# Pseudo-scalar and -vector meson transition form factors in 1+1 dimensions

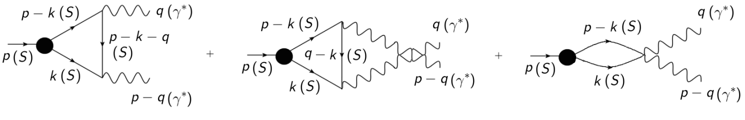
Bailing Ma

group meeting

Mar. 4, 2022

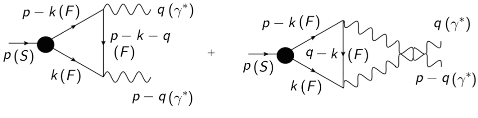
- Last time I presented the transition form factors of scalar meson in 1+1-D scalar model as well as the fermion loop calculation
- Both manifestly covariant method and light-front dynamics calculations were done, and results found to agree, DR was satisfied, although for the fermion loop case there is singularity at the threshold which is still under investigation
- For the light-front time-ordered calculation, I showed how choosing different component ($\Gamma^{++}$ vs $\Gamma^{+-}$, for example), can result in different LFTO amplitudes, however the sum of all time-ordered contributions is the same as it must be. A new definition of the each individual LFTO contribution was presented.

# Last time




(D) (C) (S)

Three diagrams contributing to the amplitude of a scalar meson going to 2 virtual photons in the scalar field theory




(D) (C)

$$F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \int_0^1 dx \int_0^{1-x} dy (1-2y) \left( \frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right)$$

Two diagrams contributing to the amplitude of a scalar meson going to 2 virtual photons in QED $_{1+1}$

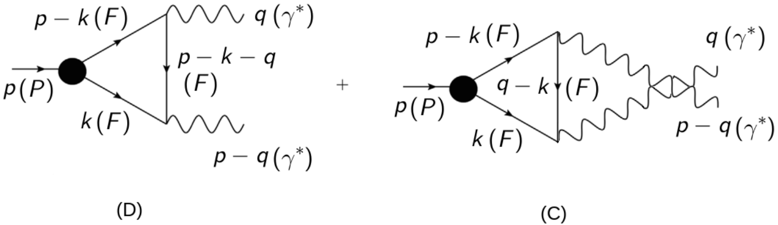
$$\boxed{\Gamma^{\mu\nu} = F(q^2, q'^2)\,(g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu})}$$

$$F(q^2, q'^2) = \frac{e^2 g N_c m}{\pi} \int_0^1 dx \int_0^{1-x} dy (-y) \left( \frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right)$$

where

$$\Delta_1 = x(x-1)q^2 + 2x(x+y-1)q \cdot q' + (x+y)(x+y-1)q'^2 + m^2$$
$$\Delta_2 = x(x-1)q'^2 + 2x(x+y-1)q \cdot q' + (x+y)(x+y-1)q^2 + m^2$$

# This time




- Now let us calculate the transition form factor of a pseudoscalar/pseudovector meson going into 2 virtual photons
- In 3+1-D this contributes to the axial anomaly

# Historical remarks on the axial anomaly

C. Adam, R.A. Bertlmann and P. Hofer, Riv. Nuovo. Cim. 16 (1993) 1

**1. – Introduction.**

1·1. *Motivation and historical survey.* – Generally speaking, a quantum field theory is called anomalous when a symmetry that is valid classically (*i.e.* fulfilled by the equations of motion) cannot be maintained on the quantum level.

The discovery of the axial anomaly has quite a long history. As soon as 1949 Steinberger [1] computed some Feynman graphs (that, in today's language, contribute to the anomaly) for the $\pi^0 \to \gamma\gamma$ decay. He found that some requirements of the axial symmetry were not fulfilled by his results, but this fact was too surprising to be accepted, and, as a consequence, he left theory and became a Nobel Laureate in experimental physics.

Later on it was considered, *e.g.*, by Schwinger in 1951 [2] or by Johnson in 1963 [3], that the conservation of the axial current — an immediate consequence of the axial symmetry — was violated for appropriately regularized current operators. But even the importance of those results was not noticed in the subsequent years.

However, in the sixties the belief in the axial symmetry caused some problems. The way this symmetry principle entered the computations in hadronic physics was named PCAC (= «partial conservation of the axial current» — partial, because when the fermions have masses there is no exact axial symmetry). For instance, using PCAC, Sutherland and Veltman stated a theorem that predicted a strong suppression of the decay of a neutral pion into two photons. But this result was in contradiction to well-known experimental facts.

So, the situation was prepared for a real start of the anomaly story, and this occurred in 1969. In this year Adler and, independently, Bell and Jackiw derived the anomaly using Feynman-graph methods for quantum electrodynamics (QED) and for the linear $\sigma$-model, respectively [4, 5] (therefore in honour of its discoverers the axial anomaly in four dimensions is often named ABJ anomaly).

The $\sigma$-model was used to describe pion-nucleon physics, and so the anomaly result of Bell and Jackiw solved the $\pi^0 \to \gamma\gamma$ puzzle: the anomaly corrected the decay rate resulting from the Sutherland-Veltman theorem by a definite amount that was in excellent agreement with the experimental results.

From that moment on the interest in the anomaly problem steadily increased. It was noticed that in chiral gauge theories like the standard model of the electroweak interactions the anomaly could spoil gauge invariance (*e.g.*, by Gross and Jackiw in 1972 [6]) thereby ruining the theory at least on a perturbative level. Avoiding this leads to severe constraints on the particle contents of a theory (*e.g.*, to the prediction of the top quark; moreover this «anomaly cancellation» mechanism requires three quark colours to work and therefore supports QCD).

When you want to re-derive the anomalous $\pi^0 \to \gamma\gamma$ result within the quark model then the number of colours must be three, too.

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# $\gamma_5$ INVARIANCE *

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It has long been argued with the use of formal $\gamma_5$ invariance, that a Lagrangian which contains no bare mass for Fermi particles implies that no mass can be generated by an interaction which "possesses" the symmetry, if the vacuum possesses the symmetry. If the vacuum does not possess the symmetry then other objectionable features arise (zero mass pseudoscalar states) [1]. Recently it has been shown that another exception can occur if the Fermion mass operator has certain anomalies in its spectrum [2]. However, the symmetry is still present in that *two* Fermi particles appear (with masses $\pm m$). This would then mean that the zero mass pseudoscalar particles could be avoided but only with the cost of a doublet of Fermions with opposite parities.

It is the purpose of this note to point out a mathematical (but nonetheless interesting) way out of the dilemma. The formal use of an invariance in the Lagrangian is very dangerous because of the loosely defined character of the operators it contains. In fact, charge conservation holds for the example of a free Fermion interacting with an external electromagnetic field only if the current is defined by very careful limiting procedures [3,4] which allow one to circumvent the so-called photon mass term in the vacuum polarisation, which gives a term in $j^\mu(x)$ proportional to $A_\mu{}^{\text{ext}}(x)$ and hence a $j^\mu(x)$ which is not conserved. Thus, the formal use of the equations of motion together with the formal expression for the current

$$j^\mu(x) = \tfrac{1}{2}\left[\bar{\psi}(x)\gamma^\mu,\ \psi(x)\right]$$

does not enable one safely to conclude that $\partial_\mu j^\mu = 0$. In this problem the current must be defined by a limiting procedure applied to the product

$$\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x)\,\left(1 - i\int_x^{x+\epsilon} d\xi^\mu\, A_\mu^{\text{ext}}(\xi)\right),$$

where $A_\mu{}^{\text{ext}}$ is the external potential. This form is locally gauge invariant and this insures conservation in the limit. Without the factor involving $A_\mu{}^{\text{ext}}$ the resulting operator is not conserved.

Since such great care is needed in this example to get an invariant theory, one might worry also about the "proof" of the conservation law for $\bar{\psi}\gamma^\mu\gamma_5\psi$. In the electrodynamic example one can circumvent the difficulty by using the terms in the *interaction* $(A_\mu)$ to generate a compensation for the singular term. One need not *always* be able to do this. In fact, a simple, exactly soluable model in one space-one time dimension [5] illustrates this very nicely and provides a counter example to the formal proof. The model is quantum electrodynamics with $m_0 = 0$ for the charged Fermi particles. The Lagrangian is formally $\gamma_5$ invariant. However, if one solves the model, one obtains for the current-current vacuum expectation value

$$\langle j^\mu(q)\ j^\nu(q)\rangle = \left(g^{\mu\nu}\frac{e^2}{\pi} + q^\mu q^\nu\right)\cdot\ \delta\left(q^2 + \frac{e^2}{\pi}\right),$$

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where $e$ is the coupling constant. One easily sees that this is consistent with charge conservation, $\boxed{q^\mu j_\mu(q) = 0.}$ However, in one space-one time dimension since

$$\gamma^5 = \gamma^0\gamma^1\,, \qquad \gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$$

the pseudovector current is

$$j_5^\mu = \epsilon^{\mu\nu}\ j\ ,$$

so

$$\langle j_5^\mu(q)\ j_5^\nu(-q)\rangle = \left(-g^{\mu\nu}\frac{e^2}{\pi} + \epsilon^{\mu\lambda}q_\lambda\epsilon^{\nu\sigma}q_\sigma\right)\delta\left(q^2 + \frac{e^2}{\pi}\right),$$

and thus,

$$\boxed{q_\mu\langle j_5^\mu(q)\ j_5^\nu(q)\rangle = -\frac{e^2}{\pi}q^\nu\ \delta\left(q^2 + \frac{e^2}{\pi}\right) \neq 0\,.}$$

Hence only in the absence of coupling is the theory $\gamma_5$ invariant. This means simply that the care required to make $j^\mu A_\mu$ a meaningful and gauge invariant coupling also yields the result that the coupling is not actually $\gamma_5$ invariant. The formal "proof" of that invariance is just incorrect.

