INTERPOLATING POINCARE GENERATORS IN DIFFERENT BASES

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Interpolating transformation Relations

- Contravariant Interpolating space time coordinates.

$$x^{\hat{\mu}} = G^{\hat{\mu}}_{\nu} x^{\nu}$$

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

- Covariant Interpolating space time coordinates

$$x_{\hat{\mu}} = g_{\hat{\mu}\hat{\nu}} x^{\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} G^{\hat{\nu}}_{\alpha} x^{\alpha} = R_{\hat{\mu}\alpha} x^{\alpha}$$

$$R_{\hat{\mu}\alpha} = g_{\hat{\mu}\hat{\nu}} G^{\hat{\nu}}_{\alpha} = \begin{pmatrix} \cos\delta & 0 & 0 & -\sin\delta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\delta & 0 & 0 & \cos\delta \end{pmatrix}$$

- Interpolating space-time matrix tensor

$$g^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix} = g_{\hat{\mu}\hat{\nu}}$$

$$a_{\hat{+}} = \mathbb{C}a^{\hat{+}} + \mathbb{S}a^{\hat{-}}; \qquad a^{\hat{+}} = \mathbb{C}a_{\hat{+}} + \mathbb{S}a_{\hat{-}}$$

$$a_{\hat{-}} = \mathbb{S}a^{\hat{+}} - \mathbb{C}a^{\hat{-}}; \qquad a^{\hat{-}} = \mathbb{S}a_{\hat{+}} - \mathbb{C}a_{\hat{-}}$$

$$a_j = -a^j, \qquad (j = 1, 2).$$

Interpolating Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

$x^{\nu} \longrightarrow x'^{\mu}$

$$(\Lambda^{\mu}_{\alpha})^{tr} \eta^{\alpha\rho} \Lambda^{\nu}_{\rho} = \eta^{\mu\nu}$$

$\eta^{\mu\nu}$ is the Mankowski matrix

- <u>Contravariant interpolation Lorentz transformation</u>

$$x'^{\hat{\mu}} = G^{\hat{\mu}}_{\nu} x'^{\nu} = G^{\hat{\mu}}_{\nu} \Lambda^{\nu}_{\alpha} x^{\alpha} = G^{\hat{\mu}}_{\nu} \Lambda^{\nu}_{\alpha} (G^{-1})^{\alpha}_{\hat{\nu}} x^{\hat{\nu}} = \Lambda^{\hat{\mu}}_{\hat{\nu},con} x^{\hat{\nu}}$$

$x^{\hat{\nu}} \longrightarrow x'^{\hat{\mu}}$

$$(\Lambda^{\hat{\mu}}_{\hat{\alpha},con})^{tr} g^{\hat{\alpha}\hat{\eta}} \Lambda^{\hat{\nu}}_{\hat{\eta},con} = g^{\hat{\mu}\hat{\nu}}$$

- <u>Covariant interpolation Lorentz transformation</u>

$$x'_{\hat{\mu}} = R_{\hat{\mu}\nu} x'^{\nu} = R_{\hat{\mu}\nu} \Lambda^{\nu}_{\alpha} x^{\alpha} = R_{\hat{\mu}\nu} \Lambda^{\nu}_{\alpha} (R^{-1})^{\hat{\nu}\alpha} x_{\hat{\nu}} = \Lambda^{\hat{\nu}}_{\hat{\mu},cov} x_{\hat{\nu}}$$

$x_{\hat{\nu}} \longrightarrow x'_{\hat{\mu}}$

$$(\Lambda^{\hat{\mu}}_{\hat{\alpha},cov})^{tr} g^{\hat{\alpha}\hat{\eta}} \Lambda^{\hat{\nu}}_{\hat{\eta},cov} = g^{\hat{\mu}\hat{\nu}}$$

New Basis

$$x^N = H.x$$

$$\delta \to 0, \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \to x^0, x^{\hat{1}} \to x^1, x^{\hat{2}} \to x^2, \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \to x^3$$

$$\delta \to \frac{\pi}{4}, \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \to \frac{x^+}{0}, x^{\hat{1}} \to x^1, x^{\hat{2}} \to x^2, \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \to \frac{x^+}{0}$$

$$\begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\sin\delta}{\sqrt{\mathbb{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\cos\delta}{\sqrt{\mathbb{C}}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Light front time $x^+ \to 0$ $\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}$ and $\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}$ goes to a finite value ($\bar{x}^+$)

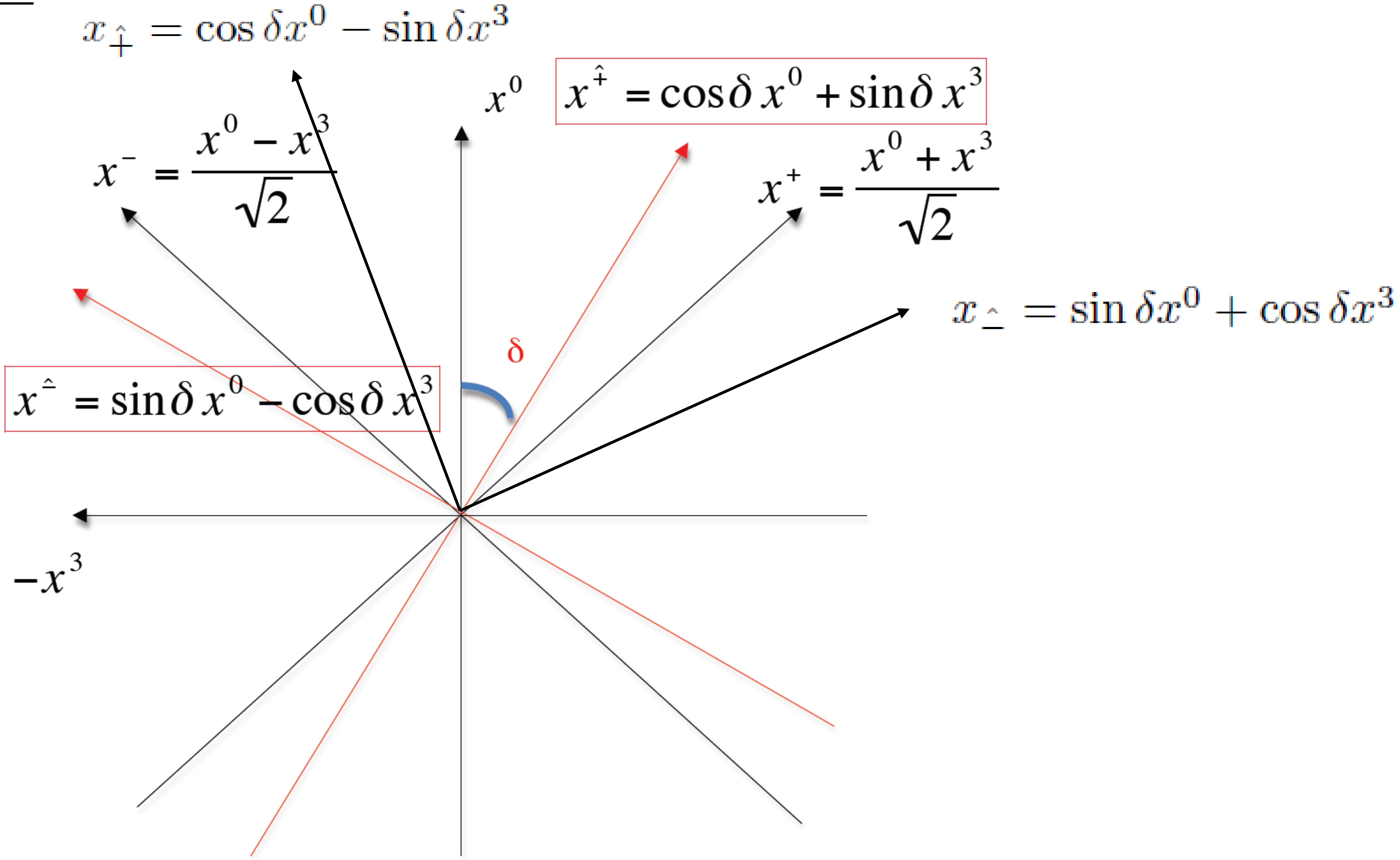
- We can see similar behavior in the momentum space light front-zero mode ,$P^+ \to 0$

