Axial Anomaly in 1+1-D QED

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group meeting

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2. – Perturbative approach to the two-dimensional anomaly (*).

2·1. *Perturbative calculations.* – At the beginning we will compute Feynman diagrams relevant for the axial anomaly in two dimensions. A naively regularized result will not fulfil the correct WI whereas the proper dimensional (or PV) regularized Feynman integral will do so.

The Lagrangian of QED_2 with one fermion is

$$L = \bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad \mu, \nu = 0, 1. \tag{1}$$

The A-field is treated as an exterior field in the following because this is sufficient for the computation of the anomaly. The currents

$$J_\mu = \bar{\Psi}\gamma_\mu\Psi, \tag{2}$$

$$J^5_\mu = \bar{\Psi}\gamma_\mu\gamma_5\Psi, \tag{3}$$

$$J^5 = \bar{\Psi}\gamma_5\Psi, \tag{4}$$

classically fulfil the identities

$$\partial^\mu J_\mu = 0 \tag{5}$$

and

$$\partial^\mu J^5_\mu = 2imJ^5. \tag{6}$$

The axial anomaly in QED_2 stems from the two-point function

$$T^5_{\mu\nu}(x-y) = \langle T(J_\mu(x)J^5_\nu(y))\rangle, \tag{7}$$

Let us now compute the said two-point function

$$T_5^{\mu\nu} = ie^2 \int \frac{d^2p}{(2\pi)^2} \frac{Tr\left[\gamma^\mu(\not{p}+m)\gamma^\nu\gamma^5(\not{p}-\not{q}+m)\right]}{[p^2-m^2]\,[(p-q)^2-m^2]}. \tag{1}$$

Using the gamma matrices relation in two dimensions

$$\gamma^\nu\gamma^5 = \varepsilon^{\nu\lambda}\gamma_\lambda, \tag{2}$$

this is nothing but

$$T_5^{\mu\nu} = ie^2 \int \frac{d^2p}{(2\pi)^2} \frac{\varepsilon^{\nu\lambda}\, Tr\left[\gamma^\mu(\not{p}+m)\gamma_\lambda(\not{p}-\not{q}+m)\right]}{[p^2-m^2]\,[(p-q)^2-m^2]} = \varepsilon^{\nu\lambda}\, T^\mu_\lambda, \tag{3}$$

where $T^{\mu\nu}$ is the vacuum polarization tensor.

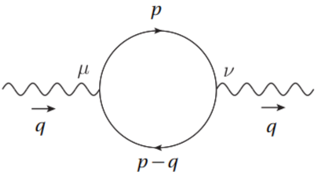


Figure: Feynman diagram for the photon self-energy at one-loop order.

Eq. (2) is generally true. We can check in light-front

$$\gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \qquad \gamma^5 = \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma^- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \qquad \varepsilon^{+-} = 1 \quad \varepsilon^{-+} = -1$$

$$\gamma^+ \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} = \varepsilon^{+-} \gamma_- = \gamma^+ \quad \checkmark$$

$$\gamma^- \gamma^5 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix} = \varepsilon^{-+} \gamma_+ = -\gamma^- \quad \checkmark$$

Now, let us compute $T^{\mu\nu}$ in the light-front. The fermion propogator in the light-front time-ordered method has two contributions, which are the on-shell and instantaneous contributions:

$$\Sigma(p) = \Sigma_{\text{on}}(p) + \Sigma_{\text{ins}}(p), \quad \Sigma_{\text{on}}(p) = \frac{\not{p}_{\text{on}} + m}{p^2 - m^2}, \quad \Sigma_{\text{ins}}(p) = \frac{\gamma^+}{2p^+}. \quad (4)$$

Thus,

$$T^{\mu\nu}(q) = \frac{ie^2}{4\pi^2} \int dp^+ \int dp^- \, Tr \left[\gamma^\mu \left(\frac{\not{p}_{\text{on}} + m}{p^2 - m^2} + \frac{\gamma^+}{2p^+} \right) \cdot \gamma^\nu \left(\frac{(\not{p} - \not{q})_{\text{on}} + m}{(p-q)^2 - m^2} + \frac{\gamma^+}{2(p-q)^+} \right) \right]. \quad (5)$$

