Analysis of Virtual Meson Production in Solvable (1+1) Dimensional Scalar Field Theory

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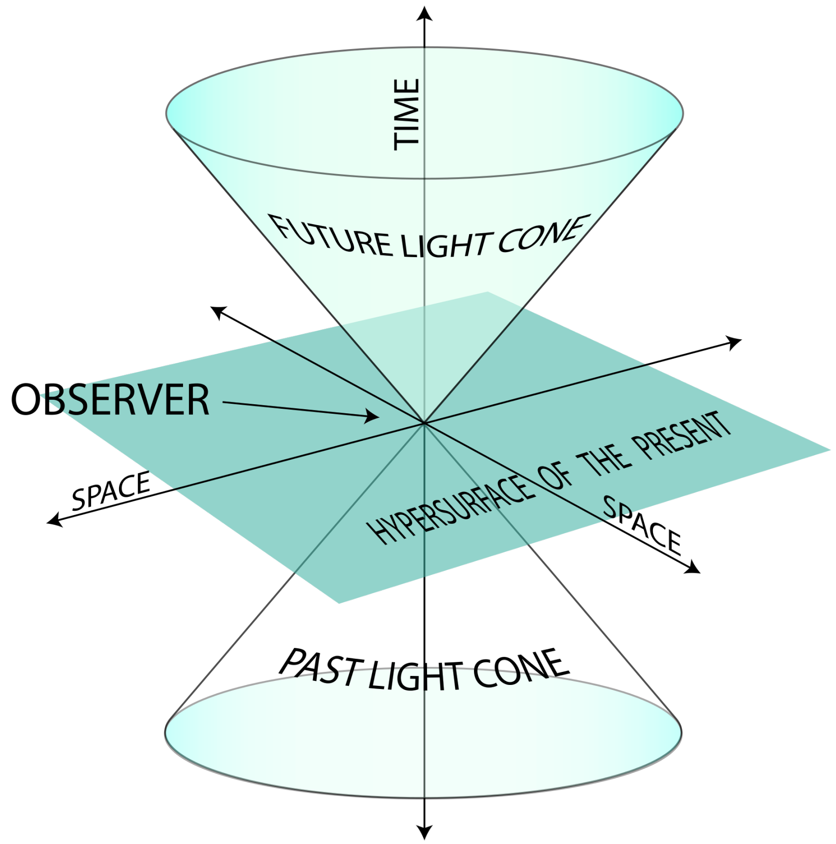
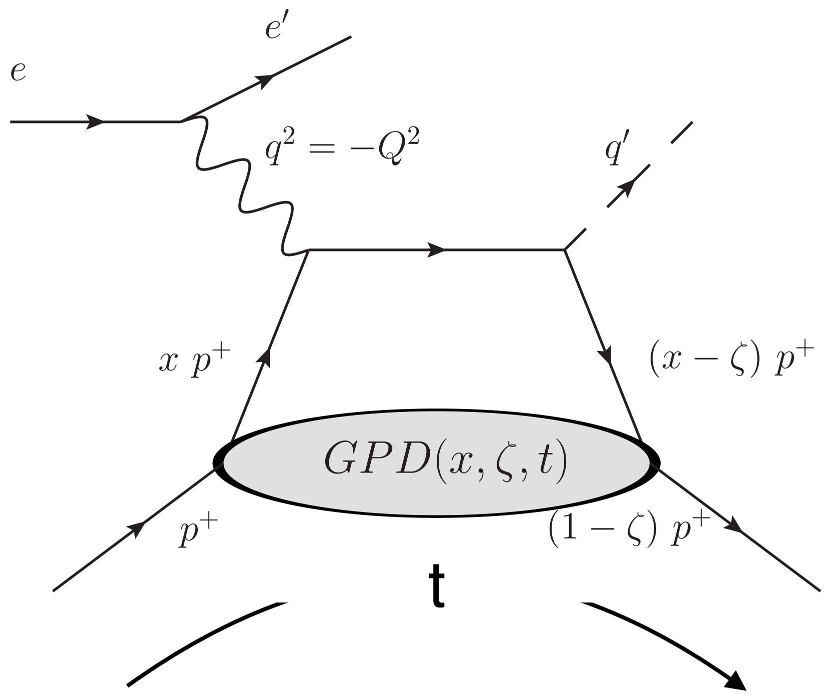


I. Review & Results
II. (1+1) & (3+1) dimensions
III. Imaginary part of scattering amplitude

Review & Results

Virtual Meson Production (VMP)



The skewed (non-forward) parton distribution :

hard part (VMP) $\otimes$ soft part (GPD)

In case $(\zeta,\ t \to 0)$, forward parton distribution :

hard part (VMP) $\otimes$ PDF

$\int dx$, elastic form factor :

F(t) = $\int dx$ GPD

p^+ : light-front(LF) longitudinal momentum of incoming target,

Q^2 : virtuality $\to$ factorable in large Q^2,

x : LF longitudinal momentum of parton,

ζ : skewness, $(p^+ - p'^+)/p^+$, asymmetry of LF longitudinal momentum of target,

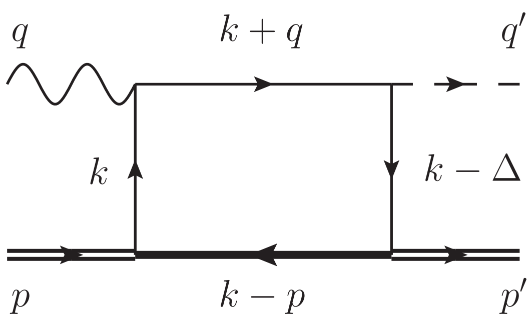
t : square momentum transfer, $\Delta^2 = (q'-q)^2 = (p-p')^2$,

"kick" transverse momentum depending on scattering angle.

$$t = -\frac{\zeta^2\ M_t^2 + \cancel{\Delta_\perp^2}}{1-\zeta}$$

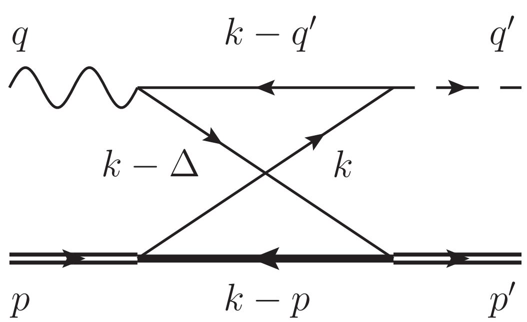
Box, Cat's ears, and Effective tree diagrams

Box-S



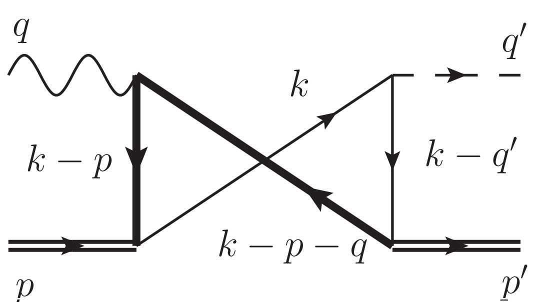
$$\mathscr{M}_s^\mu \sim i\int d^2k \frac{2k^\mu + q^\mu}{(k^2-m^2)((k+q)^2-m^2)((k-\Delta)^2-m^2)((k-p)^2-M^2)}$$

Box-U



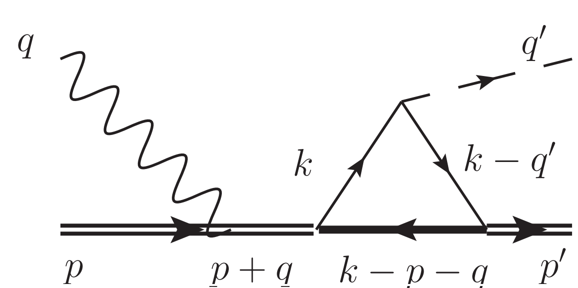
$$\mathscr{M}_u^\mu \sim i\int d^2k \frac{2k^\mu - \Delta^\mu - q'^\mu}{(k^2-m^2)((k-q')^2-m^2)((k-\Delta)^2-m^2)((k-p)^2-M^2)}$$