*References*

1) Goldstone, Salam and Weinberg, Phys. Rev. 127 (1962) 965.
2) W. Thirring, Physics Letters 4 (1963) 167.
3) J. Schwinger, Phys. Rev. Letters 3 (1959) 296.
4) K. Johnson, Nuclear Phys. 25 (1961) 431; Nuovo Cimento 20 (1961) 773.
5) J. Schwinger, Phys. Rev. 128 (1962) 2425.

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# Perturbative calculation of the axial anomaly in 3+1-D

## Ryder's QFT book



group (Zinn-Justin 1975).

### 9.10 Chiral anomalies

In QED and QCD, the only type of interaction between matter and gauge fields is a vector interaction, of the form $g_V J_\mu W^\mu$ where $W^\mu$ is the gauge field and $J_\mu \sim \bar{\psi}\gamma_\mu\psi$ is the vector current (internal indices being suppressed) of the Fermi matter field. The vector current is conserved:

$$\boxed{\partial^\mu J_\mu = 0,} \tag{9.242}$$

and this leads to a Ward identity for the vertex function: for the graph of Fig. 9.22, for example, whose amplitude is

$$W^\mu \langle p'|J_\mu|p\rangle, \tag{9.243}$$

current conservation implies

$$(p' - p)^\mu J_\mu = q^\mu J_\mu = 0 \tag{9.244}$$

which is the Ward identity.

In the Weinberg–Salam theory, however – more generally, in a gauge theory of weak interactions, commonly called *quantum flavour-dynamics* or QFD – there is also an *axial vector* coupling between matter and gauge fields, $g_A J^5_\mu W^\mu$, with

$$J^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi. \tag{9.245}$$

Simple use of the Dirac equation shows that

$$\boxed{\partial^\mu J^5_\mu = 2\mathrm{i}m\bar{\psi}\gamma_5\psi \equiv 2mJ_5.} \tag{9.246}$$

$J_5$ may be called the chiral density. The axial current is not conserved unless $m = 0$, but (9.246) does give rise to an *axial Ward identity*, even in the case $m \neq 0$. To keep matters simple, we shall consider below the situation when $m = 0$, so the axial current is exactly conserved. This simplification is amply justified because the complications which arise in the massive case remain unaffected by putting $m = 0$.

The statements above are of course $\boxed{\text{purely formal.}}$ What we now do is to set about verifying them in *perturbation theory*. For the vector coupling, this was done in Chapter 7; an expansion of the vertex function $\Gamma_\mu$ to order $e^3$ gives the



two diagrams of Fig. 9.22, and it is found that the Ward identity

$$(p' - p)^\mu \Gamma_\mu = 0$$

is indeed satisfied.

An analogous expansion of the axial vertex is shown in Fig. 9.23, up to order $e^5$. In each diagram in the expansion, the lowest vertex is $\gamma_\mu \gamma_5$, an axial coupling, and the others are vector couplings. What we find is that the last graph, containing a 'triangle' closed loop of Fermi fields, fails to satisfy the axial Ward identities, giving rise to the so-called axial, or chiral, or triangle anomaly. The serious nature of this anomaly lies in the fact that, as has been emphasised already, the Ward identity (and its appropriate generalisation in the non-Abelian case) is essential to proving the renormalisability of gauge theories; so the triangle anomaly threatens renormalisability of the Weinberg–Salam model, which would be a disaster! The only way of saving renormalisability is to ensure that the *total* contribution of the triangle graphs is zero, so the anomalies cancel. This is a condition on the fermion content of the theory, which, remarkably, turns out to be satisfied in the Weinberg–Salam model if there exist quarks as well as leptons, and if the quarks carry an additional $SU(3)$ (colour) label.

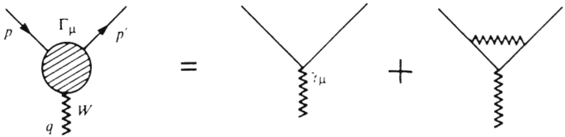



Fig. 9.22. Expansion of the vector coupling between gauge and matter fields to order $e^3$.

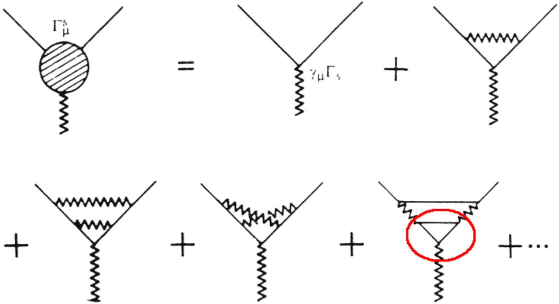



Fig. 9.23. Expansion of the axial vector vertex between gauge and matter fields to order $e^5$.

Let us now consider the fermion triangle in Fig. 9.23. (Actually, it is not only the $AVV$ triangle ($A = \gamma_\mu\gamma_5$ vertex, $V = \gamma_\mu$ vertex) which contributes to $\Gamma^5_\mu$, but also the $AAA$ triangle, and also square and pentagon configurations, which occur in higher orders. It turns out, however, that when the anomaly is cancelled in the $VVA$ graph, it disappears in all of them, so it is sufficient to consider the $VVA$ triangle. It also turns out that the anomaly is unaffected by radiative corrections. So, again, we need only consider the simplest $VVA$ graph. For details of these matters, the student is referred to the literature.) There are two contributions to it, shown in Fig. 9.24. The fermion contribution to the amplitude (i.e. we ignore the gauge field propagators) is

$$T_{\kappa\lambda\mu}(p_1, p_2) = S_{\kappa\lambda\mu}(p_1, p_2) + S_{\lambda\kappa\mu}(p_2, p_1), \tag{9.247}$$

since the second graph in Fig. 9.24 is obtained from the first by interchanging $\kappa \leftrightarrow \lambda$, $p_1 \leftrightarrow p_2$. The Feynman rules give (ignoring the coupling constants, which in a real process will depend on $\theta_\mathrm{W}$)

$$S_{\kappa\lambda\mu}(p_1, p_2) = -(-\mathrm{i})^3 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\kappa \frac{\mathrm{i}}{\not{k} - \not{p}_1 - m}\gamma_\mu\gamma_5 \frac{\mathrm{i}}{\not{k} + \not{p}_2 - m}\gamma_\lambda \frac{\mathrm{i}}{\not{k} - m}\right).$$

Putting $m = 0$ (see above) gives

$$S_{\kappa\lambda\mu} = -(2\pi)^{-4} \int \mathrm{d}^4 k \, \frac{\mathrm{Tr}\left[\gamma_\kappa(\not{k} - \not{p}_1)\gamma_\mu\gamma_5(\not{k} + \not{p}_2)\gamma_\lambda \not{k}\right]}{(k - p_1)^2 (k + p_2)^2 k^2}. \tag{9.248}$$

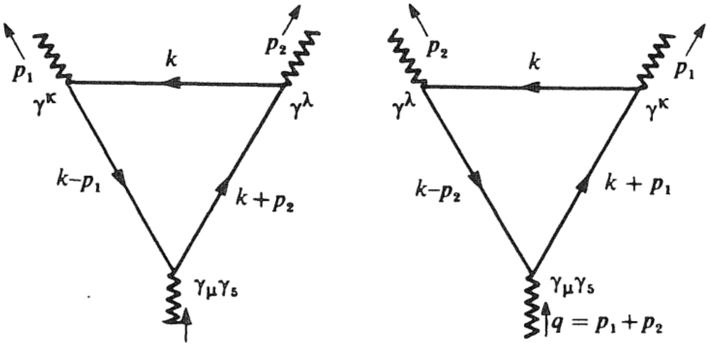



Fig. 9.24. The triangle graphs.

The expected Ward identities, following from conservation of the vector and axial currents at the three vertices, are

$$\left.\begin{aligned}(p_1 + p_2)^\mu T_{\kappa\lambda\mu} &= 0\ (A),\\ p_1^\kappa T_{\kappa\lambda\mu} &= 0\ (V),\\ p_2^\lambda T_{\kappa\lambda\mu} &= 0\ (V).\end{aligned}\right\} \qquad (9.249a, b, c)$$

The $(V)$ identities follow from conservation of charge, and what we shall show is that, if these two hold, then the $(A)$ identity (9.249$a$) does not hold, unless we impose extra conditions.