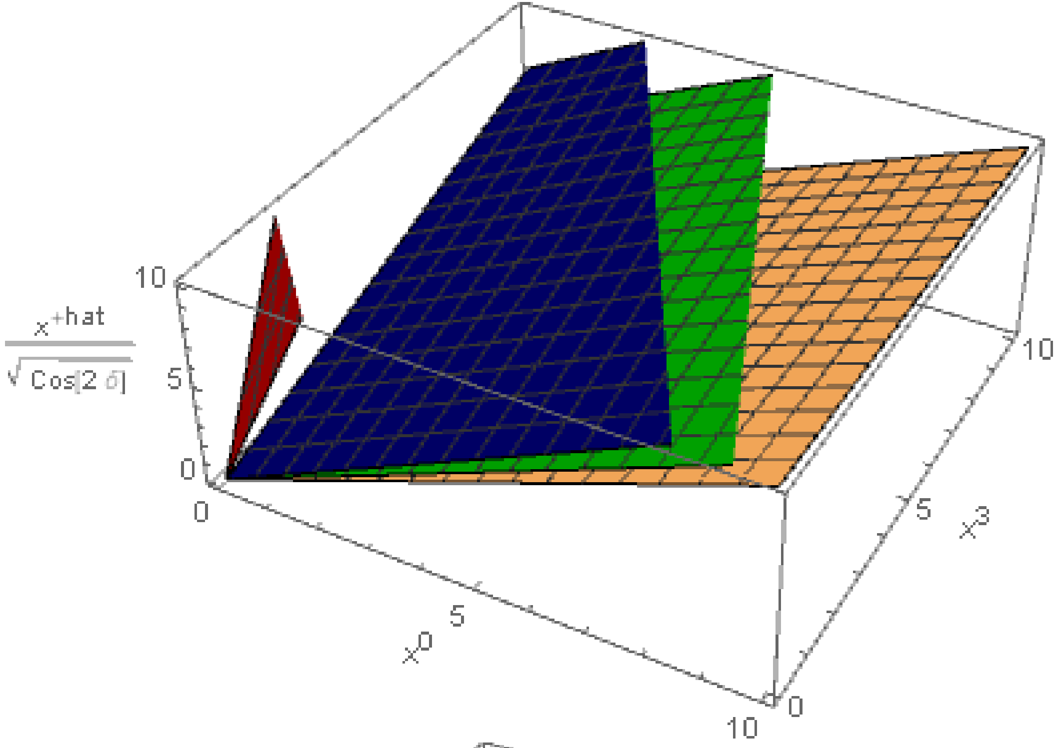
At IFD , 4 degrees of freedom, but at LFD 3 degrees of freedom

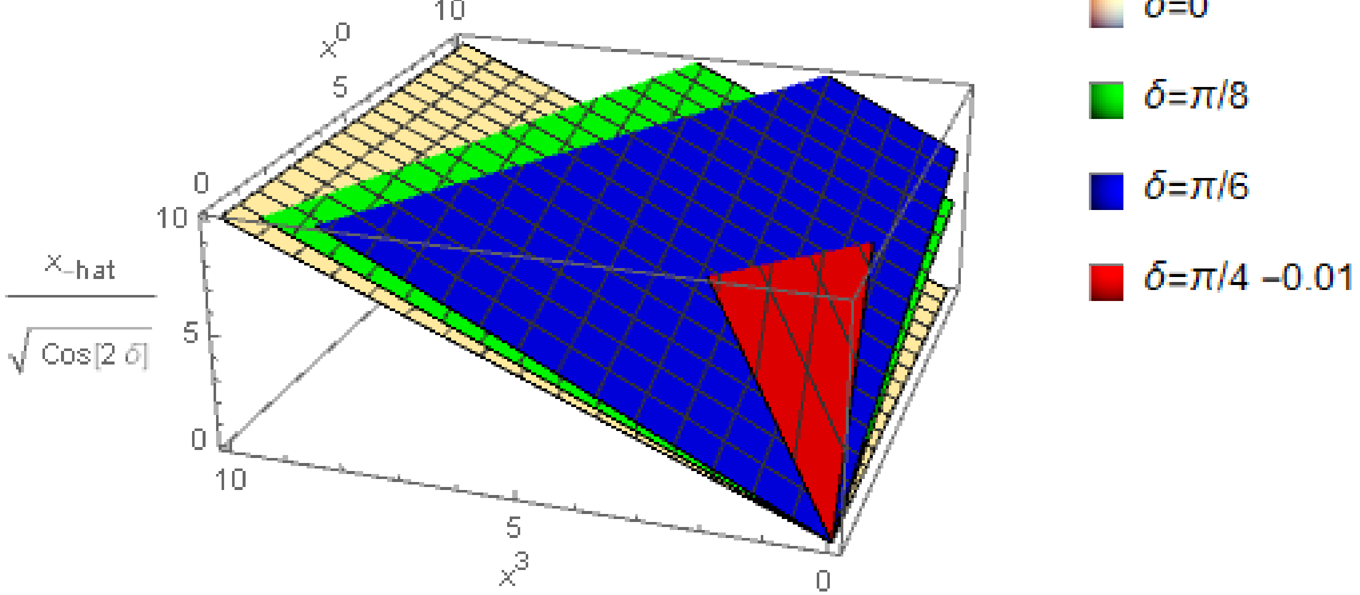
$$x'^N = Hx' = H\Lambda.x = H\Lambda(H^{-1}).x^N = \Lambda^N x^N$$