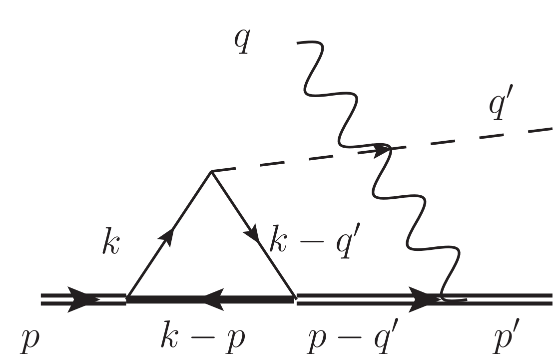
Cat's ears



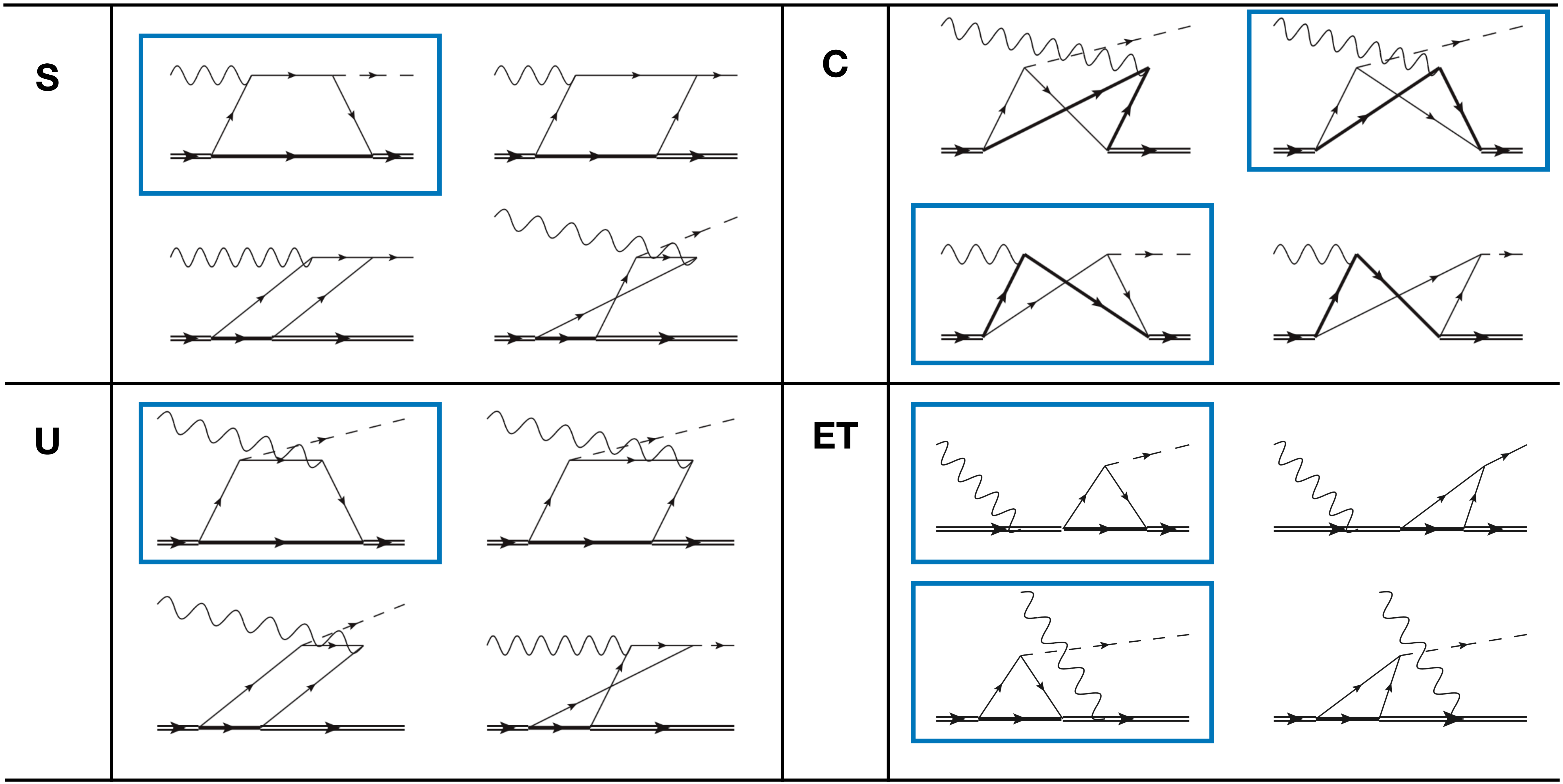
$$\mathscr{M}_c^\mu \sim i\int d^2k \frac{2k^\mu - 2p^\mu - q^\mu}{(k^2-m^2)((k-q')^2-m^2)((k-p-q)^2-M^2)((k-p)^2-M^2)}$$

For the charged target, +



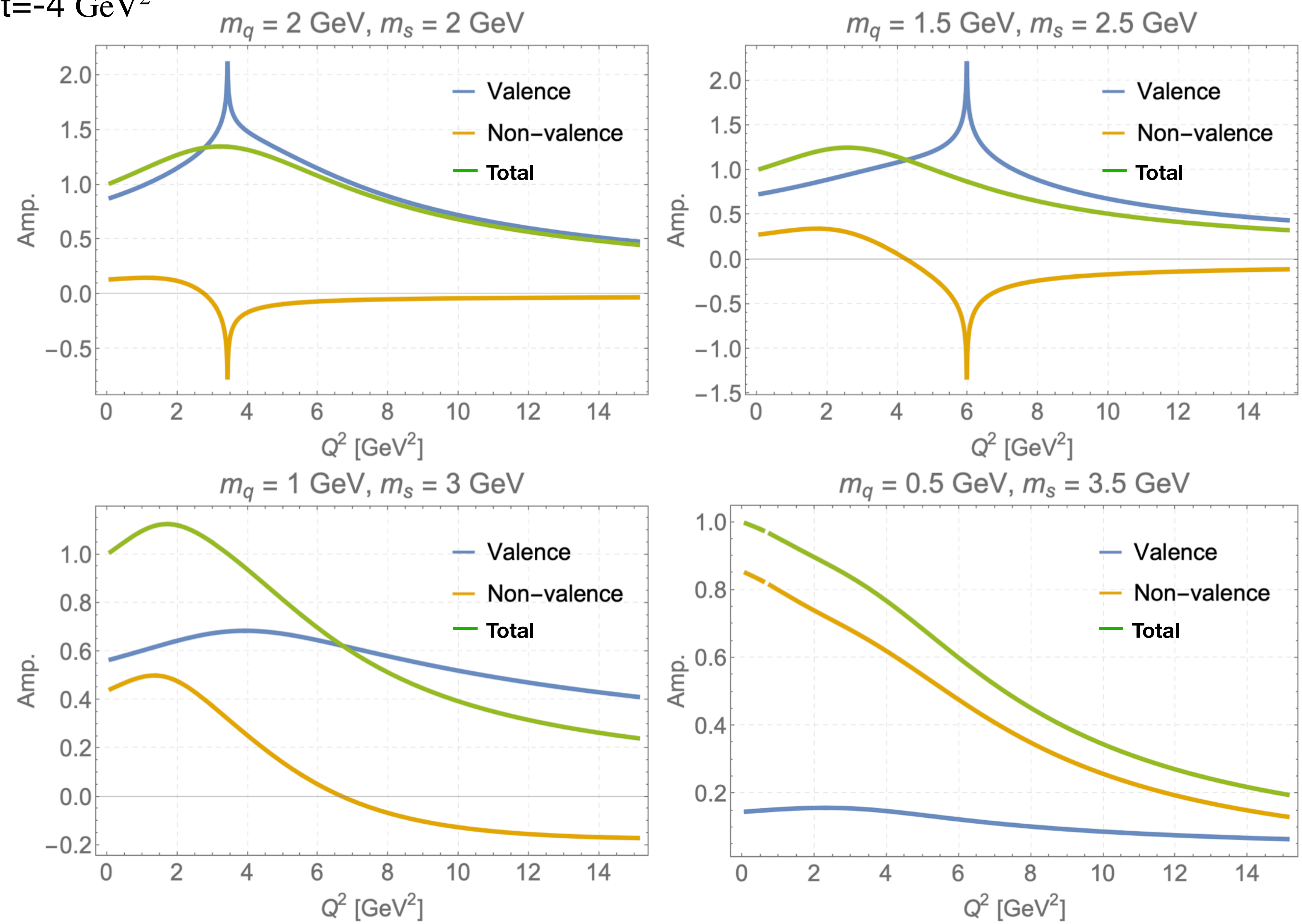


Time-ordered diagrams



Scalar Constituent Mass

t=-4 GeV^2



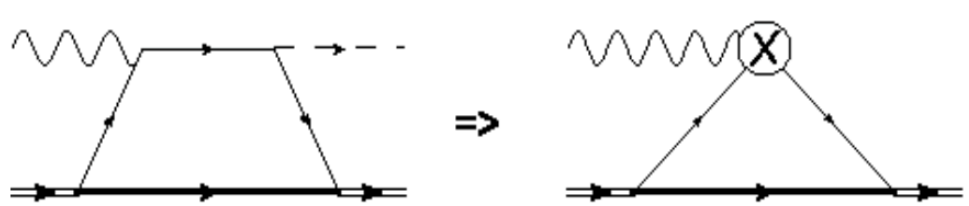
For the equal masses of constituents, the valence contribution is dominant.