Expanding this, we find four contributions, below labeled as $T^{\mu\nu}_{(1)}(q)$, $T^{\mu\nu}_{(2)}(q)$, $T^{\mu\nu}_{(3)}(q)$, and $T^{\mu\nu}_{(4)}(q)$.

$$T^{\mu\nu}_{(1)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\gamma^\nu \frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ + m}{2(p-q)^+(p-q)^- - m^2}\right],$$

$$T^{\mu\nu}_{(2)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\gamma^\nu \frac{\gamma^+}{2(p-q)^+}\right],$$

$$T^{\mu\nu}_{(3)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\gamma^\nu \frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ + m}{2(p-q)^+(p-q)^- - m^2}\right],$$

$$T^{\mu\nu}_{(4)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\gamma^\nu \frac{\gamma^+}{2(p-q)^+}\right]. \tag{6}$$

In four dimensions for the real photons, the fourth contribution vanishes, since once the photons' polarization vectors are attached, due to the light-cone gauge, $\varepsilon_-(q) = 0$ while the only surviving trace in the fourth contribution is $Tr[\gamma^-\gamma^+\gamma^-\gamma^+]=4$ ($\mu = -$ and $\nu = -$). However, in two dimensions, photon can only be virtual, as we can only have the longitudinal polarization. Thus this argument does not hold anymore. The fourth contribution in our case survives for the T^{--} component and vanishes for the other 3 components.

$$T^{--}_{(4)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4}{2p^+ 2(p-q)^+}. \tag{7}$$

For the second and third contributions, the relevant traces in addition to $Tr[\gamma^\mu\gamma^+\gamma^\nu\gamma^+]$ are $Tr[\gamma^\mu\gamma^-\gamma^\nu\gamma^+]$, and similarly, $Tr[\gamma^\mu\gamma^+\gamma^\nu\gamma^-]$. For both these traces, we see that due to $(\gamma^+)^2 = 0$ and $(\gamma^-)^2 = 0$, there is no $\mu\nu$ can make them non-zero. However in four dimensions, there is the transverse directions, resulting in a non-zero contribution for them. So, in our case, the second and third contributions survive also for the T^{--} component only and vanishes for other components.

$$T^{--}_{(2)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4p^-_{\text{on}}}{(2p^+p^- - m^2)2(p-q)^+}, \tag{8}$$

$$T^{--}_{(3)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4(p-q)^-_{\text{on}}}{2p^+(2(p-q)^+(p-q)^- - m^2)}. \tag{9}$$

Finally, for the first contribution, all four components survive

$$T_{(1)}^{++}(q) = \frac{ie^2}{4\pi^2} \int dp^+ \int dp^- \, Tr \left[\gamma^+ \frac{\gamma^- p^+}{2p^+ p^- - m^2} \gamma^+ \frac{\gamma^- (p-q)^+}{2(p-q)^+ (p-q)^- - m^2} \right] \tag{10}$$

$$T_{(1)}^{+-}(q) = \frac{ie^2}{4\pi^2} \int dp^+ \int dp^- \, Tr \left[\gamma^+ \frac{m}{2p^+ p^- - m^2} \gamma^- \frac{m}{2(p-q)^+ (p-q)^- - m^2} \right] \tag{11}$$

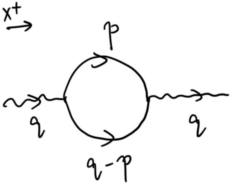
$$T_{(1)}^{-+}(q) = \frac{ie^2}{4\pi^2} \int dp^+ \int dp^- \, Tr \left[\gamma^- \frac{m}{2p^+ p^- - m^2} \gamma^+ \frac{m}{2(p-q)^+ (p-q)^- - m^2} \right] \tag{12}$$

$$T_{(1)}^{--}(q) = \frac{ie^2}{4\pi^2} \int dp^+ \int dp^- \, Tr \left[\gamma^- \frac{\gamma^+ p_{\text{on}}^-}{2p^+ p^- - m^2} \gamma^- \frac{\gamma^+ (p-q)_{\text{on}}^-}{2(p-q)^+ (p-q)^- - m^2} \right] \tag{13}$$

So, combining all of this, we have $T^{++}(q)$, $T^{+-}(q)$ and $T^{-+}(q)$ coming solely from contribution (1), while $T^{--}(q)$ have four contributions. First,

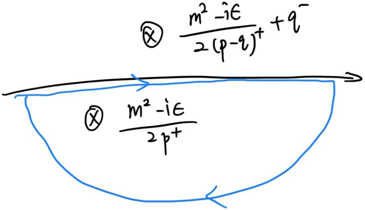
$$T^{++}(q) = T^{++}_{(1)}(q)$$
$$= \frac{ie^2}{\pi^2} \int dp^+ \int dp^- \frac{p^+(p-q)^+}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]} \tag{14}$$

For the energy integration, we have two poles



$q^+ > 0$, $p^+ > 0$, $q^+ - p^+ > 0$.