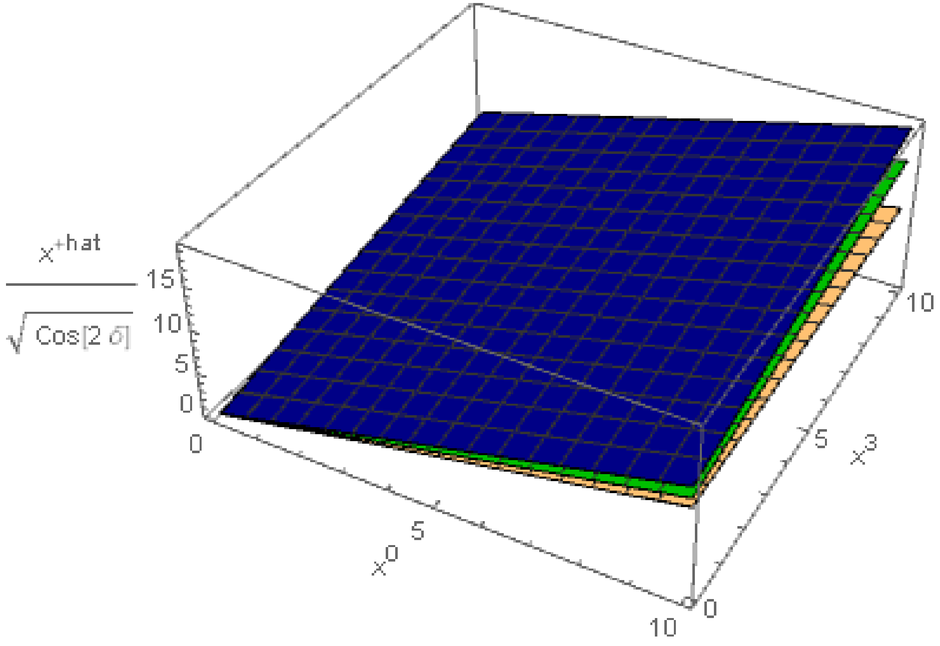
$$x^N = \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}, x^{\hat{1}}, x^{\hat{2}}, \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \quad \longrightarrow \quad x'^N = \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}}, x'^{\hat{1}}, x'^{\hat{2}}, \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}}$$

Coordinate -Space

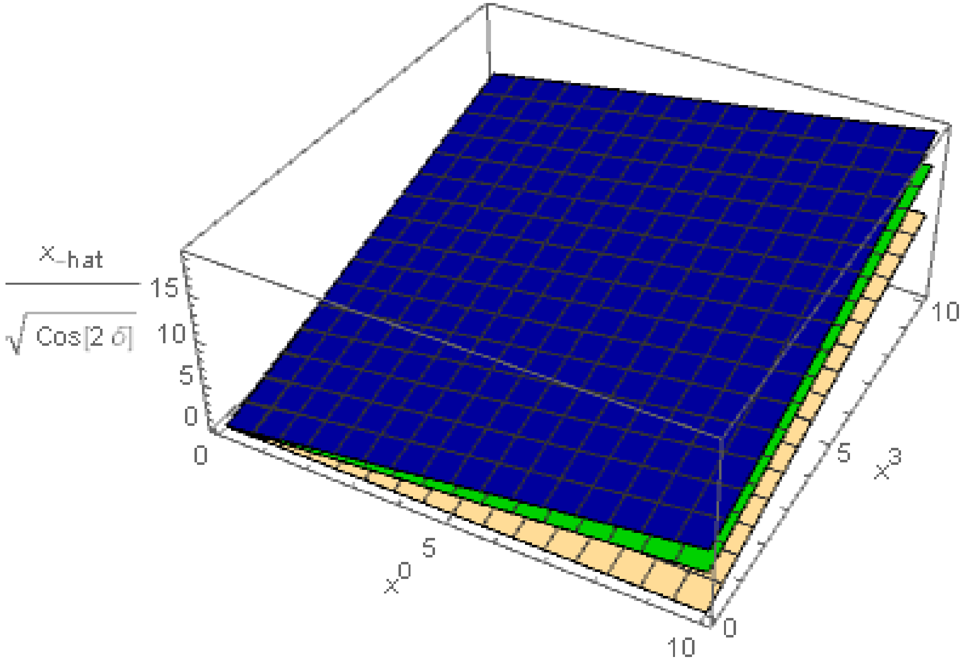




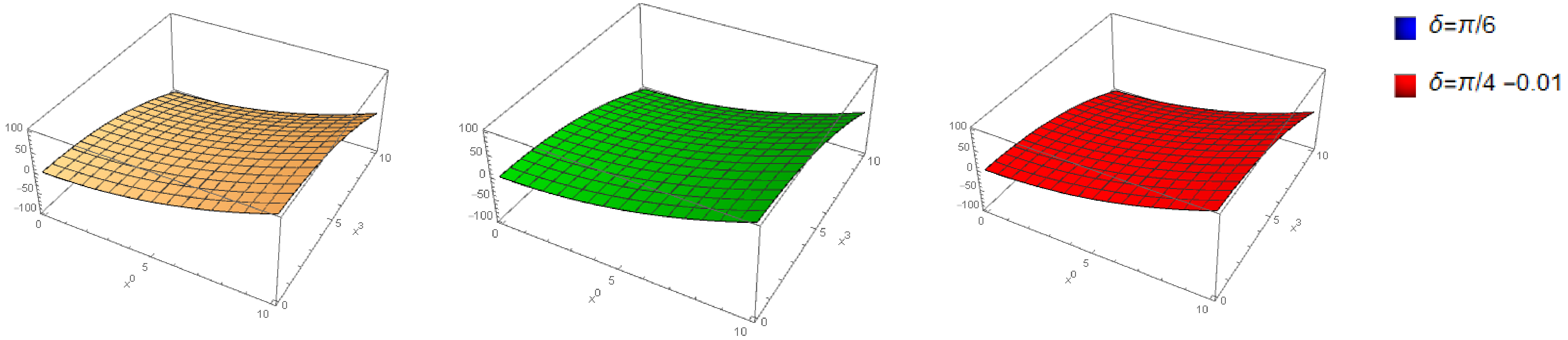




- Different Planes for different interpolation angles except at LF



$$\left(\frac{x^{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{x_{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2$$



We can prove invariant in space time interval in the new basis

$$s^2 = x^\mu x_\mu = x^{\hat{\mu}} x_{\hat{\mu}} = \left(\frac{x^{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2 - (x^1)^2 - (x^2)^2 - \left(\frac{x_{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2$$

Momentum space, For the particle of mass M, $P^\mu P_\mu$ on the mass shell is equal to M^2

$$M^2 = P^\mu P_\mu = P^{\hat{\mu}} P_{\hat{\mu}} = \left(\frac{P^{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2 - (P^1)^2 - (P^2)^2 - \left(\frac{P_{\hat{\dagger}}}{\sqrt{\mathbb{C}}}\right)^2$$

New basis in the momentum space.