In contrast, the non-valence contribution is getting larger as the mass asymmetry increases.

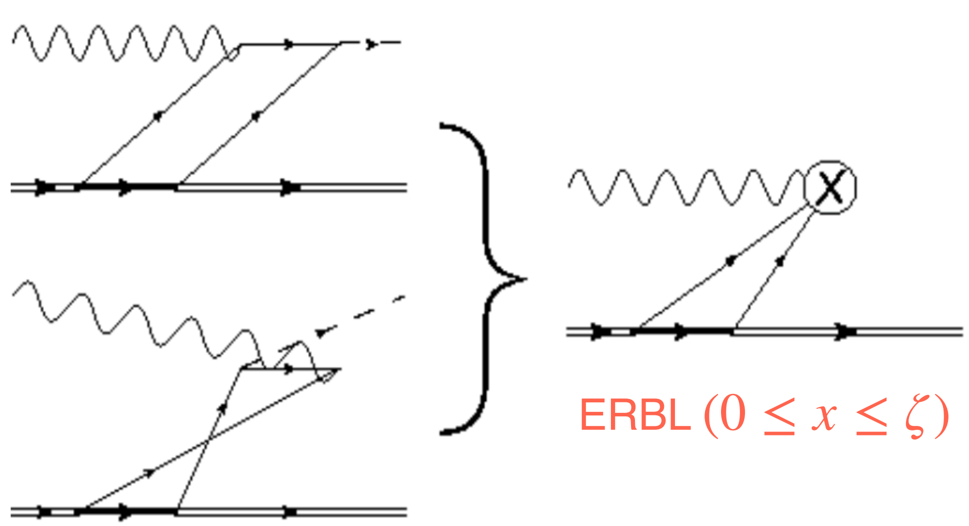
Reduced Form and GPDs Approximation

- In the deeply virtual limit, the virtuality Q^2 is very large.
  Thus q^- and $q'^{\,-}$ are dominant, leading to the reduced diagrams.

s-ch : $\dfrac{1}{(k+q)^2 - m^2} \sim \dfrac{1}{q^-\ (k^+ + q^+)} \sim \dfrac{1}{q^-\,\boxed{(x-\zeta)}\,p^+}$, u-ch : $\dfrac{1}{(k-q)^2 - m^2} \sim -\dfrac{1}{q'^-\ (k^+ - q'^+)} \sim -\dfrac{1}{q^-\,\boxed{x}\,p^+}$



DGLAP $(\zeta \le x \le 1)$



ERBL $(0 \le x \le \zeta)$

Generalized Parton Distributions

$$\mathscr{M} \sim \int_0^{\zeta} dx \left(\frac{1}{\boxed{x-\zeta}} - \frac{1}{\boxed{x}} \right) \boxed{H_{ERBL}(x,\, t)}$$

$$+ \int_{\zeta}^{1} dx \left(\frac{1}{\boxed{x-\zeta}} - \frac{1}{\boxed{x}} \right) \boxed{H_{DGLAP}(x,\, t)}$$

Parton Distribution Functions

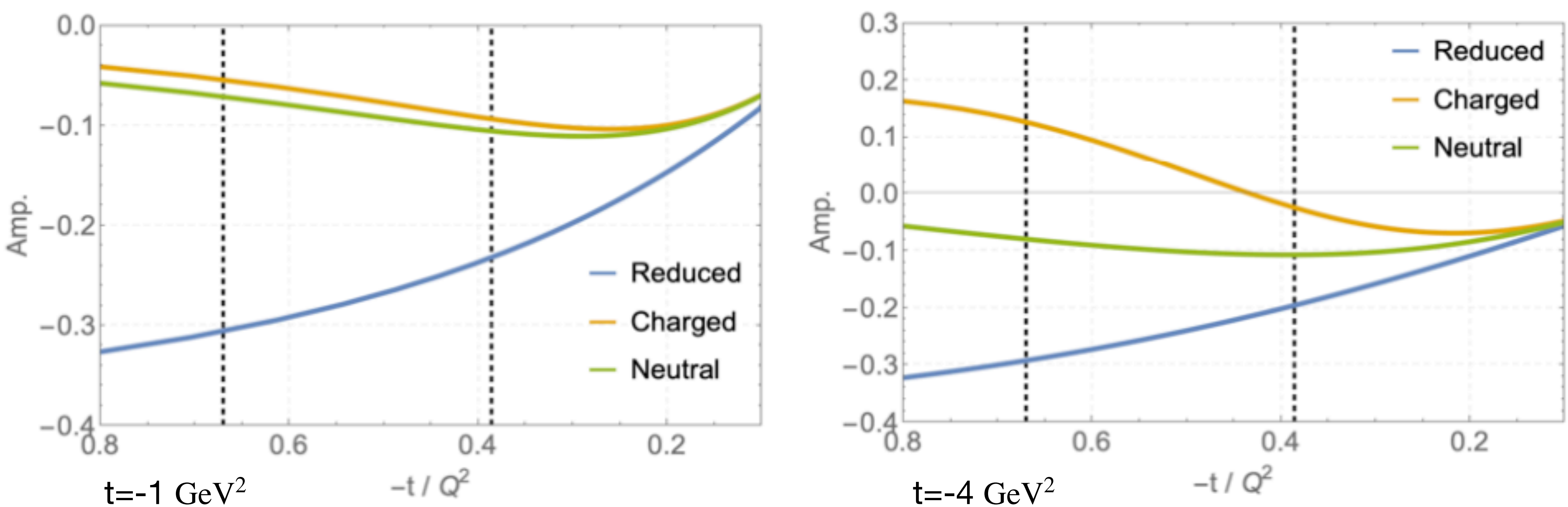
$$f(x) \sim \lim_{\zeta \to 0} H(x,\, \zeta,\, t) = H_{DGLAP}(x,\, t=0)$$

Elastic Form Factors

$$F(t) \sim \frac{1}{2-\zeta} \left[\int_0^{\zeta} dx\ H_{ERBL}(x,\, t) + \int_{\zeta}^{1} dx\ H_{DGLAP}(x,\, t) \right]$$

$-t/Q^2$ behavior

Q^2(GeV^2)	x	k|(GeV)	k' (GeV)	θ_e (°)	θ_q (°)	q'(0°) (GeV)	W^2 (GeV^2)	M (GeV)	t (GeV^2)	t_min (GeV^2)	$-t/Q^2$
1.9	0.36	5.75	2.94	19.3	18.1	2.73	4.2	3.72738	−1.06554	−0.955137	0.560813
3.	0.36	6.6	2.15	26.5	11.7	4.35	6.2	3.72738	−1.32826	−1.22078	0.442755
4.	0.36	8.8	2.88	22.9	10.3	5.83	8.	3.72738	−1.54314	−1.40201	0.385786
4.55	0.36	11.	4.26	17.9	10.8	6.65	9.	3.72738	−1.68065	−1.48463	0.369374
3.1	0.5	6.6	3.2	22.5	18.5	3.11	4.1	3.72738	−1.9983	−1.83768	0.644611
4.8	0.5	8.8	3.68	22.2	14.5	4.91	5.7	3.72738	−2.64071	−2.41298	0.550148
6.3	0.5	11	4.29	21.1	12.4	6.5	7.2	3.72738	−3.09918	−2.81838	0.491934
7.2	0.5	11.	3.32	25.6	10.2	7.46	8.1	3.72738	−3.27475	−3.02728	0.454826
5.1	0.6	8.8	4.27	21.1	17.8	4.18	4.3	3.72738	−3.41689	−3.15331	0.669978
6	0.6	8.8	3.47	25.6	14.1	4.97	4.9	3.72738	−3.74772	−3.51599	0.624621
7.7	0.6	11	4.16	23.6	13.1	6.47	6	3.72738	−4.45326	−4.12602	0.578346
9.	0.6	11	3	30.2	10.2	7.62	6.9	3.72738	−4.81139	−4.53706	0.534599



t=-1 GeV^2 t=-4 GeV^2

(1+1) & (3+1) dimensions

Kinematics

Kinematics in (1+1)-LFD

$$k = [\ xp^+,\ \ k^-\ ],$$

$$p = \left[\ p^+,\ \frac{M_t}{p^+}\ \right],$$

$$q = \left[\ \left(\mu_s\zeta' - \zeta\right) p^+,\ \ \left(\frac{1}{\zeta'} + \frac{\tau}{\zeta}\right) \frac{Q^2}{p^+}\ \right],$$