Note, incidently, that $S$ is a *linearly* divergent integral. Also, it is symmetric under the intercharge $(p_1, \kappa) \leftrightarrow (p_2, \lambda)$, so that the presence of the crossed graph in (9.247) simply results in a factor 2. Hence the identities (9.249) should apply to $S_{\kappa\lambda\mu}$ alone.

The source of the fallacy is that changing the variable of integration (as in (9.253)) in a *linearly divergent* integral changes the value of the integral by a finite amount. The divergent part of $S_{\kappa\lambda\mu}$ is

$$S_{\kappa\lambda\mu} = -(2\pi)^{-4} \int \mathrm{d}^4 k \, \frac{\mathrm{Tr}\,(\gamma_\kappa \not{k} \gamma_\mu \gamma_5 \not{k} \gamma_\lambda \not{k})}{k^6}. \tag{9.255}$$

If $k$ is shifted to $k' = k + a$, the change in $S_{\kappa\lambda\mu}$ is calculated using the following considerations

$$\int \mathrm{d}^4 k F(k) = \int \mathrm{d}^4 k' F(k' - a) = \int \mathrm{d}^4 k' \left[ F(k') - a \frac{\partial F}{\partial k} + \cdots \right].$$

The last term is changed into a surface integral by Gauss' theorem. In our calculation, $F \sim k^{-3}$, so since the hypersurface $\sim k^3$, the surface integral is non-vanishing. To find its value, putting $k'_\mu = (k + a)_\mu$ in (9.255), gives

$$S'_{\kappa\lambda\mu} = S_{\kappa\lambda\mu} + U_{\kappa\lambda\mu\nu} a^\nu \tag{9.256}$$

where

$$U_{\kappa\lambda\mu\nu} = -(2\pi)^{-4}\int \mathrm{d}^4k \frac{\partial}{\partial k^\nu}\left[\frac{\mathrm{Tr}\,(\gamma_\kappa \not{k}\gamma_\mu\gamma_5\not{k}\gamma_\lambda\not{k})}{k^6}\right]$$

$$= -(2\pi)^{-4}\int \mathrm{d}^4k \frac{\partial}{\partial k^\nu}\left[\frac{\mathrm{Tr}\,(\gamma_5\not{k}\gamma_\lambda\not{k}\gamma_\kappa\not{k}\gamma_\mu)}{k^6}\right]. \tag{9.257}$$

Now we use the trace formula



$$\mathrm{Tr}\,(\gamma_5\gamma_\rho\gamma_\lambda\gamma_\sigma\gamma_\kappa\gamma_\tau\gamma_\mu) = 4\mathrm{i}\varepsilon_{\kappa\tau\mu\alpha}(\delta^\alpha_\rho g_{\lambda\sigma} - \delta^\alpha_\lambda g_{\rho\sigma} + \delta^\alpha_\sigma g_{\rho\lambda})$$
$$- 4\mathrm{i}\varepsilon_{\rho\lambda\sigma\alpha}(\delta^\alpha_\kappa g_{\tau\mu} - \delta^\alpha_\tau g_{\kappa\mu} + \delta^\alpha_\mu g_{\kappa\tau}) \tag{9.258}$$

which gives

$$\mathrm{Tr}\,(\gamma_5\not{k}\gamma_\lambda\not{k}\gamma_\kappa\not{k}\gamma_\mu) = 4\mathrm{i}\varepsilon_{\kappa\lambda\mu\alpha}k^\alpha k^2.$$

Hence

$$U_{\kappa\lambda\mu\nu} = -\frac{4\mathrm{i}}{(2\pi)^4}\varepsilon_{\kappa\lambda\mu\alpha}\int \mathrm{d}^4 k \frac{\partial}{\partial k^\nu}\left(\frac{k^\alpha}{k^4}\right). \tag{9.259}$$

We transform this integral to a Euclidean space, so that $k_4 = \mathrm{i}k_0$. Then we observe that if $\alpha \neq \nu$ the integral vanishes, since it is odd in $k$; and if $\alpha = \nu$ we have $(k^\alpha)^2 = \frac{1}{4}k^2$ ($\alpha = 1, \ldots, 4$). So, transforming to a 3-dimensional surface integral yields (using $\oint \mathrm{d}\Omega = 2\pi^2$ for $S^3$)

$$\begin{aligned}
U_{\kappa\lambda\mu\nu} &= \frac{4}{(2\pi)^4}\varepsilon_{\kappa\lambda\mu\alpha}\int \mathrm{d}^4 k_\mathrm{E} \frac{\partial}{\partial k^\nu}\left(\frac{k^\alpha}{k^4}\right) \\
&= \frac{1}{(2\pi)^4}\varepsilon_{\kappa\lambda\mu\nu}\int \mathrm{d}^4 k_\mathrm{E} \frac{\partial}{\partial k^\alpha}\left(\frac{k^\alpha}{k^4}\right) \\
&= \frac{1}{(2\pi)^4}\varepsilon_{\kappa\lambda\mu\nu}\oint (\mathrm{d}^3 S_\mathrm{E})_\alpha \frac{k^\alpha}{k^4} \\
&= \frac{1}{(2\pi)^4}\varepsilon_{\kappa\lambda\mu\nu}\oint \frac{k_\alpha}{k}(k^3\,\mathrm{d}\Omega)\frac{k^\alpha}{k^4} \\
&= \frac{1}{8\pi^2}\varepsilon_{\kappa\lambda\mu\nu}.
\end{aligned} \tag{9.260}$$

Referring back to the change of variable introduced in (9.253), we see from (9.256) and (9.260) that $S_{\kappa\lambda\mu}$ should be replaced by

$$S'_{\kappa\lambda\mu}(p_1, p_2) = \frac{1}{8\pi^2}\varepsilon_{\kappa\lambda\mu\nu}p_2^\nu$$

and therefore the Ward identity corresponding to (9.249$b$) should now read

$$\boxed{p_1^\kappa S_{\kappa\lambda\mu} = \frac{1}{8\pi^2}\varepsilon_{\kappa\lambda\mu\nu}p_1^\kappa p_2^\nu.} \tag{9.261}$$

Similarly, (9.249$c$) becomes

$$\boxed{p_2^\lambda S_{\kappa\lambda\mu} = -\frac{1}{8\pi^2}\varepsilon_{\kappa\lambda\mu\nu}p_2^\lambda p_1^\nu,} \tag{9.262}$$

whereas the axial Ward identity (9.249$a$) remains satisfied. We see that it is impossible to satisfy both the vector and the axial vector Ward identities, but of the two the $V$ identity has more right to be regarded as sacrosanct because it corresponds to conservation of charge. In order to *retain* (9.249$b$, $c$), then, we redefine the amplitude for the triangle graph to be (cf. (9.247))

$$T'_{\kappa\lambda\mu}(p_1, p_2) = S_{\kappa\lambda\mu}(p_1, p_2) + S_{\lambda\kappa\mu}(p_2, p_1) + \boxed{\frac{1}{4\pi^2}\varepsilon_{\kappa\lambda\mu\nu}(p_1 - p_2)^\nu} \quad (9.263)$$

which satisfies the Ward identities (cf. (9.249))

$$\boxed{\left.\begin{aligned} &p_1^\kappa T'_{\kappa\lambda\mu} = 0 && (V), \\ &p_2^\lambda T'_{\kappa\lambda\mu} = 0 && (V), \\ &(p_1 + p_2)^\mu T'_{\kappa\lambda\mu} = \frac{1}{2\pi^2}\varepsilon_{\kappa\lambda\mu\nu} p_2^\mu p_1^\nu && (A). \end{aligned}\right\}} \quad (9.264a, b, c)$$

The vector Ward identities are now satisfied, whereas the axial Ward identity contains an anomaly, which no method of regularisation can avoid. Indeed, in dimensional regularisation the existence of the anomaly is already hinted at by the impossibility of defining a suitable generalisation of $\gamma_5$ in $d$ dimensions, as mentioned above.

On reworking the above calculation in the case of massive fermions ($m \neq 0$), it is seen that the axial anomaly is unchanged. On the other hand, in the massless case, the fact that the Ward identity (9.264$c$) contains an anomaly indicates that the axial current $J^5_\mu$ is not conserved, i.e. there should be an additional term in (9.246). In fact, it is not difficult to see that amending (9.246) to

$$\partial^\mu J^5_\mu = 2mJ_5 \boxed{+ \frac{e^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}} \tag{9.265}$$

where $\tilde{F}^{\mu\nu}$ (the dual of $F^{\mu\nu}$) is defined by (2.234), yields (9.264$c$), since

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) \\ &= 2\varepsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma \\ &= \partial^\mu(2\varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma). \end{aligned} \tag{9.266}$$

Further, since this term is a total divergence the ‘new’ axial current

$$\tilde{J}^5_\mu = J^5_\mu - \frac{e^2}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma \tag{9.267}$$

is divergenceless in the limit $m = 0$:

$$\partial^\mu \tilde{J}^5_\mu = 0 \quad (m = 0),$$

but is not gauge invariant, so cannot be considered as a physical current.