Boost and Rotation operators in 4-vector representation

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Poincare Matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

Interpolating Poincare Matrix

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$

$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$

$$\mathcal{D}^{\hat{1}} = -K^1 \cos\delta + J^2 \sin\delta,$$

$$\mathcal{D}^{\hat{2}} = -J^1 \sin\delta - K^2 \cos\delta.$$

- Convert all these generators into $\Lambda^{\hat{\mu}}_{\hat{\nu},con}$ and $\Lambda^{\hat{\nu}}_{\hat{\mu},cov}$ structures and see whether they satisfy Lorentz transformation conditions $(\Lambda^{\hat{\mu}}_{\hat{\alpha},con})^{tr} g^{\hat{\alpha}\hat{\eta}} \Lambda^{\hat{\nu}}_{\hat{\eta},con} = g^{\hat{\mu}\hat{\nu}}$ and $(\Lambda^{\hat{\mu}}_{\hat{\alpha},cov})^{tr} g^{\hat{\alpha}\hat{\eta}} \Lambda^{\hat{\nu}}_{\hat{\eta},cov} = g^{\hat{\mu}\hat{\nu}}$

- Convert all these generators into Λ^N structure in the new basis and understand the behavior of them in the IFD and LFD

- Consider the dot product of transformed four vectors in space-time using the above Transformations, to check the invariant of the space-time interval.

z-direction boost $e^{(-i\beta_z K_3)}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) + \sinh(\beta_z)\mathbb{S} & 0 & 0 & -\sinh(\beta_z)\mathbb{C} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\beta_z)\mathbb{C} & 0 & 0 & \cosh(\beta_z) - \sinh(\beta_z)\mathbb{S} \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) - \sinh(\beta_z)\mathbb{S} & 0 & 0 & \sinh(\beta_z)\mathbb{C} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z)\mathbb{C} & 0 & 0 & \cosh(\beta_z) + \sinh(\beta_z)\mathbb{S} \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

β_z is the rapidity

LFD

$$\bar{x}'^{+} = (Cosh(\beta_z) + Sinh(\beta_z))\, \bar{x}^{+}$$

$$\left.\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right|_{\delta \to \frac{\pi}{4}} = \bar{x}^{+}$$

$Cosh(\beta_z) = \gamma$, $Sinh(\beta_z) = \gamma\beta$
and
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v_z}{c}$

$\bar{x}'^{+}$ $\bar{x}^{+}$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$\bar{x}'^{+}$ $\bar{x}^{+}$

$$\bar{x}'^{+} = \sqrt{\frac{1+\beta}{1-\beta}}\, \bar{x}^{+}$$

Relativistic longitudinal doppler effect

$$\lambda_r = \sqrt{\frac{1+\beta}{1-\beta}}\, \lambda_s$$

- It seems this basis produce the doppler effect

x-direction boost $e^{(-i\beta_x K_1)}$

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_x) & \sinh(\beta_x) & 0 & 0 \\ \sinh(\beta_x) & \cosh(\beta_x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_x)\cos\delta^2 + \sin\delta^2 & 0 & \sinh(\beta_x)\cos\delta & (\cosh(\beta_x)-1)\mathbb{S}/2 \\ \sinh(\beta_x)\cos\delta & \cosh(\beta_x) & 0 & \sinh(\beta_x)\sin\delta \\ 0 & 0 & 1 & 0 \\ (\cosh(\beta_x)-1)\mathbb{S}/2 & \sinh(\beta_x)\sin\delta & 0 & \cosh(\beta_x)\sin\delta^2 + \cos\delta^2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_x)\cos\delta^2 + \sin\delta^2 & 0 & -\sinh(\beta_x)\cos\delta & (\cosh(\beta_x)-1)\mathbb{S}/2 \\ -\sinh(\beta_x)\cos\delta & \cosh(\beta_x) & 0 & -\sinh(\beta_x)\sin\delta \\ 0 & 0 & 1 & 0 \\ (\cosh(\beta_x)-1)\mathbb{S}/2 & -\sinh(\beta_x)\sin\delta & 0 & \cosh(\beta_x)\sin\delta^2 + \cos\delta^2 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$
\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cosh(\beta_x)+1)\mathbb{C}+(\cosh(\beta_x)-1)}{2\mathbb{C}} & \frac{\sinh(\beta_x)\cos\delta}{\sqrt{\mathbb{C}}} & 0 & -\frac{(\cosh(\beta_x)-1)\mathbb{S}}{2\mathbb{C}} \\ \frac{\sinh(\beta_x)\cos\delta}{\sqrt{\mathbb{C}}} & \cosh(\beta_x) & 0 & -\frac{\sinh(\beta_x)\sin\delta}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{(\cosh(\beta_x)-1)\mathbb{S}}{2\mathbb{C}} & \frac{\sinh(\beta_x)\sin\delta}{\sqrt{\mathbb{C}}} & 0 & \frac{(\cosh(\beta_x)+1)\mathbb{C}-(\cosh(\beta_x)-1)}{2\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}
$$