$$\Delta = q' - q = p - p' = \left[\ \zeta p^+,\ \ \frac{t}{\zeta p^+}\ \right],$$

$$p' = \left[\ (1-\zeta)\ p^+,\ \ \left(M_t^2 - \frac{t}{\zeta}\right) \frac{1}{p^+}\ \right],$$

$$q' = \left[\ \mu_s\zeta' p^+, \frac{Q^2}{\zeta' p^+}\ \right],$$

$\rightarrow$

Kinematics in (3+1)-LFD

$$k = [\ xp^+,\ \ k^-\ , \mathbf{k}_\perp],$$

$$p = \left[\ p^+,\ \frac{M_t}{p^+}\ , \mathbf{0}_\perp\right],$$

$$q = \left[\ \left(\mu_s\zeta' - \zeta\right) p^+,\ \ \left(\frac{1}{\zeta'} + \frac{\tau}{\zeta}\right) \frac{Q^2}{p^+}, \mathbf{0}_\perp\right],$$

$$\Delta = q' - q = p - p' = \left[\ \zeta p^+,\ \ \frac{t + \mathbf{\Delta}_\perp^2}{\zeta p^+}\ , \mathbf{\Delta}_\perp\right],$$

$$p' = \left[\ (1-\zeta)\ p^+,\ \ \left(M_t^2 - \frac{t}{\zeta}\right) \frac{1}{p^+}\ , -\mathbf{\Delta}_\perp\right],$$

$$q' = \left[\ \mu_s\zeta' p^+, \frac{Q^2}{M_s^2}\left(\frac{M_s^2 + \mathbf{\Delta}_\perp^2}{\zeta' p^+}\right), \mathbf{\Delta}_\perp\right],$$

where $\zeta = \dfrac{p^+ - p'^+}{p^+},\ \ \mu_s = \dfrac{M_s^2}{Q^2},\ \ \tau = \dfrac{-t}{Q^2},\ \ \Delta^2 = t.$

CR Ji, BLG Bakker,
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Compton Form Factors

- **Ward identity :** $q \cdot \mathscr{M} = 0 \;\rightarrow\; F_3 = -\,(q^2\ F_1 + \mathscr{P} \cdot q\ F_2)/(\Delta \cdot q)$
- **Two Lorentz vectors are parallel in (1+1)-dim. :** $A \cdot B = \pm\,|A|\,|B|$

$$\mathscr{M}^{\mu}_{tot} = q^{\mu}\ F_1 + \mathscr{P}^{\mu}\ F_2 + \Delta^{\mu}\ F_3$$

$$= \left(q^{\mu} - \frac{q^2}{\Delta \cdot q}\ \Delta^{\mu}\right) F_1 + \left(\ \mathscr{P}^{\mu} - \frac{\mathscr{P} \cdot q}{\Delta \cdot q}\ \Delta^{\mu}\right) F_2$$

$$= A^{\mu}\ F_1 + B^{\mu}\ F_2$$ (3+1)-dim.

$$= A^{\mu}\ (F_1 + cF_2) = A^{\mu}\ F_A$$ (1+1)-dim.

where $\mathscr{P} = p + p',$

$\Delta = p - p' = q' - q,$

$c = B^{+}/A^{+} = B^{-}/A^{-}.$

(1+1) dimensions

$$\mathscr{M}(Q^2,\, t) = \int dk^+ dk^- \; f(k^+,\, k^-)$$

$$\zeta_{(1+1)} = \frac{t + \sqrt{t^2 - 4\; t\; M_t^2}}{2M_t^2}$$

$$\mathscr{M}(Q^2,\, t) = \left(q^\mu - \frac{q^2}{\Delta \cdot q}\Delta^\mu\right) F_A$$

Because $A^\mu = \left(q^\mu - \frac{q^2}{\Delta \cdot q}\Delta^\mu\right) \parallel B^\mu = \left(\mathscr{P}^\mu - \frac{\mathscr{P} \cdot q}{\Delta \cdot q}\Delta^\mu\right)$

parallel relation : $A \cdot B = |A||B|$

(3+1) dimensions

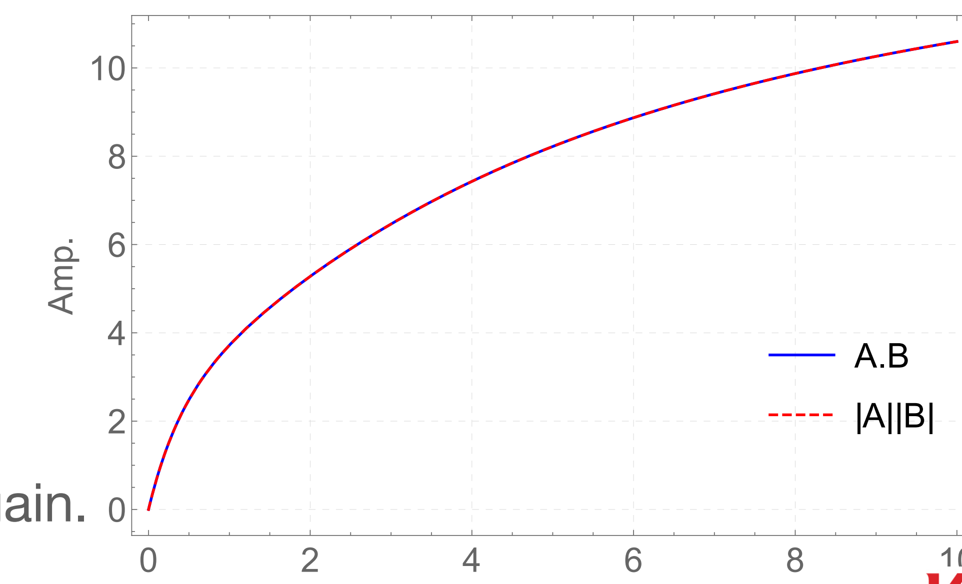
$$\mathscr{M}(Q^2,\, t,\, \zeta) = \int dk^+ dk^- d^2k_\perp \; f(k^+,\, k^-,\, \mathbf{k}_\perp)$$

$$\zeta_{3+1} = \frac{t + \sqrt{t^2 - 4\;(t + \Delta_\perp^2)\; M_t^2}}{2M_t^2}$$

$$\mathscr{M}(Q^2,\, t,\, \zeta) = \left(q^\mu - \frac{q^2}{\Delta \cdot q}\Delta^\mu\right) F_1 + \left(\mathscr{P}^\mu - \frac{\mathscr{P} \cdot q}{\Delta \cdot q}\Delta^\mu\right) F_2$$

In the (3+1) calculation, if we consider the maximum of ζ,

$$\zeta_{max} = \frac{t + \sqrt{t^2 - 4\; t\; M_t^2}}{2M_t^2} = \zeta_{(1+1)}$$



then, we encounter that the Compton form factor is reduced again.

