' &= P_+ \\ P_-' &= P_- \\ P'^1 &= P^1 \cos\alpha_3 - P^2 \sin\alpha_3 \\ P'^2 &= P^2 \cos\alpha_3 + P^1 \sin\alpha_3 \end{aligned}$$

This results confirm that R_3 is kinematical in both LFD and IFD.

Applying Kinematic and Dynamics Poincaré generators over the momentum operator components

The following table contains the transformation of each momentum operator components under all kinematic and dynamic generators in IFD and LFD:

Generators	IFD	LFD
K^3	$P'^0 = \cosh\beta_3 P^0 + \sinh\beta_3 P^3$ $P'^3 = \cosh\beta_3 P^3 + \sinh\beta_3 P^0$ $P'^j = P^j, \ (j = 1, 2)$	$P'_+ = e^{-\beta_3} P^-$ $P'_- = e^{\beta_3} P^+$ $P'^j = P^j, \ (j = 1, 2)$
J^3	$P'_0 = P_0$ $P'_3 = P_3$ $P'^1 = P^1 \cos\alpha_3 - P^2 \sin\alpha_3$ $P'^2 = P^2 \cos\alpha_3 + P^1 \sin\alpha_3$	$P'_+ = P_+$ $P'_- = P_-$ $P'^1 = P^1 \cos\alpha_3 - P^2 \sin\alpha_3$ $P'^2 = P^2 \cos\alpha_3 + P^1 \sin\alpha_3$
$\mathcal{K}^{\hat{1}}$	$P'_0 = P_0$ $P'_3 = P_3 \cos\alpha - P_1 \sin\alpha$ $P'_1 = P_1 \cos\alpha - P_3 \sin\alpha$ $P'_2 = P_2$	$P'_+ = P_+ + \alpha P_1 + \frac{\alpha^2}{2!} P_-$ $P'_- = P_-$ $P'_1 = P_1 + \alpha P_-$ $P'_2 = P_2$
$\mathcal{K}^{\hat{2}}$	$P'_0 = P_0$ $P'_3 = P_3 \cos\alpha - P_2 \sin\alpha$ $P'_1 = P_1$ $P'_2 = P_2 \cos\alpha - P_3 \sin\alpha$	$P'_+ = P_+ + \alpha P_2 + \frac{\alpha^2}{2!} P_-$ $P'_- = P_-$ $P'_1 = P_1$ $P'_2 = P_2 + \alpha P_-$
$\mathcal{D}^{\hat{1}}$	$P'_0 = P_0 \cosh\alpha + P_1 \sinh\alpha$ $P'_3 = P_3$ $P'_1 = P_1 \cosh\alpha + P_0 \sinh\alpha$ $P'_2 = P_2$	$P'_+ = P_+$ $P'_- = P_- + \alpha P_1 + \frac{\alpha^2}{2!} P_+$ $P'_1 = P_1 + \alpha P_+$ $P'_2 = P_2$
$\mathcal{D}^{\hat{2}}$	$P'_0 = P_0 \cosh\alpha + P_2 \sinh\alpha$ $P'_3 = P_3$ $P'_1 = P_1$ $P'_2 = P_2 \cosh\alpha + P_0 \sinh\alpha$	$P'_+ = P_+$ $P'_- = P_- + \alpha P_2 + \frac{\alpha^2}{2!} P_+$ $P'_1 = P_1$ $P'_2 = P_2 + \alpha P_+$

Applying $S_+ = e^{ib\mathfrak{K}_{\hat{+}}}$ over the momentum operator components

We apply $S_+ = e^{ib\mathfrak{K}_{\hat{+}}}$ to each of the momentum operator components $(\widehat{\mu} = \widehat{+}, \widehat{-}, \widehat{1}, \widehat{2})$:

$$S_+^{\dagger}\mathcal{P}_{\widehat{\mu}}S_+ = e^{ib\mathfrak{K}_{\hat{+}}}\mathcal{P}_{\widehat{\mu}}e^{-ib\mathfrak{K}_{\hat{+}}}$$

$$= \mathcal{P}_{\widehat{\mu}} + i\left[b\mathfrak{K}_{\hat{+}}, \mathcal{P}_{\widehat{\mu}}\right] + \frac{i^2}{2!}\left[b\mathfrak{K}_{\hat{+}}, \left[b\mathfrak{K}_{\hat{+}}, \mathcal{P}_{\widehat{\mu}}\right]\right] + \cdots \tag{22}$$

This yields

$$\mathcal{P}'_{\widehat{+}} = \mathcal{P}_{\widehat{+}} - 2b\mathbb{C}D + b^2\mathbb{C}\mathfrak{K}_{\hat{+}} \tag{23}$$

$$\mathcal{P}'_{\widehat{-}} = \mathcal{P}_{\widehat{-}} - 2b(\mathbb{S}D + K^3) + b^2(\mathbb{C}\mathfrak{K}_{\hat{-}}) \tag{24}$$

$$\mathcal{P}'_{\widehat{1}} = \mathcal{P}_{\widehat{1}} + 2b\mathcal{D}^1 - b^2\mathbb{C}\mathfrak{K}_{\hat{1}} \tag{25}$$

$$\mathcal{P}'_{\widehat{2}} = \mathcal{P}_{\widehat{2}} + 2b\mathcal{D}^2 - b^2\mathbb{C}\mathfrak{K}_{\hat{2}} \tag{26}$$