This structure is not independent of δ

y-direction boost $e^{(-i\beta_y K_2)}$

$$
\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_y) & 0 & \sinh(\beta_y) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(\beta_y) & 0 & \cosh(\beta_y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}
$$

$$
\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_y)\cos\delta^2 + \sin\delta^2 & 0 & \sinh(\beta_y)\cos\delta & (\cosh(\beta_y)-1)\mathbb{S}/2 \\ 0 & 1 & 0 & 0 \\ \sinh(\beta_y)\cos\delta & 0 & \cosh(\beta_y) & \sinh(\beta_y)\sin\delta \\ (\cosh(\beta_y)-1)\mathbb{S}/2 & 0 & \sinh(\beta_y)\sin\delta & \cosh(\beta_y)\sin\delta^2 + \cos\delta^2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}
$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_y)\cos\delta^2 + \sin\delta^2 & 0 & -\sinh(\beta_y)\cos\delta & (\cosh(\beta_y)-1)\mathbb{S}/2 \\ 0 & 1 & 0 & 0 \\ -\sinh(\beta_y)\cos\delta & 0 & \cosh(\beta_y) & -\sinh(\beta_y)\sin\delta \\ (\cosh(\beta_y)-1)\mathbb{S}/2 & 0 & -\sinh(\beta_y)\sin\delta & \cosh(\beta_y)\sin\delta^2 + \cos\delta^2 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cosh(\beta_y)+1)\mathbb{C}+(\cosh(\beta_y)-1)}{2\mathbb{C}} & 0 & \frac{\sinh(\beta_y)\cos\delta}{\sqrt{\mathbb{C}}} & -\frac{(\cosh(\beta_y)-1)\mathbb{S}}{2\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sinh(\beta_y)\cos\delta}{\sqrt{\mathbb{C}}} & 0 & \cosh(\beta_y) & -\frac{\sinh(\beta_y)\sin\delta}{\sqrt{\mathbb{C}}} \\ \frac{(\cosh(\beta_y)-1)\mathbb{S}}{2\mathbb{C}} & 0 & \frac{\sinh(\beta_y)\sin\delta}{\sqrt{\mathbb{C}}} & \frac{(\cosh(\beta_y)+1)\mathbb{C}-(\cosh(\beta_y)-1)}{2\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

This structure is not independent of δ

Rotation around z- axis $e^{-iJ_3\theta_z}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

This matrix does not mix with x^0 and x^3

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- We have the same transformation matrix, which is totally independent of the δ

Rotation around x- axis $e^{-iJ_1\theta_x}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta^2 + \cos\theta_x \sin\delta^2 & 0 & \sin\delta\sin\theta_x & \sin 2\delta \sin(\theta_x/2)^2 \\ 0 & 1 & 0 & 0 \\ -\sin\delta\sin\theta_x & 0 & \cos\theta_x & \cos\delta\sin\theta_x \\ \sin 2\delta\sin(\theta_x/2)^2 & 0 & -\cos\delta\sin\theta_x & \cos\delta^2\cos\theta_x + \sin\delta^2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta^2 + \cos\theta_x \sin\delta^2 & 0 & \sin\delta\sin\theta_x & \sin 2\delta \sin(\theta_x/2)^2 \\ 0 & 1 & 0 & 0 \\ -\sin\delta\sin\theta_x & 0 & \cos\theta_x & \cos\delta\sin\theta_x \\ \sin 2\delta\sin(\theta_x/2)^2 & 0 & -\cos\delta\sin\theta_x & \cos\delta^2\cos\theta_x + \sin\delta^2 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cos\delta^2 - \cos\theta_x \sin\delta^2)}{\mathbb{C}} & 0 & \frac{\sin\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & -\frac{(\sin(\theta_x/2)^2\mathbb{S})}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sin\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & 0 & \cos\theta_x & -\frac{\cos\delta\sin\theta_x}{\sqrt{\mathbb{C}}} \\ \frac{(\sin(\theta_x/2)^2\mathbb{S})}{\mathbb{C}} & 0 & \frac{\cos\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & \frac{(\cos\delta^2\cos\theta_x - \sin\delta^2)}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

This structure is not independent of δ

Rotation around y- axis $\quad e^{-iJ_2\theta_y}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta^2 + \cos\theta_y \sin\delta^2 & -\sin\delta\sin\theta_y & 0 & \sin 2\delta \sin(\theta_y/2)^2 \\ \sin\delta\sin\theta_y & \cos\theta_y & 0 & -\cos\delta\sin\theta_y \\ 0 & 0 & 1 & 0 \\ \sin 2\delta \sin(\theta_y/2)^2 & \cos\delta\sin\theta_y & 0 & \cos\delta^2\cos\theta_y + \sin\delta^2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta^2 + \cos\theta_y \sin\delta^2 & -\sin\delta\sin\theta_y & 0 & \sin 2\delta \sin(\theta_y/2)^2 \\ \sin\delta\sin\theta_y & \cos\theta_y & 0 & -\cos\delta\sin\theta_y \\ 0 & 0 & 1 & 0 \\ \sin 2\delta \sin(\theta_y/2)^2 & \cos\delta\sin\theta_y & 0 & \cos\delta^2\cos\theta_y + \sin\delta^2 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cos\delta^2 - \cos\theta_y \sin\delta^2)}{\mathbb{C}} & -\frac{\sin\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & 0 & -\frac{(\sin(\theta_y/2)^2\mathbb{S})}{\mathbb{C}} \\ -\frac{\sin\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & \cos\theta & 0 & \frac{\cos\delta\sin\theta_y}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{(\sin(\theta_y/2)^2\mathbb{S})}{\mathbb{C}} & -\frac{\cos\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & 0 & \frac{(\cos\delta^2\cos\theta_y - \sin\delta^2)}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

This structure is not independent of δ

Kinematic Generator $\mathcal{K}^{\hat{1}}$ $\quad e^{i\beta_1 \mathcal{K}^{\hat{1}}}$ $\quad \alpha_1 = \sqrt{\beta_1^2 \cos 2\delta} = \sqrt{\beta_1^2 \mathbb{C}}$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_1 \sin\delta^2}{\mathbb{C}} & \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ \frac{\sin\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & \cos\alpha_1 & 0 & \frac{\cos\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ -\frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} & -\frac{\cos\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2 \cos\alpha_1 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$\mathcal{K}^{\hat{1}} \to -J^2$ IFD

$\mathcal{K}^{\hat{1}} \to -E_1$ LFD

$$e^{-i\beta_1 J^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}[\beta 1] & 0 & \mathrm{Sin}[\beta 1] \\ 0 & 0 & 1 & 0 \\ 0 & -\mathrm{Sin}[\beta 1] & 0 & \mathrm{Cos}[\beta 1] \end{pmatrix}$$

β_1 is an angle

$$e^{-i\beta_1 E_1} = \begin{pmatrix} \frac{1}{4}(4+\beta 1^2) & \frac{\beta 1}{\sqrt{2}} & 0 & \frac{\beta 1^2}{4} \\ \frac{\beta 1}{\sqrt{2}} & 1 & 0 & \frac{\beta 1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ -\frac{\beta 1^2}{4} & -\frac{\beta 1}{\sqrt{2}} & 0 & 1-\frac{\beta 1^2}{4} \end{pmatrix}$$