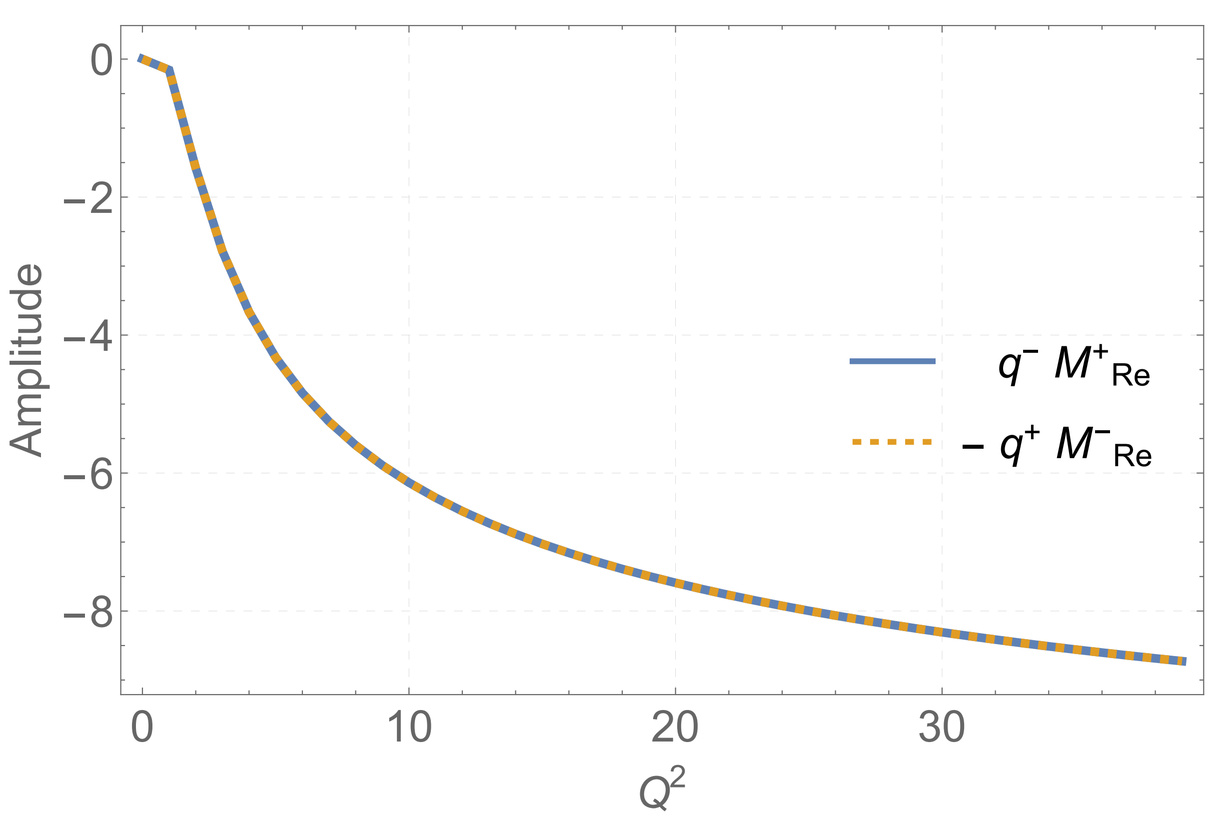
Ward Identity - (3+1) dimensions

Ward-Takahashi identity : $q \cdot \mathscr{M} = 0$

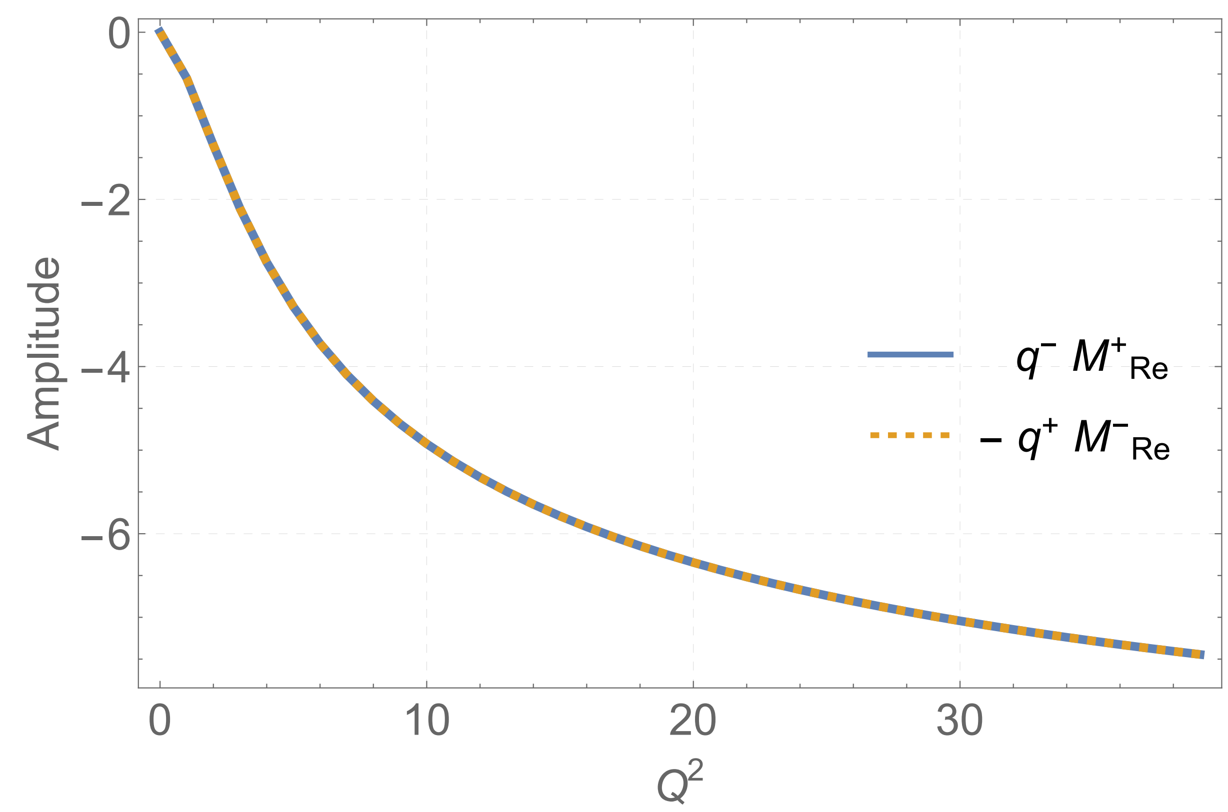
We set $q = (q^+, q^-, \mathbf{0}_\perp)$ in the kinematics, thus $q^+\mathscr{M}^- + q^-\mathscr{M}^+ = 0$

The identity does not hold without cat's ears and is satisfied with arbitrary t and ζ.

Charged



Neutral



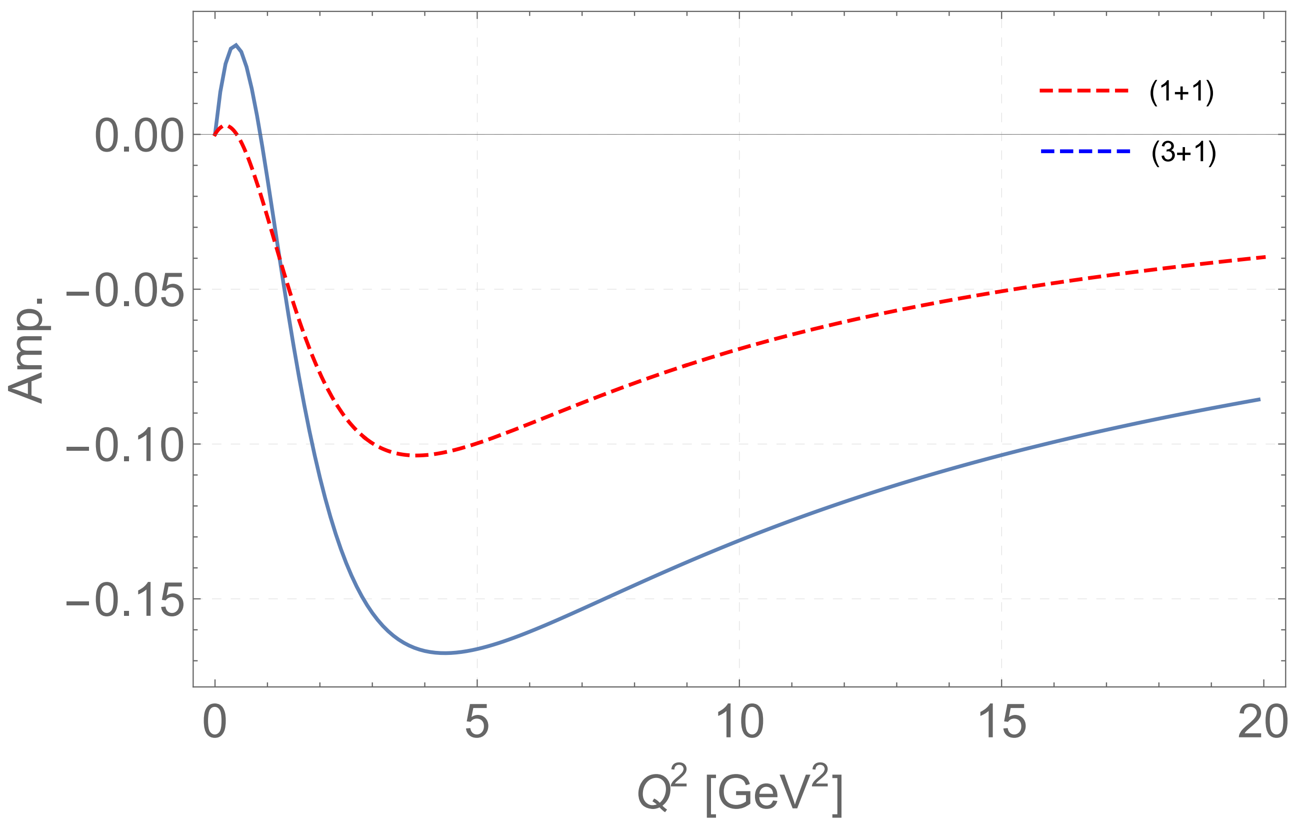
parameter : Mt=3.7, Mm=0.98, ms=2, mq=2, t=-1