# Anomaly-free condition

$$\sum_i Q_{\mathrm{L}}^i = 0; \tag{9.274}$$

that is, the sum of the electric charges of the left-handed fermions should be zero. In the Weinberg–Salam model with leptons only, this is not satisfied. When hadrons – quarks – are included as well, however, we have, for the case of one generation, and quarks with three colours

$$Q_e + Q_{\nu_e} + 3(Q_u + Q_d) = -1 + 0 + 3(\tfrac{2}{3} - \tfrac{1}{3}) = 0; \tag{9.275}$$

the chiral anomaly disappears, and renormalisability is restored. This condition clearly remains intact if other generations of particles are added with the same charges as the first generation. This is the case for $(\mu^-, \nu_\mu, c, s)$ and $(\tau^-, \nu_\tau, t, b)$. The anomaly-free condition (9.274), then, appears to shed some light on lepton–hadron symmetry, but allows for an arbitrary number of generations. A solution to the generation problem must be sought elsewhere.

# Pseudoscalar triangle diagram in 2 dimensions

The total amplitude $\Gamma^{\mu\nu}$ is of the form

$$\Gamma^{\mu\nu} = F(q^2, q'^2)\, T^{\mu\nu\alpha\beta} q_\alpha q'_\beta$$

In 4 dimensions, $T^{\mu\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}$, but here in 2 dimensions, $T^{\mu\nu\alpha\beta}$ is taken to be

$$T^{\mu\nu\alpha\beta} = -\left(g^{\mu\nu}\varepsilon^{\alpha\beta} + \varepsilon^{\mu\nu} g^{\alpha\beta}\right),$$

which satisfies

$$g_{\mu\alpha} T^{\mu\nu\alpha\beta} = 0; \;\; g_{\nu\beta} T^{\mu\nu\alpha\beta} = 0.$$

So that current conservation

$$q_\mu \Gamma^{\mu\nu} = 0$$

and

$$q'_\nu \Gamma^{\mu\nu} = 0$$

are satisfied. The total amplitude is

$$\Gamma^{\mu\nu} = F(q^2, q'^2)\left(-g^{\mu\nu}\varepsilon^{\alpha\beta} q_\alpha q'_\beta - \varepsilon^{\mu\nu} q \cdot q'\right).$$

So, the form factor is

$$F(q^2, q'^2) = \frac{\Gamma^{\mu\nu}}{-g^{\mu\nu}\varepsilon^{\alpha\beta}q_\alpha q'_\beta - \varepsilon^{\mu\nu}q \cdot q'}.$$

Contracting the symmetric and anti-symmetric tensors $g_{\mu\nu}$ and $\varepsilon_{\mu\nu}$ to the amplitude respectively, we get

$$F(q^2, q'^2) = \frac{g_{\mu\nu}\Gamma^{\mu\nu}}{-2\varepsilon^{\alpha\beta}q_\alpha q'_\beta}$$

and

$$F(q^2, q'^2) = \frac{\varepsilon_{\mu\nu}\Gamma^{\mu\nu}}{2q \cdot q'}.$$

The amplitude $\Gamma^{\mu\nu}$ is calculated as such, following the Feynman rules.

$$
\begin{aligned}
\Gamma^{\mu\nu} =& \Gamma_D^{\mu\nu} + \Gamma_C^{\mu\nu} \\
=& ie^2 g N_c \int \frac{d^2k}{(2\pi)^2} \\
& \left\{ \frac{Tr\left[\gamma^5(\not{p} - \not{k} + m)\gamma^\mu(\not{p} - \not{k} - \not{q} + m)\gamma^\nu(-\not{k} + m)\right]}{\left((p-k-q)^2 - m^2\right)\left((p-k)^2 - m^2\right)\left(k^2 - m^2\right)} \right. \\
& \left. + \frac{Tr\left[\gamma^5(\not{p} - \not{k} + m)\gamma^\nu(\not{q} - \not{k} + m)\gamma^\mu(-\not{k} + m)\right]}{\left((p-k)^2 - m^2\right)\left(k^2 - m^2\right)\left((q-k)^2 - m^2\right)} \right\}
\end{aligned}
$$

We have

$$
Tr\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] = -2(g^{\mu\nu}\varepsilon^{\rho\sigma} + g^{\rho\sigma}\varepsilon^{\mu\nu}).
$$

We obtain

$$
\begin{aligned}
\Gamma^{\mu\nu} =& ie^2 g N_c \int \frac{d^2 k}{(2\pi)^2}(-2m)\Big\{\Big[-(p-k)^\mu \varepsilon^{\nu\delta}(p+k-q)_\delta \\
& + q^\mu \varepsilon^{\nu\delta} k_\delta - \varepsilon^{\mu\delta}(p-k)_\delta (p-k-q)^\nu \\
& + \varepsilon^{\mu\delta} q_\delta k^\nu + \varepsilon^{\mu\nu} m^2\Big] \\
& \cdot \left[\left((p-k-q)^2 - m^2\right)\left((p-k)^2 - m^2\right)\left(k^2 - m^2\right)\right]^{-1} \\
& + \Big[-(p-k)^\nu \varepsilon^{\mu\delta} q_\delta + k^\mu \varepsilon^{\nu\delta}(2p-k-q)_\delta - (q-k)^\nu \varepsilon^{\mu\delta} k_\delta \\
& - q^\mu \varepsilon^{\nu\delta}(p-k)_\delta - \varepsilon^{\mu\nu} m^2\Big] \\
& \cdot \left[\left((p-k)^2 - m^2\right)\left(k^2 - m^2\right)\left((q-k)^2 - m^2\right)\right]^{-1}\Big\}.
\end{aligned}
$$

After doing the Feynman parametrization, this becomes

$$\Gamma^{\mu\nu} = \frac{-ie^2 g N_c m}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times \right.$$
$$\left[ -\left((1-x)q + (1-x-y)q'\right)^\mu \varepsilon^{\nu\delta} \left(xq + (1+x+y)q'\right)_\delta \right.$$
$$+ q^\mu \varepsilon^{\nu\delta} \left(xq + (x+y)q'\right)_\delta$$
$$- \varepsilon^{\mu\delta} \left((1-x)q + (1-x-y)q'\right)_\delta \left(-xq + (1-x-y)q'\right)^\nu$$
$$+ \varepsilon^{\mu\delta} q_\delta \left((x+y)q' + xq\right)^\nu$$
$$\left. +\varepsilon^{\mu\nu} m^2 + l_1^\mu \varepsilon^{\nu\delta} l_{1\delta} - l_1^\nu \varepsilon^{\mu\delta} l_{1\delta} \right]$$
$$+ \int d^2 l_2 \frac{2}{(l_2^2 - \Delta_2)^3} \times \left[ -\left((1-x-y)q + (1-x)q'\right)^\nu \varepsilon^{\mu\delta} q_\delta \right.$$
$$+ \left((x+y)q + xq'\right)^\mu \varepsilon^{\nu\delta} \left((1-x-y)q + (2-x)q'\right)_\delta$$
$$- \left((1-x-y)q - xq'\right)^\nu \varepsilon^{\mu\delta} \left((x+y)q + xq'\right)_\delta$$
$$- q^\mu \varepsilon^{\nu\delta} \left((1-x-y)q + (1-x)q'\right)_\delta$$
$$\left.\left. -\varepsilon^{\mu\nu} m^2 - l_2^\mu \varepsilon^{\nu\delta} l_{2\delta} + l_2^\nu \varepsilon^{\mu\delta} l_{2\delta} \right] \right\},$$

where for $\Gamma_D^{\mu\nu}$ we made the momentum substitution $k \to l_1 + (x+y)p - yq$ and for $\Gamma_C^{\mu\nu}$ we made the substitution $k \to l_2 + xp + yq$,
$\Delta_1 = x(x-1)q^2 + 2x(x+y-1)q \cdot q' + (x+y)(x+y-1)q'^2 + m^2$, and
$\Delta_2 = x(x-1)q'^2 + 2x(x+y-1)q \cdot q' + (x+y)(x+y-1)q^2 + m^2$.