β_1 is the rapidity

Apply to a rest particle of mass M with spin S

$$P = \{M, 0,0,0\} \qquad S = \{0,0,0,1\}$$

IFD

$$P\,' = \{M, 0,0,0\}$$

$$S' = \{0, \mathrm{Sin}[\beta1], 0, \mathrm{Cos}[\beta1]\}$$

LFD

$$P\,' = \left\{\frac{1}{4} M(4+\beta1^2), \frac{M\beta1}{\sqrt{2}}, 0, -\frac{M\beta1^2}{4}\right\}$$

$$S' = \left\{\frac{\beta1^2}{4}, \frac{\beta1}{\sqrt{2}}, 0, 1-\frac{\beta1^2}{4}\right\}$$

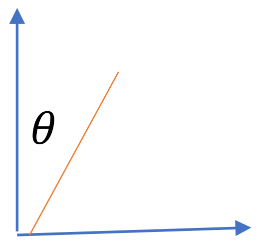
$$\mathrm{Tan}[\theta] = -\frac{2\sqrt{2}}{\beta1}$$

$$Cos[\theta a] = \frac{4-\beta1^2}{\sqrt{16+\beta1^4}}$$

$$Cos[\theta a] = \frac{1-2\mathrm{Cot}[\theta]^2}{\sqrt{1+4\mathrm{Cot}[\theta]^4}}$$

Apply to a rest particle of mass M with spin S $P = \{M, 0,0,0\}$ $S = \{0,0,0,1\}$

$$\begin{pmatrix} P'^0 \\ P'^1 \\ P'^2 \\ P'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_1 \sin\delta^2}{\mathbb{C}} & \frac{\sin\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ \frac{\sin\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & \cos\alpha_1 & 0 & \frac{\cos\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ -\frac{\sin(\alpha_1/2)^2\mathbb{S}}{\mathbb{C}} & -\frac{\cos\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2\cos\alpha_1 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}$$



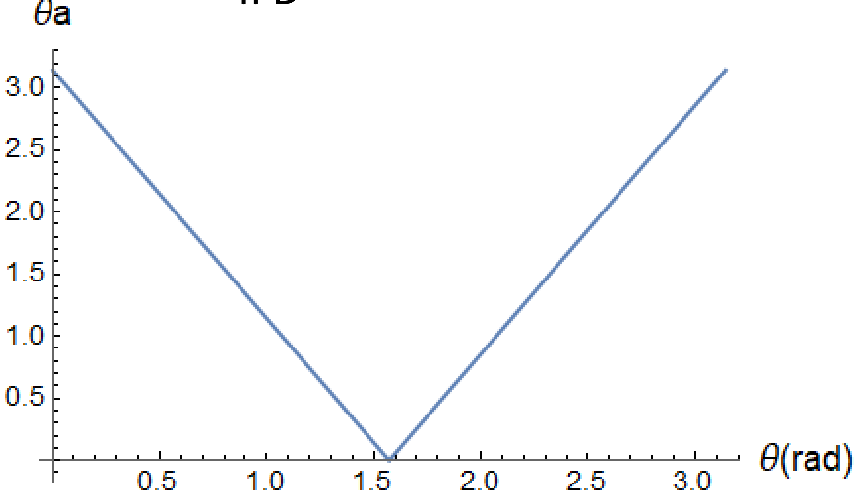
$$P' = \left\{M - M\left(-1 + \mathrm{Cos}\left[\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]\right)\mathrm{Sec}[2\delta]\mathrm{Sin}[\delta]^2, M\sqrt{\mathrm{Sec}[2\delta]}\mathrm{Sin}[\delta]\mathrm{Sin}\left[\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right], 0, -M\,\mathrm{Sin}\left[\frac{1}{2}\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]^2 \mathrm{Tan}[2\delta]\right\}$$

$$S' = \left\{\mathrm{Sin}\left[\frac{1}{2}\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]^2 \mathrm{Tan}[2\delta], \mathrm{Cos}[\delta]\sqrt{\mathrm{Sec}[2\delta]}\mathrm{Sin}\left[\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right], 0, 1 + \mathrm{Cos}[\delta]^2\left(-1 + \mathrm{Cos}\left[\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]\right)\mathrm{Sec}[2\delta]\right\}$$

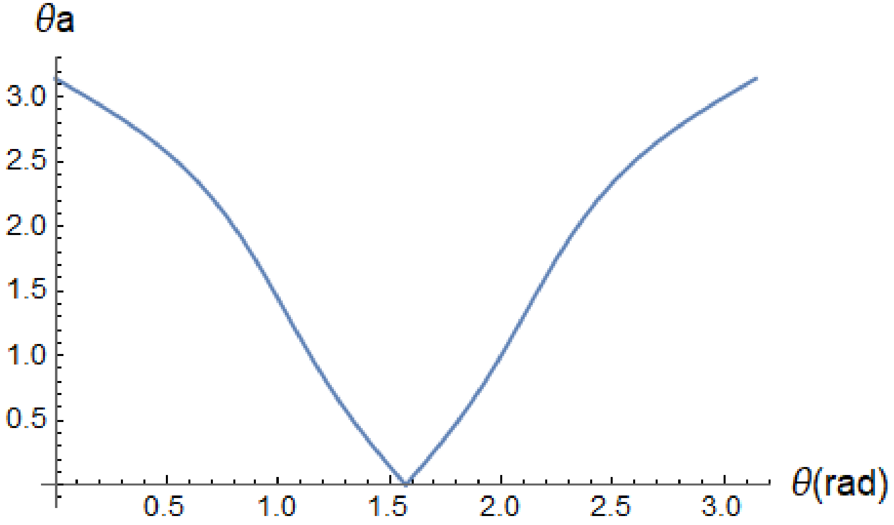
$$\mathrm{Tan}[\theta] = -\frac{\mathrm{Cot}\left[\frac{1}{2}\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]\mathrm{Sec}[\delta]}{\sqrt{\mathrm{Sec}[2\delta]}} \qquad \mathrm{Cos}[\theta_a] = \frac{1 + \mathrm{Cos}[\delta]^2\left(-1 + \mathrm{Cos}\left[\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]\right)\mathrm{Sec}[2\delta]}{\sqrt{1 + \mathrm{Sin}\left[\frac{1}{2}\beta 1\sqrt{\mathrm{Cos}[2\delta]}\right]^4 \mathrm{Tan}[2\delta]^2}}$$