Compare (3+1) with (1+1) dimensions

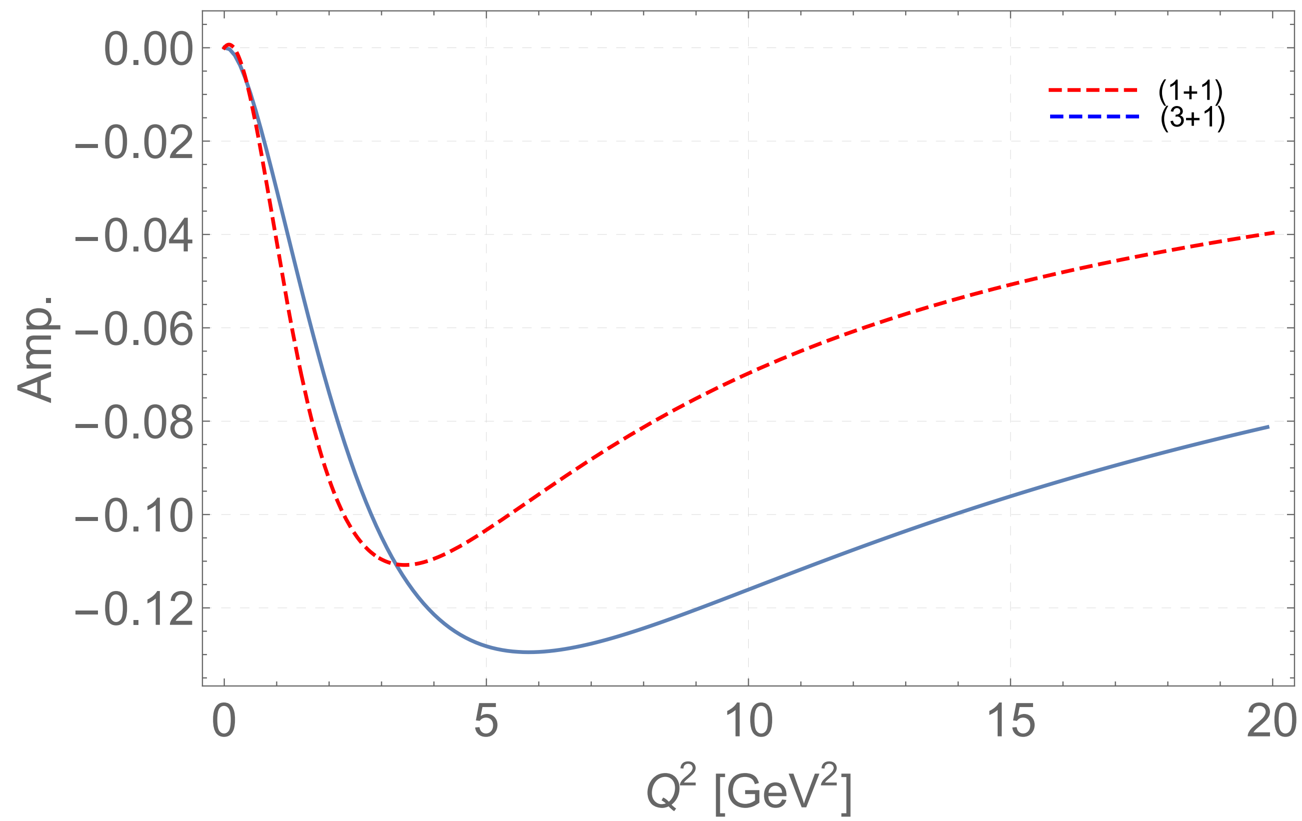
At first, we expected the (3+1) calculation to be completely consistent with the (1+1) result, if the (3+1) computation could be sorted in one direction, $\zeta = \zeta_{max}$ case. But, it was not.

Even if we can obtain the (3+1) results only in the z-direction, the calculation is integrated with $\mathbf{k}_\perp$.
On the other hand, the (1+1) calculation is evaluated excluding $\mathbf{k}_\perp$ itself.
These two things have completely different meanings. Thus, **two results cannot be the same.**

Charged



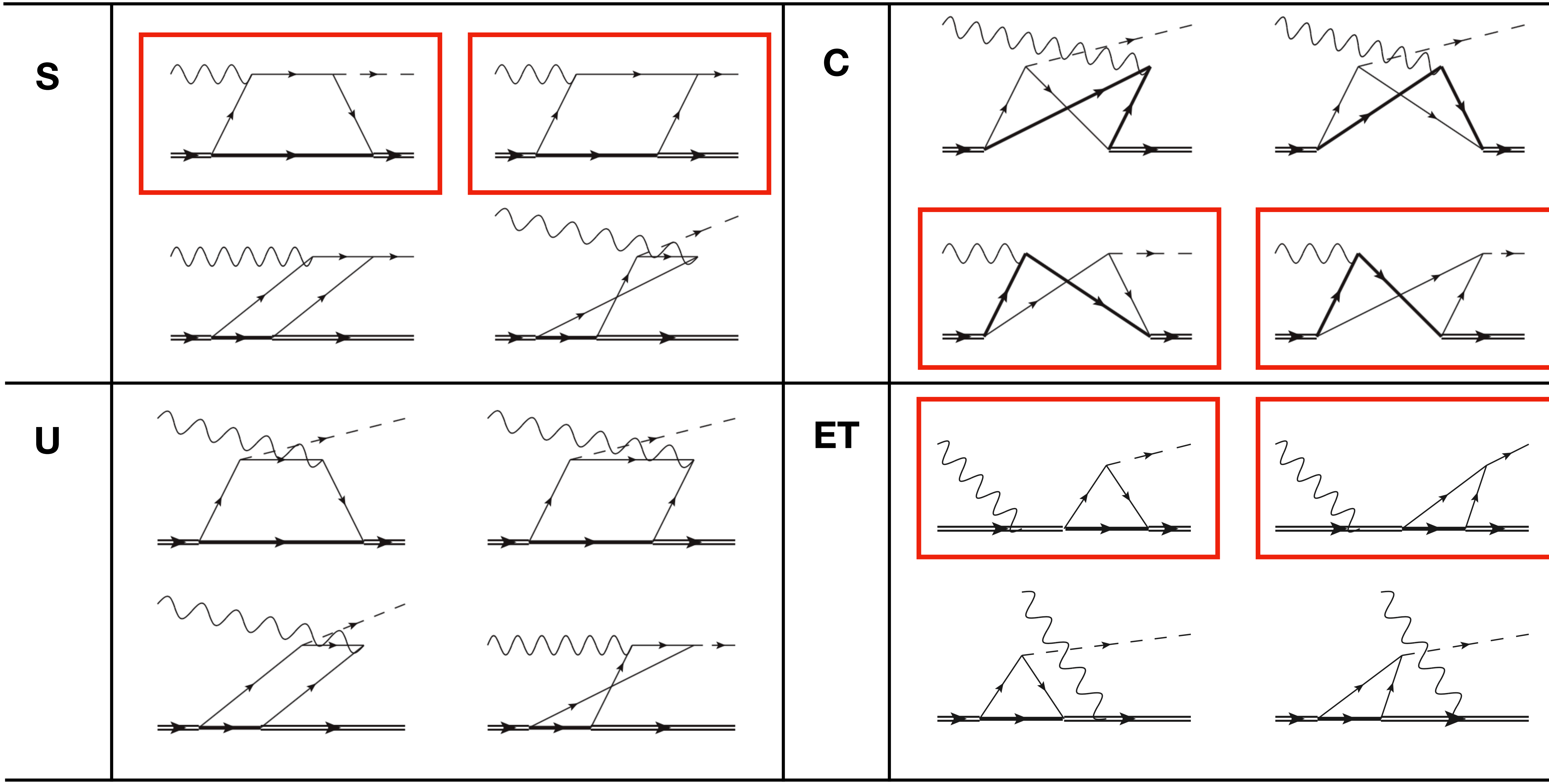
Neutral



parameter : Mt=3.7, Mm=0.98, ms=2, mq=2, t=-1, $\zeta = \zeta_{max} = 0.236204$

Imaginary part of scattering amplitude

Diagrams occurring the imaginary values



if we can draw the line across only the top and bottom constituents,
then the imaginary value can be generated.