Let $x = \frac{p^+}{q^+}$, $0 < x < 1$.



We get

$$T^{++}(q) = \frac{ie^2}{\pi^2}(-2\pi i)q^+ \int_0^1 dx \frac{p^+(p-q)^+}{2p^+ 2(p-q)^+ \left(\frac{m^2}{2p^+} - \frac{m^2}{2(p-q)^+} - \frac{q^2}{2q^+}\right)}$$
$$= -\frac{e^2}{\pi} q^+ q^+ \int_0^1 dx \frac{1}{\frac{m^2}{x(x-1)} + q^2}$$
$$= -\frac{e^2}{2\pi}\left(2q^+ q^+\right) \int_0^1 dx \frac{x(x-1)}{x(x-1)q^2 + m^2}. \tag{15}$$

Next,

$$T^{+-}(q) = T^{+-}_{(1)}(q)$$
$$= \frac{ie^2 m^2}{4\pi^2} \int dp^+ \int dp^- \frac{2}{[2p^+ p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]}$$

This is same pole structure.

We get

$$T^{+-}(q) = \frac{ie^2 m^2}{4\pi^2}(-2\pi i)q^+ \int_0^1 dx \frac{2}{2p^+2(p-q)^+\left(\frac{m^2}{2p^+} - \frac{m^2}{2(p-q)^+} - \frac{q^2}{2q^+}\right)}$$

$$= -\frac{e^2 m^2}{2\pi}\int_0^1 dx \frac{1}{x(x-1)q^2 + m^2}. \tag{16}$$

The answer is the same for $T^{-+}(q)$. Now,

$$T^{--}(q) = T^{--}_{(1)}(q) + T^{--}_{(2)}(q) + T^{--}_{(3)}(q) + T^{--}_{(4)}(q)$$

$$= \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4p^-_{\text{on}}(p-q)^-_{\text{on}}}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]}$$

$$+ \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4p^-_{\text{on}}}{(2p^+p^- - m^2)2(p-q)^+}$$

$$+ \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4(p-q)^-_{\text{on}}}{2p^+(2(p-q)^+(p-q)^- - m^2)}$$

$$+ \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{4}{2p^+2(p-q)^+}$$

$T^{--}_{(1)}(q)$ has the same pole structure as before. So it is easily seen

$$T^{--}_{(1)}(q) = \frac{ie^2}{\pi^2}(-2\pi i)q^+ \int_0^1 dx \frac{\frac{m^2}{2p^+}\frac{m^2}{2(p-q)^+}}{2p^+ 2(p-q)^+ \left(\frac{m^2}{2p^+} - \frac{m^2}{2(p-q)^+} - \frac{q^2}{2q^+}\right)}$$

$$= -\frac{e^2 m^4}{4\pi q^+ q^+} \int_0^1 dx \frac{\frac{1}{x(x-1)}}{m^2 + x(x-1)q^2}$$

$$= -\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right) \int_0^1 dx \frac{\frac{1}{x(x-1)}\left(\frac{m^2}{q^2}\right)^2}{x(x-1)q^2 + m^2} \tag{17}$$

For $T^{--}_{(2)}(q)$, we enclose the upper half plane where there is no pole and note the arc contribution $\int_{\text{real axis}} + \int_{\text{arc}} = 0$.

$$\begin{aligned}
T^{--}_{(2)}(q) &= \frac{ie^2}{\pi^2}\int dp^+ \int dp^- \frac{\frac{m^2}{2p^+}}{(2p^+p^- - m^2)2(p-q)^+} \\
&= -\frac{ie^2}{4\pi^2}q^+ \int_0^1 dx \frac{m^2}{2p^+p^+(p-q)^+}\int_0^{\pi} iRe^{i\theta}d\theta \frac{1}{Re^{i\theta}} \\
&= \frac{e^2m^2}{8\pi q^+q^+}\int_0^1 dx \frac{1}{x^2(x-1)}
\end{aligned}$$