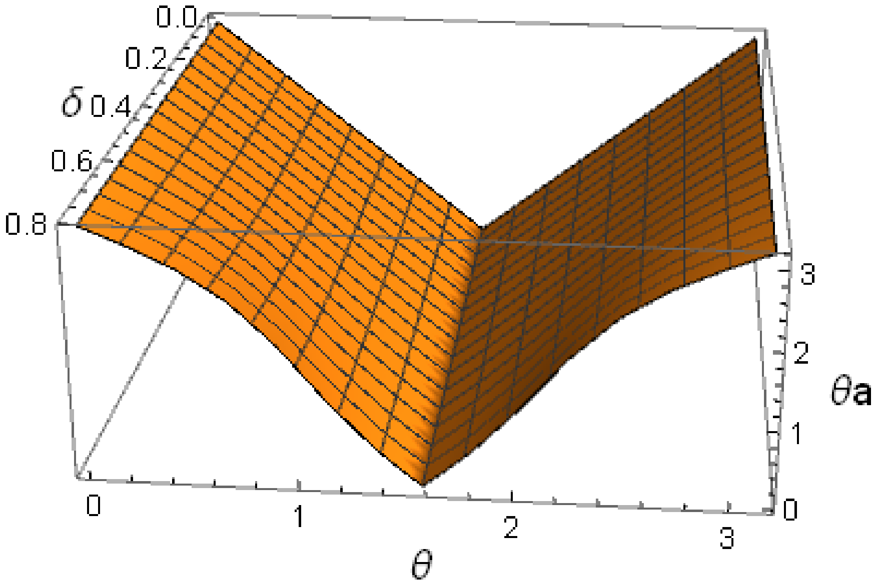
$$Cos[\theta a] = \frac{1 - \frac{2\mathrm{Cot}[\theta]^2}{1+\mathrm{Cos}[2\delta]\mathrm{Cot}[\theta]^2\mathrm{Sec}[\delta]^2}}{\sqrt{1+\frac{4\mathrm{Cot}[\theta]^4\mathrm{Tan}[\delta]^2}{(1+\mathrm{Cos}[2\delta]\mathrm{Cot}[\theta]^2\mathrm{Sec}[\delta]^2)^2}}}$$

IFD



LFD





Kinematic Generator $\mathcal{K}^{\hat{1}}$ $e^{i\beta_1\mathcal{K}^{\hat{1}}}$ $\alpha_1 = \sqrt{\beta_1^2\cos 2\delta} = \sqrt{\beta_1^2\mathbb{C}}$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\sin\alpha_1\mathbb{S}}{\sqrt{\mathbb{C}}} & \cos\alpha_1 & 0 & -\sin\alpha_1\sqrt{\mathbb{C}} \\ 0 & 0 & 1 & 0 \\ -\frac{(-1+\cos\alpha_1)\mathbb{S}}{\mathbb{C}} & \frac{\sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\sin\alpha_1\mathbb{S}}{\sqrt{\mathbb{C}}} & 0 & -\frac{(-1+\cos\alpha_1)\mathbb{S}}{\mathbb{C}} \\ 0 & \cos\alpha_1 & 0 & -\frac{\sin\alpha_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\mathbb{C}}\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix}} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- Similar to the $e^{-i\beta_1 J^2}$, but here we have $e^{-i\alpha_1 J^2}$
- It seems $\mathcal{K}^{\hat{1}}$ play the role of rotation around y axis starting from the IFD to LF-zero mode with this new basis

Kinematic operator $\mathcal{K}^{\hat{1}}$ exclusively independent of interpolation angle in the new basis

Kinematic Generator $\mathcal{K}^{\hat{2}}$ $\quad e^{i\beta_2 \mathcal{K}^{\hat{2}}}$

$$\alpha_2 = \sqrt{\beta_1^2 \cos 2\delta} = \sqrt{\beta_2^2 \mathbb{C}}$$

$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_2 \sin\delta^2}{\mathbb{C}} & 0 & \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sin\delta \sin\alpha_2}{\sqrt{\mathbb{C}}} & 0 & \cos\alpha_2 & \frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} \\ -\frac{\sin(\alpha_2/2)^2 \mathbb{S}}{\mathbb{C}} & 0 & -\frac{\cos\delta \sin\alpha_2}{\sqrt{\mathbb{C}}} & \frac{\cos\delta^2 \cos\alpha_2 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sin\alpha_2 \mathbb{S}}{\sqrt{\mathbb{C}}} & 0 & \cos\alpha_2 & -\sin\alpha_2 \sqrt{\mathbb{C}} \\ -\frac{(-1+\cos\alpha_2)\mathbb{S}}{\mathbb{C}} & 0 & \frac{\sin\alpha_2}{\sqrt{\mathbb{C}}} & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{\sin\alpha_2 \mathbb{S}}{\sqrt{\mathbb{C}}} & -\frac{(-1+\cos\alpha_1)\mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\frac{\sin\alpha_2}{\sqrt{\mathbb{C}}} \\ 0 & 0 & \sqrt{\mathbb{C}} \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & \sin\alpha_2 \\ 0 & 0 & -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- Similar to the $e^{i\beta_2 J^1}$, but here we have $e^{i\alpha_2 J^1}$
- It seems $\mathcal{K}^{\hat{2}}$ play the role of rotation around y axis starting from the IFD to LF-zero mode with this new basis
- Kinematic operator $\mathcal{K}^{\hat{2}}$ exclusively independent of interpolation angle in the new basis

Dynamic Generator $\mathcal{D}^{\hat{1}}$ $\quad e^{i\eta_1\mathcal{D}^{\hat{1}}}$ $\quad \rho_1=\sqrt{\eta_1^2\cos 2\delta}=\sqrt{\eta_1^2\mathbb{C}}$

$$\mathcal{D}^{\hat{1}}=-K^1\cos\delta+J^2\sin\delta$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2\cosh\rho_1-\sin\delta^2}{\mathbb{C}} & \frac{\cos\delta\sinh\rho_1}{\sqrt{\mathbb{C}}} & 0 & -\frac{\sinh(\rho_1/2)^2\mathbb{S}}{\mathbb{C}} \\ \frac{\cos\delta\sinh\rho_1}{\sqrt{\mathbb{C}}} & \cosh\rho_1 & 0 & -\frac{\sin\delta\sinh\rho_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{\sinh(\rho_1/2)^2\mathbb{S}}{\mathbb{C}} & \frac{\sin\delta\sinh\rho_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2-\cosh\rho_1\sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\rho_1 & \frac{\sinh\rho_1}{\sqrt{\mathbb{C}}} & 0 & \frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}} \\ \sinh\rho_1\sqrt{\mathbb{C}} & \cosh\rho_1 & 0 & \frac{\sinh\rho_1\mathbb{S}}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\rho_1 & -\sinh\rho_1\sqrt{\mathbb{C}} & 0 & 0 \\ -\frac{\sinh\rho_1}{\sqrt{\mathbb{C}}} & \cosh\rho_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}} & -\frac{\sinh\rho_1\mathbb{S}}{\sqrt{\mathbb{C}}} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1+\frac{(-1+\cosh\rho_1)}{\mathbb{C}^2} & \frac{\sinh\rho_1}{\mathbb{C}} & 0 & -\frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} \\ \frac{\sinh\rho_1}{\mathbb{C}} & \cosh\rho_1 & 0 & -\frac{\sinh\rho_1\mathbb{S}}{\mathbb{C}} \\ 0 & 0 & 1 & 0 \\ \frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} & \frac{\sinh\rho_1\mathbb{S}}{\mathbb{C}} & 0 & 1-\frac{(-1+\cosh\rho_1)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