$$\mathscr{M}_s^+ \sim i\int d^2k \frac{2k^+ + q^+}{(k^2-m^2)((k+q)^2-m^2)((k-\Delta)^2-m^2)((k-p)^2-M^2)}$$

$$= \frac{i}{2}\int dk^+ dk^- \frac{2k^+ + q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k^- - k_i^-)(k^- - k_t^-)(k^- - k_f^-)(k^- - k_b^-)}$$

$$k_i^- = \frac{m^2}{k^+} - i\epsilon\frac{1}{k^+}$$

$$k_t^- = -q^- + \frac{m^2}{k^+ + q^+} - i\epsilon\frac{1}{k^+ + q^+}$$

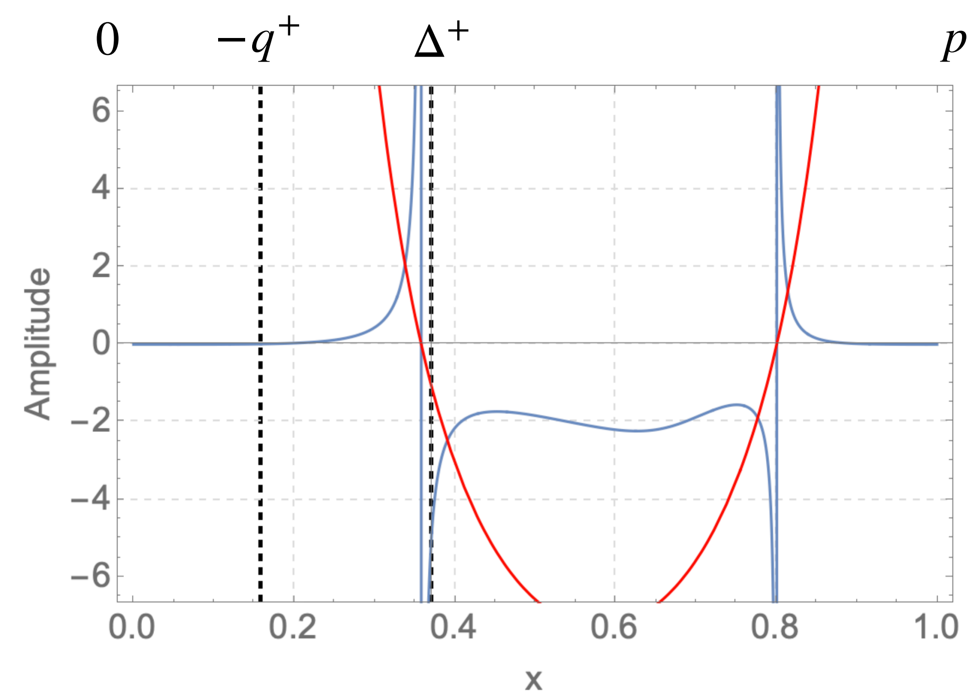
$$k_f^- = \Delta^- + \frac{m^2}{k^+ - \Delta^+} - i\epsilon\frac{1}{k^+ - \Delta^+}$$

$$k_b^- = p^- + \frac{M^2}{k^+ - p^+} - i\epsilon\frac{1}{k^+ - p^+}$$

For the handbag and open diamond diagrams

$$\mathscr{M}^+_{handbag} = -2\pi\int_{\Delta^+}^{p^+} dk^+ \frac{2k^+ + q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)\boxed{(k_b^- - k_t^-)}(k_b^- - k_f^-)}$$

$$\mathscr{M}^+_{open\ dia.} = -2\pi\int_{-q^+}^{\Delta^+} dk^+ \frac{-2k^+ - q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)\boxed{(k_b^- - k_t^-)}(k_f^- - k_t^-)}$$



$$k_b^- - k_t^- = p^- + q^- + \frac{M^2}{k^+ - p^+} - \frac{m^2}{k^+ + q^+} = \frac{1}{p^+}\left(M_t^2 + \frac{Q^2}{\zeta'} - \frac{t}{\zeta} + \frac{M^2}{x-1} - \frac{m^2}{x + \mu_s\zeta' - \zeta}\right) = 0$$

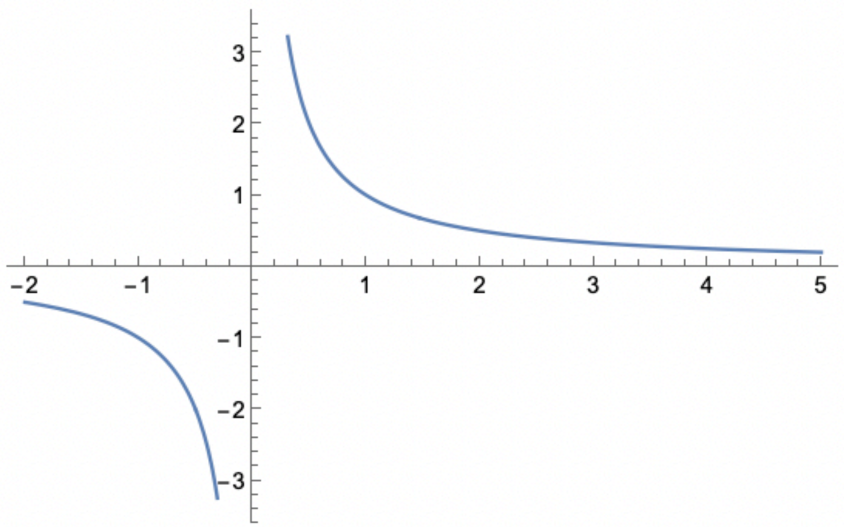
: the equation depends on Q^2 and t

When there are singularities in integrand, the imaginary values are occurred

Example 1

In[10]:= Plot[$\frac{1}{x}$, {x, -2, 5}]

Out[10]=



In[1]:= Integrate[$\frac{1}{x}$, x]

Out[1]= Log[x]

In[5]:= N[(Log[x] /. x → 5) - (Log[x] /. x → -2)]

Out[5]= 0.916291 - 3.14159 i

In[6]:= NIntegrate[$\frac{1}{x}$, {x, -2, 5}]

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

NIntegrate: NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {0.00965683}. NIntegrate obtained 0.3162187623014931 5` and 3.172832183744568`

Out[6]= 0.316219

In[9]:= NIntegrate[$\frac{1}{x}$, {x, -2, 5}, Method → $\boxed{\text{"PrincipalValue"}}$, Exclusions → x == 0]

NIntegrate: Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyG

Out[9]= 0.916291

The imaginary values cannot be evaluated by NIntegrate of Mathematica

Example 2

```
In[11]:= Integrate[1/(x (x - 4)), x]

Out[11]= 1/4 Log[4 - x] - Log[x]/4

In[12]:= N[(1/4 Log[4 - x] - Log[x]/4 /. x → 5) - (1/4 Log[4 - x] - Log[x]/4 /. x → -2)]

Out[12]= -0.677013 + 1.5708 i
```

$$\frac{1}{x \pm i\epsilon} = \mathscr{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

By using the relation, the imaginary values can be obtained, but, note that we do not know how to choose the sign of $i\epsilon$.

```
In[14]:= Integrate[1/4 (1/(x - 4) - 1/x), x]

Out[14]= 1/4 Log[4 - x] - Log[x]/4

In[23]:= 1/4 (1/(x - 4 + I ϵ)) - 1/4 (1/(x + I ϵ))

In[24]:= N[Integrate[-I π 1/4 DiracDelta[x - 4], {x, -2, 5}] - Integrate[-I π 1/4 DiracDelta[x], {x, -2, 5}]]

Out[24]= 0.

1/4 (1/(x - 4 - I ϵ)) - 1/4 (1/(x + I ϵ))

In[25]:= N[Integrate[I π 1/4 DiracDelta[x - 4], {x, -2, 5}] - Integrate[-I π 1/4 DiracDelta[x], {x, -2, 5}]]

Out[25]= 0. + 1.5708 i

1/(x (x - 4) - I ϵ)

In[26]:= N[Integrate[I π DiracDelta[x (x - 4)], {x, -2, 5}]]

Out[26]= 0. + 1.5708 i
```