And for $T^{--}_{(3)}(q)$, we enclose the lower half plane where there is no pole and note the arc contribution $\int_{\text{real axis}} + \int_{\text{arc}} = 0$.

$$\begin{aligned}
T^{--}_{(3)}(q) &= \frac{ie^2}{\pi^2}\int dp^+ \int dp^- \frac{\frac{m^2}{2(p-q)^+}}{2p^+(2(p-q)^+(p-q)^- - m^2)} \\
&= -\frac{ie^2}{4\pi^2}q^+ \int_0^1 dx \frac{m^2}{2p^+(p-q)^+(p-q)^+}\int_0^{-\pi} iRe^{i\theta}d\theta \frac{1}{Re^{i\theta}} \\
&= -\frac{e^2m^2}{8\pi q^+q^+}\int_0^1 dx \frac{1}{x(x-1)^2}
\end{aligned}$$

Thus,

$$
\begin{aligned}
&T^{--}_{(2)}(q) + T^{--}_{(3)}(q) \\
=&\frac{e^2 m^2}{8\pi q^+ q^+}\int_0^1 dx \frac{1}{x(x-1)}\left(\frac{1}{x} - \frac{1}{x-1}\right) \\
=&-\frac{e^2 m^2}{8\pi q^+ q^+}\int_0^1 dx \frac{1}{x^2(x-1)^2} \\
=&-\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right)\int_0^1 dx \frac{\frac{x(x-1)q^2+m^2}{2x^2(x-1)^2 m^2}\left(\frac{m^2}{q^2}\right)^2}{x(x-1)q^2+m^2}
\end{aligned} \tag{18}
$$

Lastly,

$$
\begin{aligned}
&T^{--}_{(4)}(q) \\
=&\frac{ie^2}{\pi^2}\int dp^+ \int dp^- \frac{1}{2p^+2(p-q)^+} \\
=&\frac{ie^2}{4\pi^2 q^+}\int_0^1 dx \frac{1}{x(x-1)}\int dp^-
\end{aligned}
\tag{19}
$$

This is infinity because there is no p^- in the integrand. But infinity aside, let us add $T^{--}_{(1)}(q)$ and $T^{--}_{(2)}(q) + T^{--}_{(3)}(q)$.

$$
\begin{aligned}
&T^{--}_{(1)}(q) + T^{--}_{(2)}(q) + T^{--}_{(3)}(q) \\
=&-\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right)\int_0^1 dx \frac{\left(\frac{1}{x(x-1)} + \frac{x(x-1)q^2+m^2}{2x^2(x-1)^2m^2}\right)\left(\frac{m^2}{q^2}\right)^2}{x(x-1)q^2+m^2} \\
=&-\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right)\int_0^1 dx \frac{\left(\frac{(2x^2-2x+1)m^2+x(x-1)q^2}{2x^2(x-1)^2m^2}\right)\left(\frac{m^2}{q^2}\right)^2}{x(x-1)q^2+m^2}
\end{aligned}
\tag{20}
$$

The result from the covariant computation with the proper dimensional regularization is

$$T^{\mu\nu}(q) = -\frac{e^2}{2\pi}\int_0^1 dx\,\frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2+m^2} + \frac{e^2}{2\pi}g^{\mu\nu}.$$

i.e.

$$T^{++}(q) = -\frac{e^2}{2\pi}\left(2q^+q^+\right)\int_0^1 dx\,\frac{x(x-1)}{x(x-1)q^2+m^2}, \tag{21}$$

$$T^{+-}(q) = T^{-+}(q) = -\frac{e^2}{2\pi}\int_0^1 dx\,\frac{m^2}{x(x-1)q^2+m^2} + \frac{e^2}{2\pi}, \tag{22}$$

$$T^{--}(q) = -\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right)\int_0^1 dx\,\frac{x(x-1)}{x(x-1)q^2+m^2}. \tag{23}$$