Applying $S_+ = e^{ib\mathfrak{K}_{\hat{+}}}$ over the momentum operator components

Taking the limit $\delta \to 0$ in Eq., we get

$$\begin{aligned} P_0' &= P_0 - 2bD + b^2\mathfrak{K}_0 \\ P_3' &= P_3 - 2bK^3 + b^2\mathfrak{K}_3 \\ P_1' &= P_1 - 2bK^1 - b^2\mathfrak{K}_1 \\ P_2' &= P_2 - 2bK^2 - b^2\mathfrak{K}_2 \end{aligned} \tag{27}$$

Taking the limit $\delta \to \frac{\pi}{4}$, on the other hand, we get

$$\begin{aligned} P_+' &= P_+ \\ P_-' &= P_- - 2b(D + K^3) \\ P_1' &= P_1 - 2bF^1 \\ P_2' &= P_2 - 2bF^2 \end{aligned} \tag{28}$$

Applying $S_+ = e^{ib\mathfrak{K}_{\hat{+}}}$ over the momentum operator components

The following table contains the transformation of each momentum operator components under SCT generators in IFD and LFD:

Generators	IFD	LFD
$\mathfrak{K}_{\hat{+}}$	$P'_0 = P_0 - 2bD + b^2\mathfrak{K}_0$ $P'_3 = P_3 - 2bK^3 + b^2\mathfrak{K}_3$ $P'_1 = P_1 - 2bK^1 - b^2\mathfrak{K}_1$ $P'_2 = P_2 - 2bK^2 - b^2\mathfrak{K}_2$	$P'_+ = P_+$ $P'_- = P_- - 2b(D + K^3)$ $P'_1 = P_1 - 2bF^1$ $P'_2 = P_2 - 2bF^2$
$\mathfrak{K}_{\hat{-}}$	$P'_0 = P_0 + 2bK^3 + b^2\mathfrak{K}_3$ $P'_3 = P_3 - 2bD + b^2\mathfrak{K}_0$ $P'_1 = P_1 - 2bJ^2 + b^2\mathfrak{K}_1$ $P'_2 = P_2 + 2bJ^1 + b^2\mathfrak{K}_2$	$P'_+ = P_+ - 2b(D - K^3)$ $P'_- = P_-$ $P'_1 = P_1 - 2bE^1$ $P'_2 = P_2 - 2bE^2$
$\mathfrak{K}_{\hat{1}}$	$P'_0 = P_0 + 2bK^1 + b^2\mathfrak{K}_0$ $P'_3 = P_3 + 2bJ^2 + b^2\mathfrak{K}_3$ $P'_1 = P_1 + 2bD - b^2\mathfrak{K}_1$ $P'_2 = P_2 + 2bJ^3 - b^2\mathfrak{K}_2$	$P'_+ = P_+ + 2bF^1 + b^2\mathfrak{K}_+$ $P'_- = P_- + 2bE^1 + b^2\mathfrak{K}_-$ $P'_1 = P_1 + 2bD - b^2\mathfrak{K}_1$ $P'_2 = P_2 + 2bJ^3 + b^2\mathfrak{K}_2$
$\mathfrak{K}_{\hat{2}}$	$P'_0 = P_0 - 2bJ^1 + b^2\mathfrak{K}_0$ $P'_3 = P_3 - 2bJ^1 + b^2\mathfrak{K}_3$ $P'_1 = P_1 - 2bJ^3 + b^2\mathfrak{K}_1$ $P'_2 = P_2 - 2bD - b^2\mathfrak{K}_2$	$P'_+ = P_+ + 2bE^2 + b^2\mathfrak{K}_+$ $P'_- = P_- + 2bE^2 + b^2\mathfrak{K}_-$ $P'_1 = P_1 - 2bJ^3 + b^2\mathfrak{K}_1$ $P'_2 = P_2 - 2bD + b^2\mathfrak{K}_2$

Since $[\mathfrak{K}^{\hat{+}}, P^{\hat{+}}] = 2i(g^{\hat{+}\hat{+}}D - L^{\hat{+}\hat{+}}) = 2i\mathbb{C}D \to 0$ as $\delta \to \pi/4$, the conformal generator (LF time component) $\mathfrak{K}_-$ is Kinematic in LFD.
And $[D, P^{\hat{+}}] = iP^{\hat{+}}$, so D is always Kinematic in both IFD and LFD. [h]

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2,\ \mathcal{K}^{\hat{2}} = J^1,\ J^3, P^1, P^2, P^3,\ D$	$\mathcal{D}^{\hat{1}} = -K^1,\ \mathcal{D}^{\hat{2}} = -K^2,\ K^3, P^0, \mathfrak{K}_0, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$
$0 \leq \delta < \pi/4$	$\mathcal{K}^1, \mathcal{K}^2, J^3, P^1, P^2, P_{\hat{-}},\ D$	$\mathcal{D}^1, \mathcal{D}^2, K^3, P_{\hat{+}}, \mathfrak{K}_{\hat{+}}, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_{\hat{-}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1,\ \mathcal{K}^{\hat{2}} = -E^2,\ J^3, K^3, P^1, P^2, P_-,\ D, \mathfrak{K}_-$	$\mathcal{D}^{\hat{1}} = -F^1,\ \mathcal{D}^{\hat{2}} = -F^2,\ P_+, \mathfrak{K}_+, \mathfrak{K}_1, \mathfrak{K}_2$