Solution

We already have $i\epsilon$ in propagators of the covariant calculation such as $\dfrac{i}{k^2 - m^2 + i\epsilon}$

$$\mathcal{M}^+_{handbag} = -2\pi \int_{\Delta^+}^{p^+} dk^+ \frac{2k^+ + q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)\boxed{(k_b^- - k_t^-)}(k_b^- - k_f^-)}$$

$$k_b^- = p^- + \frac{M^2}{k^+ - p^+} - \boxed{i\epsilon}\frac{1}{k^+ - p^+}$$

$$k_t^- = -q^- + \frac{m^2}{k^+ + q^+} - \boxed{i\epsilon}\frac{1}{k^+ + q^+}$$

$$\boxed{k_b^- - k_t^-} = p^- + q^- + \frac{M^2}{k^+ - p^+} - \frac{m^2}{k^+ + q^+} + i\epsilon\left(\frac{1}{k^+ + q^+} - \frac{1}{k^+ - p^+}\right)$$

$$= \frac{1}{(k^+ - p^+)(k^+ + q^+)}\left[(k^+ - p^+)(k^+ + q^+)(p^- + q^-) + (k^+ + q^+)M^2 - (k^+ - p^+)m^2 + i\epsilon\Big(k^+ - p^+ - (k^+ + q^+)\Big)\right]$$

$$= \frac{p^- + q^-}{(k^+ - p^+)(k^+ + q^+)}\left[(k^+ - p^+)(k^+ + q^+) + \frac{k^+ + q^+}{p^- + q^-}M^2 - \frac{k^+ - p^+}{p^- + q^-}m^2 + i\epsilon\left(\boxed{-\frac{p^+ + q^+}{p^- + q^-}}\right)\right] < 0$$

$$= \frac{(p^- + q^-)}{(k^+ - p^+)(k^+ + q^+)}\Big((k^+ - k_l^+)(k^+ - k_u^+) - i\epsilon\Big)$$

where $$k^+_{u,l} = \frac{1}{2(p^- + q^-)}\left[m^2 - M^2 + (p^- + q^-)\left(p^+ - q^+ \pm \sqrt{\frac{((m - M)^2 - (p^- + q^-)(p^+ + q^+)((m + M)^2 - (p^- + q^-)(p^+ + q^+))}{(p^- + q^-)^2}}\right)\right]$$

$$\mathscr{M}^{+}_{handbag} = -2\pi \int_{\Delta^+}^{p^+} dk^+ \frac{2k^+ + q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)(k_b^- - k_f^-)} \frac{(k^+ - p^+)(k^+ + q^+)}{(p^- + q^-)} \frac{1}{(k^+ - k_l^+)(k^+ - k_u^+) - i\epsilon}$$

Using $\dfrac{1}{x \pm i\epsilon} = \mathscr{P}\left(\dfrac{1}{x}\right) \mp i\pi\delta(x)$

$$\mathscr{M}^{+}_{handbag} = \mathrm{Re}[\mathscr{M}^{+}_{handbag}] + i\pi(-2\pi) \int_{\Delta^+}^{p^+} dk^+ \frac{2k^+ + q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)(k_b^- - k_f^-)} \frac{(k^+ - p^+)(k^+ + q^+)}{(p^- + q^-)} \delta((k^+ - k_l^+)(k^+ - k_u^+))$$

$$= \mathrm{Re}[\mathscr{M}^{+}_{handbag}] + \boxed{i\pi(-2\pi) \int_{\Delta^+}^{p^+} dk^+ \frac{2k^+ + q^+}{k^+(k^+ - \Delta^+)(p^- + q^-)} \frac{\delta((k^+ - k_l^+)(k^+ - k_u^+))}{(k_b^- - k_i^-)(k_b^- - k_f^-)}}$$

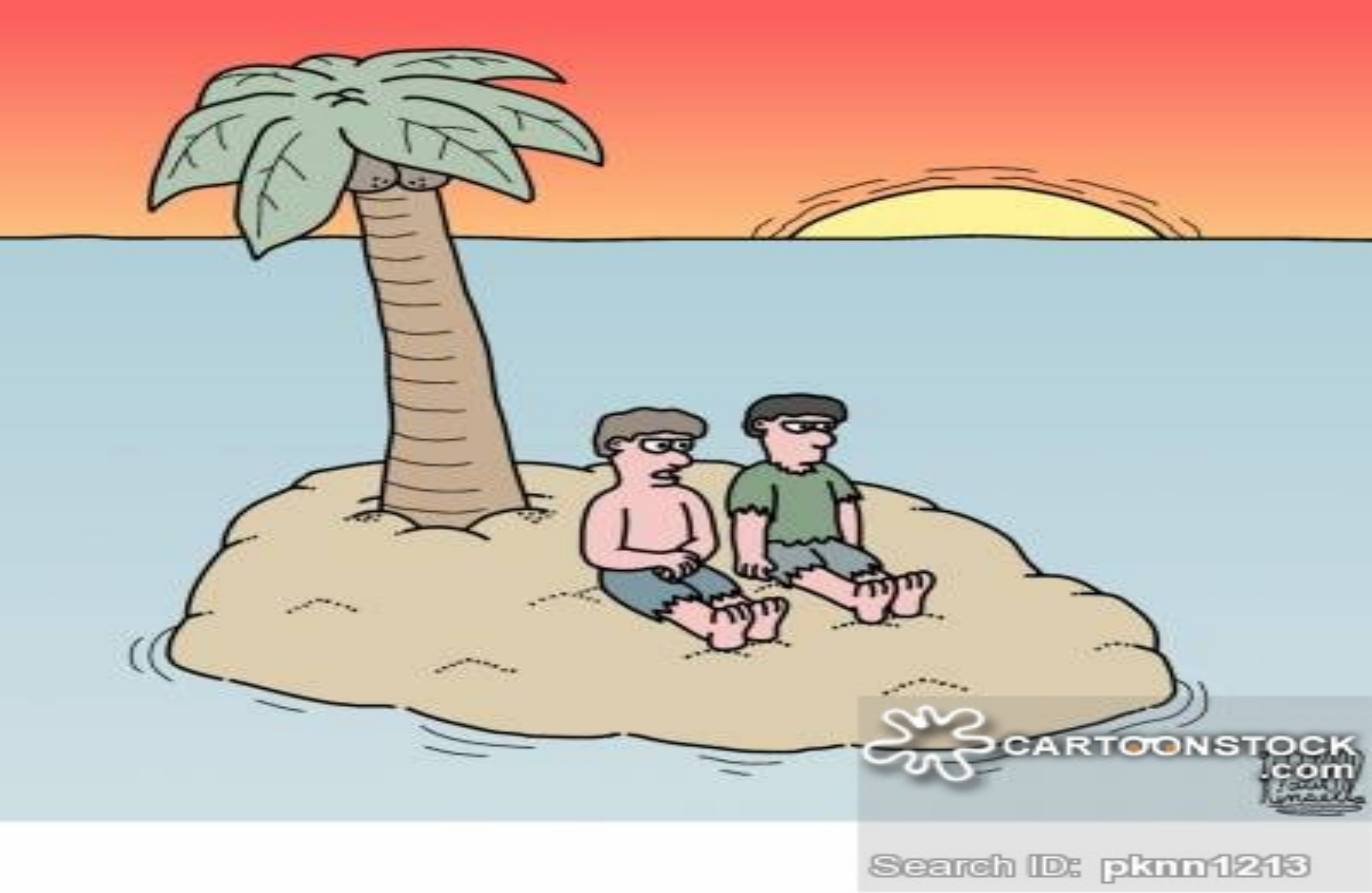
$$\mathscr{M}^{+}_{open\ dia.} = \mathrm{Re}[\mathscr{M}^{+}_{open\ dia.}] + i\pi(-2\pi) \int_{-q^+}^{\Delta^+} dk^+ \frac{-2k^+ - q^+}{k^+(k^+ + q^+)(k^+ - \Delta^+)(k^+ - p^+)} \frac{1}{(k_b^- - k_i^-)(k_f^- - k_t^-)} \frac{(k^+ - p^+)(k^+ + q^+)}{(p^- + q^-)} \delta((k^+ - k_l^+)(k^+ - k_u^+))$$

$$= \mathrm{Re}[\mathscr{M}^{+}_{open\ dia.}] + \boxed{i\pi(-2\pi) \int_{-q^+}^{\Delta^+} dk^+ \frac{-2k^+ - q^+}{k^+(k^+(k^+ - \Delta^+)(p^- + q^-)} \frac{\delta((k^+ - k_l^+)(k^+ - k_u^+))}{(k_b^- - k_i^-)(k_f^- - k_t^-)}}$$

Additionally, $\delta[g(x)] = \sum_i \dfrac{\delta(x - x_i)}{|g'(x_i)|}$, Now just consider that k_l^+ and k_u^+ are included in integral domain.

Conclusion

- ✓ The non-valence contribution is getting larger as the asymmetry of masses increase.
- ✓ For given experimental data, GPD-like approximated result is not well fit with the result of considering whole diagrams.
- ✓ In (3+1) calculation, even if the scattering amplitude is sorted in one direction, it does not produce the same results as (1+1) dimensional calculation.
- ✓ When dealing with singular integral, we have to be careful about the sign of the imaginary part, which can be selected through $i\epsilon$ in the propagators.



"Thank you for listening."