To calculate the form factor from $F(q^2, q'^2) = \frac{g_{\mu\nu}\Gamma^{\mu\nu}}{-2\varepsilon^{\alpha\beta} q_\alpha q'_\beta}$, we do

$$
\begin{aligned}
g_{\mu\nu}\Gamma^{\mu\nu} = & \frac{-ie^2 g N_c m}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times \right. \\
& \left[ -\varepsilon^{\nu\delta} \left( (1-x)q + (1-x-y)q' \right)_\nu \left( xq + (1+x+y)q' \right)_\delta \right. \\
& + \varepsilon^{\nu\delta} q_\nu \left( xq + (x+y)q' \right)_\delta \\
& - \varepsilon^{\mu\delta} \left( (1-x)q + (1-x-y)q' \right)_\delta \left( -xq + (1-x-y)q' \right)_\mu \\
& + \varepsilon^{\mu\delta} q_\delta \left( (x+y)q' + xq \right)_\mu \\
& \left. + \varepsilon^{\nu\delta} l_{1\nu} l_{1\delta} - \varepsilon^{\mu\delta} l_{1\mu} l_{1\delta} \right] \\
& + \int d^2 l_2 \frac{2}{(l_2^2 - \Delta_2)^3} \times \left[ -\varepsilon^{\mu\delta} \left( (1-x-y)q + (1-x)q' \right)_\mu q_\delta \right. \\
& + \varepsilon^{\nu\delta} \left( (x+y)q + xq' \right)_\nu \left( (1-x-y)q + (2-x)q' \right)_\delta \\
& - \varepsilon^{\mu\delta} \left( (1-x-y)q - xq' \right)_\mu \left( (x+y)q + xq' \right)_\delta \\
& - \varepsilon^{\nu\delta} q_\nu \left( (1-x-y)q + (1-x)q' \right)_\delta \\
& \left. \left. - \varepsilon^{\nu\delta} l_{2\nu} l_{2\delta} + \varepsilon^{\mu\delta} l_{2\mu} l_{2\delta} \right] \right\} \\
= & \frac{e^2 g N_c m}{2\pi} \left( -2\varepsilon^{\alpha\beta} q_\alpha q'_\beta \right) \int_0^1 dx \int_0^{1-x} dy (-y) \left( \frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2} \right) = 0.
\end{aligned}
$$

To calculate the form factor from $F(q^2, q'^2) = \frac{\varepsilon_{\mu\nu}\Gamma^{\mu\nu}}{2q\cdot q'}$, we do

$$
\begin{aligned}
\varepsilon_{\mu\nu}\Gamma^{\mu\nu} = \frac{-ie^2 g N_c m}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \Big\{ \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times \\
& \big[- \big((1-x)q + (1-x-y)q'\big) \cdot \big(xq + (1+x+y)q'\big) \\
& + q \cdot \big(xq + (x+y)q'\big) \\
& + \big((1-x)q + (1-x-y)q'\big) \cdot \big(-xq + (1-x-y)q'\big) \\
& - q \cdot \big((x+y)q' + xq\big) \\
& -2m^2 + 2l_1^2\big] \\
+ \int d^2 l_2 \frac{2}{(l_2^2 - \Delta_2)^3} \times \big[\big((1-x-y)q + (1-x)q'\big) \cdot q \\
& + \big((x+y)q + xq'\big) \cdot \big((1-x-y)q + (2-x)q'\big) \\
& + \big((1-x-y)q - xq'\big) \cdot \big((x+y)q + xq'\big) \\
& - q \cdot \big((1-x-y)q + (1-x)q'\big) \\
& +2m^2 - 2l_2^2\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-e^2 g N_c m}{2\pi} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{\Delta_1^2} \times \right. \\
&\Big[ - \big((1-x)q + (1-x-y)q'\big) \cdot \big(xq + (1+x+y)q'\big) \\
&+ q \cdot \big(xq + (x+y)q'\big) \\
&+ \big((1-x)q + (1-x-y)q'\big) \cdot \big(-xq + (1-x-y)q'\big) \\
&- q \cdot \big((x+y)q' + xq\big) \\
&-2m^2 - 2\Delta_1 \Big] \\
+&\frac{1}{\Delta_2^2} \times \Big[ \big((1-x-y)q + (1-x)q'\big) \cdot q \\
&+ \big((x+y)q + xq'\big) \cdot \big((1-x-y)q + (2-x)q'\big) \\
&+ \big((1-x-y)q - xq'\big) \cdot \big((x+y)q + xq'\big) \\
&- q \cdot \big((1-x-y)q + (1-x)q'\big) \\
&\left. +2m^2 + 2\Delta_2 \Big] \right\}
\end{aligned}
$$

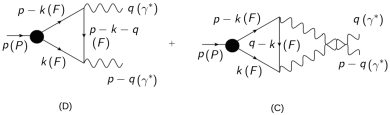
$$= \frac{-e^2 g N_c m}{\pi} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{\Delta_1^2} \left(-yq \cdot q' - 2m^2\right) \right.$$
$$\left. + \frac{1}{\Delta_2^2} \left(yq \cdot q' + 2m^2\right) \right\} = 0.$$

So, since both the symmetric and anti-symmetric parts of the total amplitude is zero, we conclude that the pseudoscalar transition amplitude itself is zero

$$\Gamma^{\mu\nu} = 0.$$

- Future work on this: calculate in light-front time-ordered formulation to confirm the results.

# Axial vector triangle diagram in 2 dimensions (AVV)




Now let us calculate the transition form factor of an axial vector meson going through the fermion triangle loop, and transitioning into 2 virtual photons. The amplitude $\Gamma^{\rho\mu\nu}$ is calculated as such, following the Feynman rules.

$$
\begin{aligned}
\Gamma^{\rho\mu\nu} =& \Gamma_D^{\rho\mu\nu} + \Gamma_C^{\rho\mu\nu} \\
=& ie^2 g N_c \int \frac{d^2k}{(2\pi)^2} \\
& \left\{ \frac{Tr\left[\gamma^\rho\gamma^5(\not{p}-\not{k}+m)\gamma^\mu(\not{p}-\not{k}-\not{q}+m)\gamma^\nu(-\not{k}+m)\right]}{((p-k-q)^2-m^2)\,((p-k)^2-m^2)\,(k^2-m^2)} \right. \\
& \left. + \frac{Tr\left[\gamma^\rho\gamma^5(\not{p}-\not{k}+m)\gamma^\nu(\not{q}-\not{k}+m)\gamma^\mu(-\not{k}+m)\right]}{((p-k)^2-m^2)\,(k^2-m^2)\,((q-k)^2-m^2)} \right\}.
\end{aligned}
$$

Using in 2 dimensions

$$
\begin{aligned}
&Tr\left[\gamma^5\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\theta\gamma^\rho\right]\\
=&-2g^{\alpha\mu}\varepsilon^{\beta\nu}g^{\theta\rho}-2\varepsilon^{\alpha\mu}g^{\beta\nu}g^{\theta\rho}-2g^{\alpha\mu}g^{\beta\nu}\varepsilon^{\theta\rho}+2g^{\alpha\beta}g^{\mu\nu}\varepsilon^{\theta\rho}-2g^{\alpha\nu}g^{\mu\beta}\varepsilon^{\theta\rho},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\rho\mu\nu}=&\frac{-ie^2gN_c}{2\pi^2}\int d^2k\Big\{\Big[\Big(g^{\alpha\mu}\varepsilon^{\beta\nu}g^{\theta\rho}+\varepsilon^{\alpha\mu}g^{\beta\nu}g^{\theta\rho}+g^{\alpha\mu}g^{\beta\nu}\varepsilon^{\theta\rho}-g^{\alpha\beta}g^{\mu\nu}\varepsilon^{\theta\rho}\\
&+g^{\alpha\nu}g^{\mu\beta}\varepsilon^{\theta\rho}\Big)\cdot(p-k)_\alpha(p-k-q)_\beta(-k)_\theta+m^2(g^{\alpha\mu}\varepsilon^{\nu\rho}+\varepsilon^{\alpha\mu}g^{\nu\rho})(p-k)_\alpha\\
&+m^2(g^{\mu\beta}\varepsilon^{\nu\rho}+\varepsilon^{\mu\beta}g^{\nu\rho})(p-k-q)_\beta+m^2(g^{\mu\nu}\varepsilon^{\theta\rho}+\varepsilon^{\mu\nu}g^{\theta\rho})(-k)_\theta\Big]\\
&\cdot\Big[\Big((p-k-q)^2-m^2\Big)\Big((p-k)^2-m^2\Big)\Big(k^2-m^2\Big)\Big]^{-1}\\
+&\Big[\Big(g^{\alpha\nu}\varepsilon^{\beta\mu}g^{\theta\rho}+\varepsilon^{\alpha\nu}g^{\beta\mu}g^{\theta\rho}+g^{\alpha\nu}g^{\beta\mu}\varepsilon^{\theta\rho}-g^{\alpha\beta}g^{\nu\mu}\varepsilon^{\theta\rho}\\
&+g^{\alpha\mu}g^{\nu\beta}\varepsilon^{\theta\rho}\Big)\cdot(p-k)_\alpha(q-k)_\beta(-k)_\theta+m^2(g^{\alpha\nu}\varepsilon^{\mu\rho}+\varepsilon^{\alpha\nu}g^{\mu\rho})(p-k)_\alpha\\
&+m^2(g^{\nu\beta}\varepsilon^{\mu\rho}+\varepsilon^{\nu\beta}g^{\mu\rho})(q-k)_\beta+m^2(g^{\nu\mu}\varepsilon^{\theta\rho}+\varepsilon^{\nu\mu}g^{\theta\rho})(-k)_\theta\Big]\\
&\cdot\Big[\Big((p-k)^2-m^2\Big)\Big(k^2-m^2\Big)\Big((q-k)^2-m^2\Big)\Big]^{-1}\Big\}.
\end{aligned}
$$