But we get

$$T^{++}_{\rm LF}(q) = -\frac{e^2}{2\pi}\left(2q^+q^+\right)\int_0^1 dx\,\frac{x(x-1)}{x(x-1)q^2+m^2}, \tag{24}$$

$$T^{+-}_{\rm LF}(q) = T^{-+}_{\rm LF}(q) = -\frac{e^2}{2\pi}\int_0^1 dx\,\frac{m^2}{x(x-1)q^2+m^2}, \tag{25}$$

$$T^{--}_{\rm LF}(q) = -\frac{e^2}{2\pi}\left(2\frac{q^2}{2q^+}\frac{q^2}{2q^+}\right)\int_0^1 dx\,\frac{\left(\frac{(2x^2-2x+1)m^2+x(x-1)q^2}{2x^2(x-1)^2m^2}\right)\left(\frac{m^2}{q^2}\right)^2}{x(x-1)q^2+m^2} + i\infty. \tag{26}$$

As an example of success, let us now turn to the computation of the two-point function P_5^μ, defined as

$$P_5^\mu(q) = ie^2 \int \frac{d^2p}{(2\pi)^2} \frac{Tr\left[\gamma^\mu(\not{p}+m)\gamma^5(\not{p}-\not{q}+m)\right]}{[p^2-m^2]\left[(p-q)^2-m^2\right]}. \tag{27}$$

Similarly there are four contributions

$$P^\mu_{5(1)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\gamma^5 \frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ + m}{2(p-q)^+(p-q)^- - m^2}\right],$$

$$P^\mu_{5(2)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\gamma^5 \frac{\gamma^+}{2(p-q)^+}\right],$$

$$P^\mu_{5(3)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\gamma^5 \frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ + m}{2(p-q)^+(p-q)^- - m^2}\right],$$

$$P^\mu_{5(4)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\gamma^5 \frac{\gamma^+}{2(p-q)^+}\right]. \tag{28}$$

Commuting γ^5,

$$P^{\mu}_{5(1)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ - m}{2(p-q)^+(p-q)^- - m^2}\gamma^5\right],$$

$$P^{\mu}_{5(2)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+ p^-_{\text{on}} + \gamma^- p^+ + m}{2p^+p^- - m^2}\frac{\gamma^+}{2(p-q)^+}\gamma^5\right],$$
$$P^{\mu}_{5(3)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\frac{\gamma^+(p-q)^-_{\text{on}} + \gamma^-(p-q)^+ - m}{2(p-q)^+(p-q)^- - m^2}\gamma^5\right],$$
$$P^{\mu}_{5(4)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \, Tr\left[\gamma^\mu \frac{\gamma^+}{2p^+}\frac{\gamma^+}{2(p-q)^+}\gamma^5\right]. \tag{29}$$

Contribution (4) vanishes due to $(\gamma^+)^2 = 0$. Others simplify due to trace of γ^5 with odd number of γ's is 0.

$$P^{\mu}_{5(1)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{mTr\left[\gamma^\mu\left(\gamma^+(p-q)^-_{\rm on} + \gamma^-(p-q)^+\right)\gamma^5\right] - mTr\left[\gamma^\mu\left(\gamma^+ p^-_{\rm on} + \gamma^- p^+\right)\gamma^5\right]}{\left[2p^+p^- - m^2\right]\left[2(p-q)^+(p-q)^- - m^2\right]} \quad (30)$$

$$P^{\mu}_{5(2)}(q) = -\frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{mTr\left[\gamma^\mu\gamma^+\gamma^5\right]}{[2p^+p^- - m^2]\,2(p-q)^+},$$

$$P^{\mu}_{5(3)}(q) = \frac{ie^2}{4\pi^2}\int dp^+ \int dp^- \frac{mTr\left[\gamma^\mu\gamma^+\gamma^5\right]}{2p^+\left[2(p-q)^+(p-q)^- - m^2\right]}. \quad (31)$$