After doing the Feynman parametrization, this becomes

$$\Gamma^{\rho\mu\nu}$$

$$= \frac{ie^2 g N_c}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times$$

$$\Big[ \left((1-x)q + (1-x-y)q'\right)^\mu \varepsilon^{\beta\nu} \left(-xq + (1-x-y)q'\right)_\beta \left(xq + (x+y)q'\right)^\rho$$

$$+ \varepsilon^{\alpha\mu} \left((1-x)q + (1-x-y)q'\right)_\alpha \left(-xq + (1-x-y)q'\right)^\nu \left(xq + (x+y)q'\right)^\rho$$

$$+ \left((1-x)q + (1-x-y)q'\right)^\mu \left(-xq + (1-x-y)q'\right)^\nu \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta$$

$$- \left((1-x)q + (1-x-y)q'\right) \cdot \left(-xq + (1-x-y)q'\right) g^{\mu\nu} \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta$$

$$+ \left((1-x)q + (1-x-y)q'\right)^\nu \left(-xq + (1-x-y)q'\right)^\mu \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta$$

$$
\begin{aligned}
&+ l_1^\mu \varepsilon^{\beta\nu} l_{1\beta} \left(xq + (x+y)q'\right)^\rho - l_1^\mu \varepsilon^{\beta\nu} \left(-xq + (1-x-y)q'\right)_\beta l_1^\rho \\
&- \left((1-x)q + (1-x-y)q'\right)^\mu \varepsilon^{\beta\nu} l_{1\beta} l_1^\rho \\
&+ \varepsilon^{\alpha\mu} l_{1\alpha} l_1^\nu \left(xq + (x+y)q'\right)^\rho - \varepsilon^{\alpha\mu} l_{1\alpha} \left(-xq + (1-x-y)q'\right)^\nu l_1^\rho \\
&- \varepsilon^{\alpha\mu} \left((1-x)q + (1-x-y)q'\right)_\alpha l_1^\nu l_1^\rho \\
&+ l_1^\mu l_1^\nu \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta - l_1^\mu \left(-xq + (1-x-y)q'\right)^\nu \varepsilon^{\theta\rho} l_{1\theta} \\
&- \left((1-x)q + (1-x-y)q'\right)^\mu l_1^\nu \varepsilon^{\theta\rho} l_{1\theta} \\
&- g^{\mu\nu} l_1^2 \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta + l_1 \cdot \left(-xq + (1-x-y)q'\right) g^{\mu\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&+ \left((1-x)q + (1-x-y)q'\right) \cdot l_1 g^{\mu\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&+ l_1^\nu l_1^\mu \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta - l_1^\nu \left(-xq + (1-x-y)q'\right)^\mu \varepsilon^{\theta\rho} l_{1\theta} \\
&- \left((1-x)q + (1-x-y)q'\right)^\nu l_1^\mu \varepsilon^{\theta\rho} l_{1\theta} \\
&- m^2 \left((1-2x)q + 2(1-x-y)q'\right)^\mu \varepsilon^{\nu\rho} + m^2 \varepsilon^{\mu\alpha} q_\alpha g^{\nu\rho} \\
&+ m^2 g^{\mu\nu} \varepsilon^{\theta\rho} \left(xq + (x+y)q'\right)_\theta + m^2 \varepsilon^{\mu\nu} \left(xq + (x+y)q'\right)^\rho \Big] \\
+ &\left\{\mu \leftrightarrow \nu \ \& \ q \leftrightarrow q'\right\}.
\end{aligned}
$$

Let us check first the axial vector current conservation

$$(q + q')_\rho \Gamma^{\rho\mu\nu} \quad (A)$$

$$
\begin{aligned}
&(q + q')_\rho \Gamma^{\rho\mu\nu} \\
&= \frac{ie^2 g N_c}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times \\
&\Big[ \Big( (1-x)q + (1-x-y)q' \Big)^\mu \varepsilon^{\beta\nu} \Big( -xq + (1-x-y)q' \Big)_\beta (q+q') \cdot \Big( xq + (x+y)q' \Big) \\
&+ \varepsilon^{\alpha\mu} \Big( (1-x)q + (1-x-y)q' \Big)_\alpha \Big( -xq + (1-x-y)q' \Big)^\nu (q+q') \cdot \Big( xq + (x+y)q' \Big) \\
&+ \Big( (1-x)q + (1-x-y)q' \Big)^\mu \Big( -xq + (1-x-y)q' \Big)^\nu \varepsilon^{\theta\rho} \Big( xq + (x+y)q' \Big)_\theta (q+q')_\rho \\
&- \Big( (1-x)q + (1-x-y)q' \Big) \cdot \Big( -xq + (1-x-y)q' \Big) g^{\mu\nu} \varepsilon^{\theta\rho} \Big( xq + (x+y)q' \Big)_\theta (q+q')_\rho \\
&+ \Big( (1-x)q + (1-x-y)q' \Big)^\nu \Big( -xq + (1-x-y)q' \Big)^\mu \varepsilon^{\theta\rho} \Big( xq + (x+y)q' \Big)_\theta (q+q')_\rho
\end{aligned}
$$

$$
\begin{aligned}
&+ l_1^{\mu}\varepsilon^{\beta\nu}l_{1\beta}(q+q')\cdot\left(xq+(x+y)q'\right) - l_1^{\mu}\varepsilon^{\beta\nu}\left(-xq+(1-x-y)q'\right)_{\beta}(q+q')\cdot l_1 \\
&- \left((1-x)q+(1-x-y)q'\right)^{\mu}\varepsilon^{\beta\nu}l_{1\beta}(q+q')\cdot l_1 \\
&+ \varepsilon^{\alpha\mu}l_{1\alpha}l_1^{\nu}(q+q')\cdot\left(xq+(x+y)q'\right) - \varepsilon^{\alpha\mu}l_{1\alpha}\left(-xq+(1-x-y)q'\right)^{\nu}(q+q')\cdot l_1 \\
&- \varepsilon^{\alpha\mu}\left((1-x)q+(1-x-y)q'\right)_{\alpha}l_1^{\nu}(q+q')\cdot l_1 \\
&+ l_1^{\mu}l_1^{\nu}\varepsilon^{\theta\rho}\left(xq+(x+y)q'\right)_{\theta}(q+q')_{\rho} - l_1^{\mu}\left(-xq+(1-x-y)q'\right)^{\nu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&- \left((1-x)q+(1-x-y)q'\right)^{\mu}l_1^{\nu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&- g^{\mu\nu}l_1^2\varepsilon^{\theta\rho}\left(xq+(x+y)q'\right)_{\theta}(q+q')_{\rho} + l_1\cdot\left(-xq+(1-x-y)q'\right)g^{\mu\nu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&+ \left((1-x)q+(1-x-y)q'\right)\cdot l_1 g^{\mu\nu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&+ l_1^{\nu}l_1^{\mu}\varepsilon^{\theta\rho}\left(xq+(x+y)q'\right)_{\theta}(q+q')_{\rho} - l_1^{\nu}\left(-xq+(1-x-y)q'\right)^{\mu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&- \left((1-x)q+(1-x-y)q'\right)^{\nu}l_1^{\mu}\varepsilon^{\theta\rho}l_{1\theta}(q+q')_{\rho} \\
&- m^2\left((1-2x)q+2(1-x-y)q'\right)^{\mu}\varepsilon^{\nu\rho}(q+q')_{\rho} + m^2\varepsilon^{\mu\alpha}q_{\alpha}(q+q')^{\nu} \\
&+ m^2 g^{\mu\nu}\varepsilon^{\theta\rho}\left(xq+(x+y)q'\right)_{\theta}(q+q')_{\rho} + m^2\varepsilon^{\mu\nu}(q+q')\cdot\left(xq+(x+y)q'\right)\Big] \\
&+ \left\{\mu\leftrightarrow\nu\ \&\ q\leftrightarrow q'\right\}.
\end{aligned}
$$

We get

$$
\begin{aligned}
& g_{\mu\nu}(q+q')_\rho \Gamma^{\rho\mu\nu} \\
=& \frac{e^2 g N_c}{\pi} m^2 (2\varepsilon^{\alpha\beta} q_\alpha q'_\beta) \int_0^1 dx \int_0^{1-x} dy(-y) \left( \frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2} \right) = 0.
\end{aligned}
$$

and

$$
\begin{aligned}
& \varepsilon_{\mu\nu}(q+q')_\rho \Gamma^{\rho\mu\nu} \\
=& \frac{-e^2 g N_c}{\pi} m^2 (q+q')^2 \int_0^1 dx \int_0^{1-x} dy \left( \frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2} \right) = 0.
\end{aligned}
$$

So the axial vector current conservation is satisfied.