For contributions (2) and (3), μ can take $-$ only. Thus,

$$P^+_5(q) = P^+_{5(1)}(q),$$

$$P^-_5(q) = P^-_{5(1)}(q) + P^-_{5(2)}(q) + P^-_{5(3)}(q). \quad (32)$$

$$
\begin{aligned}
P^{+}_{5(1)}(q) &= -\frac{ie^2 m}{4\pi^2}\int dp^+ \int dp^- \frac{Tr[\gamma^+\gamma^-\gamma^5](p-q)^+ - Tr[\gamma^+\gamma^-\gamma^5]p^+}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]} \\
&= -\frac{ie^2 m}{2\pi^2}\int dp^+ \int dp^- \frac{q^+}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]} \\
&= -\frac{ie^2 m}{2\pi^2}(-2\pi i)q^+ \int_0^1 dx \frac{q^+}{2p^+2(p-q)^+\left(\frac{m^2}{2p^+} - \frac{m^2}{2(p-q)^+} - \frac{q^2}{2q^+}\right)} \\
&= -\frac{e^2 m}{2\pi} q^+ \int_0^1 dx \frac{1}{x(x-1)\left(\frac{m^2}{x} - \frac{m^2}{x-1} - q^2\right)} \\
&= \frac{e^2 m}{2\pi} q^+ \int_0^1 dx \frac{1}{x(x-1)q^2 + m^2} \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
P^{-}_{5(1)}(q) &= -\frac{ie^2 m}{4\pi^2}\int dp^+ \int dp^- \frac{Tr[\gamma^-\gamma^+\gamma^5]\frac{m^2}{2(p-q)^+} - Tr[\gamma^-\gamma^+\gamma^5]\frac{m^2}{2p^+}}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]} \\
&= -\frac{ie^2 m}{2\pi^2}\int dp^+ \int dp^- \frac{\frac{m^2}{2(p-q)^+} - \frac{m^2}{2p^+}}{[2p^+p^- - m^2]\,[2(p-q)^+(p-q)^- - m^2]} \\
&= -\frac{ie^2 m}{2\pi^2}(-2\pi i)q^+ \int_0^1 dx \frac{\frac{m^2}{2(p-q)^+} - \frac{m^2}{2p^+}}{2p^+2(p-q)^+\left(\frac{m^2}{2p^+} - \frac{m^2}{2(p-q)^+} - \frac{q^2}{2q^+}\right)} \\
&= -\frac{e^2 m}{2\pi}\frac{q^2}{2q^+}\int_0^1 dx \frac{\frac{m^2}{x-1} - \frac{m^2}{x}}{q^2 x(x-1)\left(\frac{m^2}{x} - \frac{m^2}{x-1} - q^2\right)} \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
P^{-}_{5(2)}(q) &= -\frac{ie^2 m}{2\pi^2}\int dp^+ \int dp^- \frac{1}{[2p^+p^- - m^2]\,2(p-q)^+} \\
&= \frac{ie^2 m}{2\pi^2} q^+ \int_0^1 dx \frac{1}{2p^+ 2(p-q)^+} \int_0^{\pi} iRe^{i\theta} d\theta \frac{1}{Re^{i\theta}} \\
&= -\frac{e^2 m}{4\pi}\frac{q^2}{2q^+}\int_0^1 dx \frac{1}{q^2 x(x-1)} \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
P^{-}_{5(3)}(q) &= \frac{ie^2 m}{2\pi^2}\int dp^+ \int dp^- \frac{1}{2p^+\,[2(p-q)^+(p-q)^- - m^2]} \\
&= -\frac{ie^2 m}{2\pi^2} q^+ \int_0^1 dx \frac{1}{2p^+ 2(p-q)^+} \int_0^{-\pi} iRe^{i\theta} d\theta \frac{1}{Re^{i\theta}} \\
&= -\frac{e^2 m}{4\pi}\frac{q^2}{2q^+}\int_0^1 dx \frac{1}{q^2 x(x-1)} \qquad (36)
\end{aligned}
$$

Thus,

$$P_5^-(q) = -\frac{e^2 m}{2\pi} q^- \int_0^1 dx \frac{\frac{m^2}{x} - \frac{m^2}{x-1} - q^2 + \frac{m^2}{x-1} - \frac{m^2}{x}}{q^2 x(x-1)\left(\frac{m^2}{x} - \frac{m^2}{x-1} - q^2\right)}$$
$$= -\frac{e^2 m}{2\pi} q^- \int_0^1 dx \frac{1}{x(x-1)q^2 + m^2}. \tag{37}$$

Both this and Eq. (33) are exactly what we expect from the covariant result

$$P_5^\mu(q) = \frac{e^2 m}{2\pi} \int_0^1 dx \frac{\varepsilon^{\mu\nu} q_\nu}{x(x-1)q^2 + m^2}. \tag{38}$$