$$
p_\rho \Gamma^{\rho\mu\nu} = 0.
$$

Let us check another current conservation

$$q_\mu \Gamma^{\rho\mu\nu} \quad (V)$$

$$
\begin{aligned}
& q_\mu \Gamma_D^{\rho\mu\nu} \\
& = \frac{ie^2 g N_c}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^2 l_1 \frac{2}{(l_1^2 - \Delta_1)^3} \times \\
& \Big[ q \cdot \left( (1-x)q + (1-x-y)q' \right) \varepsilon^{\beta\nu} \left( -xq + (1-x-y)q' \right)_\beta \left( xq + (x+y)q' \right)^\rho \\
& + \varepsilon^{\alpha\mu} \left( (1-x)q + (1-x-y)q' \right)_\alpha q_\mu \left( -xq + (1-x-y)q' \right)^\nu \left( xq + (x+y)q' \right)^\rho \\
& + q \cdot \left( (1-x)q + (1-x-y)q' \right) \left( -xq + (1-x-y)q' \right)^\nu \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_\theta \\
& - \left( (1-x)q + (1-x-y)q' \right) \cdot \left( -xq + (1-x-y)q' \right) q^\nu \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_\theta \\
& + \left( (1-x)q + (1-x-y)q' \right)^\nu q \cdot \left( -xq + (1-x-y)q' \right) \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ q \cdot l_1 \varepsilon^{\beta\nu} l_{1\beta} \left( xq + (x+y)q' \right)^{\rho} - q \cdot l_1 \varepsilon^{\beta\nu} \left( -xq + (1-x-y)q' \right)_{\beta} l_1^{\rho} \\
&- q \cdot \left( (1-x)q + (1-x-y)q' \right) \varepsilon^{\beta\nu} l_{1\beta} l_1^{\rho} \\
&+ \varepsilon^{\alpha\mu} l_{1\alpha} q_{\mu} l_1^{\nu} \left( xq + (x+y)q' \right)^{\rho} - \varepsilon^{\alpha\mu} l_{1\alpha} q_{\mu} \left( -xq + (1-x-y)q' \right)^{\nu} l_1^{\rho} \\
&- \varepsilon^{\alpha\mu} \left( (1-x)q + (1-x-y)q' \right)_{\alpha} q_{\mu} l_1^{\nu} l_1^{\rho} \\
&+ q \cdot l_1 l_1^{\nu} \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_{\theta} - q \cdot l_1 \left( -xq + (1-x-y)q' \right)^{\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&- q \cdot \left( (1-x)q + (1-x-y)q' \right) l_1^{\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&- q^{\nu} l_1^2 \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_{\theta} + l_1 \cdot \left( -xq + (1-x-y)q' \right) q^{\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&+ \left( (1-x)q + (1-x-y)q' \right) \cdot l_1 q^{\nu} \varepsilon^{\theta\rho} l_{1\theta} \\
&+ l_1^{\nu} q \cdot l_1 \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_{\theta} - l_1^{\nu} q \cdot \left( -xq + (1-x-y)q' \right) \varepsilon^{\theta\rho} l_{1\theta} \\
&- \left( (1-x)q + (1-x-y)q' \right)^{\nu} q \cdot l_1 \varepsilon^{\theta\rho} l_{1\theta} \\
&- m^2 q \cdot \left( (1-2x)q + 2(1-x-y)q' \right) \varepsilon^{\nu\rho} + m^2 \varepsilon^{\mu\alpha} q_{\alpha} q_{\mu} g^{\nu\rho} \\
&+ m^2 q^{\nu} \varepsilon^{\theta\rho} \left( xq + (x+y)q' \right)_{\theta} + m^2 \varepsilon^{\mu\nu} q_{\mu} \left( xq + (x+y)q' \right)^{\rho} \Big] .
\end{aligned}
$$

And

$$
\begin{aligned}
& q_\mu \Gamma_C^{\rho\mu\nu} \\
& = \frac{ie^2 gN_c}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \int d^2 l_2 \frac{2}{(l_2^2 - \Delta_2)^3} \times \\
& \Big[ \Big((1-x)q' + (1-x-y)q\Big)^\nu \varepsilon^{\beta\mu} \Big(-xq' + (1-x-y)q\Big)_\beta q_\mu \Big(xq' + (x+y)q\Big)^\rho \\
& + \varepsilon^{\alpha\nu} \Big((1-x)q' + (1-x-y)q\Big)_\alpha q \cdot \Big(-xq' + (1-x-y)q\Big) \Big(xq' + (x+y)q\Big)^\rho \\
& + \Big((1-x)q' + (1-x-y)q\Big)^\nu q \cdot \Big(-xq' + (1-x-y)q\Big) \varepsilon^{\theta\rho} \Big(xq' + (x+y)q\Big)_\theta \\
& - \Big((1-x)q' + (1-x-y)q\Big) \cdot \Big(-xq' + (1-x-y)q\Big) q^\nu \varepsilon^{\theta\rho} \Big(xq' + (x+y)q\Big)_\theta \\
& + q \cdot \Big((1-x)q' + (1-x-y)q\Big) \Big(-xq' + (1-x-y)q\Big)^\nu \varepsilon^{\theta\rho} \Big(xq' + (x+y)q\Big)_\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ l_2^\nu \varepsilon^{\beta\mu} l_{2\beta} q_\mu \left(xq' + (x+y)q\right)^\rho - l_2^\nu \varepsilon^{\beta\mu} \left(-xq' + (1-x-y)q\right)_\beta q_\mu l_2^\rho \\
&- \left((1-x)q' + (1-x-y)q\right)^\nu \varepsilon^{\beta\mu} l_{2\beta} q_\mu l_2^\rho \\
&+ \varepsilon^{\alpha\nu} l_{2\alpha} q \cdot l_2 \left(xq' + (x+y)q\right)^\rho - \varepsilon^{\alpha\nu} l_{2\alpha} q \cdot \left(-xq' + (1-x-y)q\right) l_2^\rho \\
&- \varepsilon^{\alpha\nu} \left((1-x)q' + (1-x-y)q\right)_\alpha q \cdot l_2 l_2^\rho \\
&+ l_2^\nu q \cdot l_2 \varepsilon^{\theta\rho} \left(xq' + (x+y)q\right)_\theta - l_2^\nu q \cdot \left(-xq' + (1-x-y)q\right) \varepsilon^{\theta\rho} l_{2\theta} \\
&- \left((1-x)q' + (1-x-y)q\right)^\nu q \cdot l_2 \varepsilon^{\theta\rho} l_{2\theta} \\
&- q^\nu l_2^2 \varepsilon^{\theta\rho} \left(xq' + (x+y)q\right)_\theta + l_2 \cdot \left(-xq' + (1-x-y)q\right) q^\nu \varepsilon^{\theta\rho} l_{2\theta} \\
&+ \left((1-x)q' + (1-x-y)q\right) \cdot l_2 q^\nu \varepsilon^{\theta\rho} l_{2\theta} \\
&+ q \cdot l_2 l_2^\nu \varepsilon^{\theta\rho} \left(xq' + (x+y)q\right)_\theta - q \cdot l_2 \left(-xq' + (1-x-y)q\right)^\nu \varepsilon^{\theta\rho} l_{2\theta} \\
&- q \cdot \left((1-x)q' + (1-x-y)q\right) l_2^\nu \varepsilon^{\theta\rho} l_{2\theta} \\
&- m^2 \left((1-2x)q' + 2(1-x-y)q\right)^\nu \varepsilon^{\mu\rho} q_\mu + m^2 \varepsilon^{\nu\alpha} q'_\alpha q^\rho \\
&+ m^2 q^\nu \varepsilon^{\theta\rho} \left(xq' + (x+y)q\right)_\theta + m^2 \varepsilon^{\nu\mu} q_\mu \left(xq' + (x+y)q\right)^\rho \Big] .
\end{aligned}
$$

We get

$$g_{\rho\nu} q_\mu \Gamma^{\rho\mu\nu}_D = g_{\rho\nu} q_\mu \Gamma^{\rho\mu\nu}_C = 0$$

and

$$\begin{aligned}
\varepsilon_{\rho\nu} q_\mu \Gamma^{\rho\mu\nu} =& \frac{e^2 g N_c}{2\pi} \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta_1^2} \times \\
& \left[ x(1-x-y) \left[ (q \cdot q')^2 - q^2 q'^2 - (\varepsilon^{\alpha\beta} q_\alpha q'_\beta)^2 \right] \right. \\
& \left. - \ 2m^2 q \cdot \big( (1-2x) q + 2(1-x-y) q' \big) \right] \\
+& \frac{e^2 g N_c}{2\pi} \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta_2^2} \times \\
& \left[ -xy \left[ (q \cdot q')^2 - q^2 q'^2 - (\varepsilon^{\alpha\beta} q_\alpha q'_\beta)^2 \right] \right. \\
& \left. + 2m^2 q \cdot \big( 2x q' + (2x + 2y - 1) q \big) \right] .
\end{aligned}$$

But

$$(q \cdot q')^2 - q^2 q'^2 - (\varepsilon^{\alpha\beta} q_\alpha q'_\beta)^2 = 0.$$

And from numerical calculation, both the $2m^2q^2$ term

$$\int_0^1 dx \int_0^{1-x} dy \left( \frac{-(1-2x)}{\Delta_1^2} + \frac{2x+2y-1}{\Delta_2^2} \right)$$

and the $(2m^2)(2q \cdot q')$ term

$$\int_0^1 dx \int_0^{1-x} dy \left( \frac{-(1-x-y)}{\Delta_1^2} + \frac{x}{\Delta_2^2} \right)$$

are 0 for general values of $q^2$ and $q'^2$. So the vector current conservation is satisfied.

$$q_\mu \Gamma^{\rho\mu\nu} = 0.$$

- Future work on this: Calculate two-point function to find axial anomaly.

# Not triangle diagram

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We now set up the framework for calculating this anomaly. The Lagrangian of $QED_2$ with one flavour fermion in Euclidean space is read as follows [4]:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu}, \qquad \mu, \nu = 1, 2, \tag{1}$$

where the $\gamma$-matrices are chosen as the two-component form

$$\gamma_1 = \sigma_2, \quad \gamma_2 = -\sigma_1, \quad \gamma_5 = -i\gamma_1\gamma_2 = \sigma_3. \tag{2}$$

Classically, the following vector and axial currents

2

$$j_\mu = \bar{\psi}\gamma_\mu\psi, \quad j^5_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi \tag{3}$$

satisfy the relations

$$\partial_\mu j_\mu = 0, \quad \partial_\mu j^5_\mu = 2imj_5 \tag{4}$$

with $j_5$ being the pseudo-scalar current, $j_5 \equiv \bar{\psi}\gamma_5\psi$.

In $QED_2$ the chiral anomaly comes from the two-point function [9]

$$\Pi^5_{\mu\nu}(x-y) = \langle T[j_\mu(x)j^5_\nu(y)]\rangle. \tag{5}$$

This is contrary to the four-dimensional case, where the chiral anomaly comes from the triangle composed of the axial and vector currents. Due to the explicit relation $i\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma_\nu$, $T^5$ is relevant to the vacuum polarization tensor

# Overview on the Anomaly and Schwinger Term in Two-Dimensional QED (*).

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The simplest of all theories that show the axial anomaly is quantum electrodynamics in two-dimensional space-time, $QED_2$. One reason for us to do all computations in this article for that model is, of course, its simplicity. The different modern approaches to the anomaly we mentioned above in principle need quite an advanced mathematical apparatus and tedious calculations. But in the simple model of $QED_2$, when we present different treatments of the anomaly, the computations usually will not exceed several lines.

$QED_2$, however, is interesting as a model for its own, too, although it has only little direct physical significance (when space-time is taken to be Euclidean, $QED_2$ may be identified with an electron restricted to a plane moving in a magnetic field perpendicular to that plane [22]).

First of all, it was observed by Schwinger in 1962 [23] that when the fermion of the theory (the «electron») is massless, the model is exactly solvable. For this reason massless $QED_2$ is called Schwinger model.

In fact, the model is equivalent to the theory of a free, massive scalar particle of mass $\mu = e/\sqrt{\pi}$, where $e$ is the electron charge [23, 24-26]. This massive mode may be interpreted as a fermion-antifermion bound state — that fails to separate — or as a mass contribution to the photon. (We want to emphasize that those different interpretations by no means change the physics and merely reflect different approaches to the model. In an operator solution of the model the bound-state interpretation is natural, whereas in the path integral approach, which we will prefer in most of the article, the photon mass term will occur. And this photon mass, too, will emerge as an immediate consequence of the anomaly.) As a consequence, there are no asymptotical fermion states in the theory and you may say that the fermions are «confined». This feature is not too surprising, because in two dimensions the electric potential obeys a linear power law, and some potential models which are used to describe the confinement property of QCD in an effective way use linear potentials, too.

Some further properties that are expected to be important for QCD are present in the Schwinger model. Instantons exist because the boundary of space-time, $S^1$, equals the gauge group manifold, $U(1) \sim S^1$. Due to the occurrence of instanton configurations, vacuum condensates are formed. For instance, the non-vanishing value of the vacuum expectation value of the scalar density $\langle \bar{\Psi}\Psi \rangle \neq 0$ is precisely caused by fields with non-zero instanton number, as can be nicely demonstrated within the path integral approach [27, 28].

Summing up, we see that the Schwinger model, in spite of its simplicity, in many respects is an interesting model for features of QCD. Besides, it may be used to test new methods that are developed to deal with more complicated models like QCD [29-31].

When the fermion acquires a mass, not all surprising features of the model survive and the model cannot be solved any longer. Nevertheless, it remains an interesting field of research (see *e.g.* [32, 22]).

Concerning our main theme, the axial anomaly of $QED_2$, there is no difference between the massive and massless cases. The expression for the anomaly is completely independent of a possible fermion mass term.

**1**˙2. *Some introductory remarks.* – Before we start the computations, some considerations are in order as a guide through the article. The anomaly we are dealing with is associated to the axial symmetry. Axial transformations only exist on fermions in an even space-time dimension. Further, when the axial symmetry is spoiled by quantization of the fermions, we have to examine loop diagrams because only loops reflect quantum properties. Loop diagrams only contribute in interacting theories, so we choose the simplest gauge interaction in the smallest possible space-time dimension, $QED_2$.

Now we should look at the concept of symmetry more closely. It is easy to understand what symmetry means on a classical level. The Lagrangian or equations of motion are simply invariant under a certain transformation of the basic fields. But how can a symmetry be transferred into quantum field theory? One way of realizing a symmetry on the quantum level is the concept of Ward identities (WI). WI are identities between different $n$-point functions of a theory and can be derived in a formal way from a classical conservation equation ($n$-point function or Green's function just means the vacuum expectation value of a time-ordered operator product).

Let us choose as a simple example a free spinor field with Lagrangian $L = \bar{\Psi} i \partial\!\!\!/ \Psi$. Because of the symmetry with respect to a pure phase transformation the free-fermion current $J_\mu = \bar{\Psi}\gamma_\mu \Psi$ is conserved, $\partial^\mu J_\mu = 0$. Next, let us recall some simple properties of time-ordered operator products like

$$T(A(x)B(y)) = \Theta(x_0 - y_0)A(x)B(y) + \Theta(y_0 - x_0)B(y)A(x)$$

and

$$\partial_x^\mu T(A(x)B(y)) = \delta_{0\mu}\delta(x_0 - y_0)[A(x), B(y)] + T(\partial_x^\mu A(x)B(y)).$$

If we believe in the conservation of the quantized current operator and in the canonical (anti-) commutation relations, we may formally derive identities like

$$\boxed{\partial_x^\mu \langle 0 | T(J_\mu(x) J_\nu(y)) | 0 \rangle = 0}$$

or

$$\partial_x^\mu \langle 0 | T(J_\mu(x)\Psi(y)\bar{\Psi}(z)) | 0 \rangle = -\langle 0 | T(\Psi(y)\bar{\Psi}(z)) | 0 \rangle \delta(x-y) + \\ + \langle 0 | T(\Psi(y)\bar{\Psi}(z)) | 0 \rangle \delta(x-z),$$

which are called $\boxed{\text{WI.}}$ We want to emphasize, however, that neither of the two requirements stated above needs to be true in general (this will be demonstrated in later sections).

A real quantum check of such WI is done by computing Feynman graphs contributing to the $n$-point functions in perturbation theory (see sect. **2**). There the following situation occurs: as long as the contributing graphs are finite, they



lead to a unique result and fulfil the expected WI. When divergent diagrams contribute to a Green's function, however, they need regularization and therefore do not lead to a unique result. More precisely, the Green's function may be changed (in momentum space) by polynomials in the exterior momenta (which are called «seagulls»), where the degree of the polynomial equals the degree of divergence of the most divergent diagram.

So the addition of a seagull may destroy or restore a WI. This may lead into troubles if we suppose for an $n$-point function more than one WI to hold. It may happen that no seagull can be chosen as to fulfil all required WI. When this situation occurs we speak of an anomaly. One usually prefers one symmetry on physical grounds and associates the anomaly entirely with the other one.

The axial anomaly in $\mathrm{QED}_2$ stems from the two-point function

$$T^5_{\mu\nu}(x-y) = \langle T(J_\mu(x) J^5_\nu(y)) \rangle , \tag{7}$$

whose fermion loop is depicted in fig. 1.

In two dimensions the relation

(*) Our conventions are as in [49].



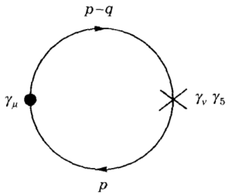


Fig. 1. – Fermion loop containing a vector and axial current.

$$q^{\mu}\, T^{5,\,\mathrm{R}}_{\mu\nu}(q) = 0\,,$$

$$q^{\nu}\, T^{5,\,\mathrm{R}}_{\mu\nu}(q) = 2mP^{5}_{\mu}(q) + \frac{1}{\pi}\, q^{\lambda}\, \epsilon_{\lambda\mu}\,,$$