This structure is not independent of δ

Dynamic Generator $\mathcal{D}^{\hat{2}}$ $\quad e^{i\eta_2 \mathcal{D}^{\hat{2}}} \quad \rho_2 = \sqrt{\eta_2^2 \cos 2\delta} = \sqrt{\eta_2^2 \mathbb{C}}$

$$\mathcal{D}^{\hat{2}} = -K^2 \cos\delta - J^1 \sin\delta$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 \cosh\rho_2 - \sin\delta^2}{\mathbb{C}} & 0 & \frac{\cos\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & -\frac{\sinh(\rho_2/2)^2 \mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\cos\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & 0 & \cosh\rho_2 & -\frac{\sin\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} \\ \frac{\sinh(\rho_2/2)^2 \mathbb{S}}{\mathbb{C}} & 0 & \frac{\sin\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & \frac{\cos\delta^2 - \cosh\rho_2 \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\rho_2 & 0 & \frac{\sinh\rho_2}{\sqrt{\mathbb{C}}} & \frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \sinh\rho_2 \sqrt{\mathbb{C}} & 0 & \cosh\rho_2 & \frac{\sinh\rho_2 \mathbb{S}}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\rho_2 & 0 & -\sinh\rho_2 \sqrt{\mathbb{C}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sinh\rho_2}{\sqrt{\mathbb{C}}} & 0 & \cosh\rho_2 & 0 \\ \frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}} & 0 & -\frac{\sinh\rho_2 \mathbb{S}}{\sqrt{\mathbb{C}}} & 1 \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{(-1+\cosh\rho_2)}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2}{\mathbb{C}} & -\frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} \\ 0 & 1 & 0 & 0 \\ \frac{\sinh\rho_2}{\mathbb{C}} & 0 & \cosh\rho_2 & -\frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} \\ \frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} & 1 - \frac{(-1+\cosh\rho_2)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

This structure is not independent of δ

- <u>Special operators in the new basis</u>

$$e^{(-i\beta_z K_3)}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{P'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{-iJ_3\theta_z}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{i\beta_1\mathcal{K}^{\hat{1}}}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{i\beta_2\mathcal{K}^{\hat{2}}}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & \sin\alpha_2 \\ 0 & 0 & -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}$ are dynamic operators in all interpolation angles.

$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}$ and J_3 are kinematic operators in all interpolation angles and K^3 is a kinematic operator exactly at the LF.

We can safely say that all Kinematic operators exclusively independent of interpolation angle in the new basis

The T transformation operator (Combination of special operators)

$$T = T_{12}T_3 = e^{i\beta_1 \mathcal{K}^{\hat{1}} + i\beta_2 \mathcal{K}^{\hat{2}}} e^{-i\beta_3 K^3}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta B_1 - \sin\delta\cos\alpha B_2}{\mathbb{C}} & \frac{\beta_1 \sin\delta\sin\alpha}{\alpha} & \frac{\beta_2 \sin\delta\sin\alpha}{\alpha} & \frac{\cos\delta B_2 - \sin\delta\cos\alpha B_1}{\mathbb{C}} \\ \frac{\beta_1 \sin\alpha B_2}{\alpha} & \frac{\beta_2^2 + \beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha - 1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1 \sin\alpha B_1}{\alpha} \\ \frac{\beta_2 \sin\alpha B_2}{\alpha} & \frac{\beta_1\beta_2(\cos\alpha - 1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2 + \beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_2 \sin\alpha B_1}{\alpha} \\ \frac{\cos\delta\cos\alpha B_2 - \sin\delta B_1}{\mathbb{C}} & -\frac{\beta_1\cos\delta\sin\alpha}{\alpha} & -\frac{\beta_2\cos\delta\sin\alpha}{\alpha} & \frac{\cos\delta\cos\alpha B_1 - \sin\delta B_2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\alpha = \sqrt{(\beta_1^2+\beta_2^2)\mathbb{C}}, \quad B_1 = \sin\delta\sinh\beta_3 + \cos\delta\cosh\beta_3, \; B_2 = \sin\delta\cosh\beta_3 + \cos\delta\sinh\beta_3$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_3 + \mathbb{S}\sinh\beta_3 & 0 & 0 & -\sinh\beta_3\mathbb{C} \\ \frac{\beta_1\sin\alpha(\cosh\beta_3\mathbb{S}+\sinh\beta_3)}{\alpha} & \frac{\beta_2^2+\beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & -\frac{\beta_1\cosh\beta_3\sin\alpha\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} \\ \frac{\beta_2\sin\alpha(\cosh\beta_3\mathbb{S}+\sinh\beta_3)}{\alpha} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & -\frac{\beta_2\cosh\beta_3\sin\alpha\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} \\ \frac{\mathbb{S}(\cosh\beta_3+\mathbb{S}\sinh\beta_3)-\cos\alpha(\cosh\beta_3\mathbb{S}+\sinh\beta_3)}{\mathbb{C}} & \frac{\beta_1\sin\alpha}{\alpha} & \frac{\beta_2\sin\alpha}{\alpha} & \cos\alpha\cosh\beta_3 - \sinh\beta_3\mathbb{S} \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$\begin{pmatrix} x'_{\hat{+}} \\ x'_{\hat{1}} \\ x'_{\hat{2}} \\ x'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_3 - \cos\alpha\mathbb{S}\sinh\beta_3 & -\frac{\beta_1\sin\alpha\mathbb{S}}{\alpha} & -\frac{\beta_2\sin\alpha\mathbb{S}}{\alpha} & \frac{(\cosh\beta_3\mathbb{S}+\sinh\beta_3)-\cos\alpha\mathbb{S}(\cosh\beta_3+\mathbb{S}\sinh\beta_3)}{\mathbb{C}} \\ -\frac{\beta_1\sin\alpha\sinh\beta_3\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_2^2+\beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & -\frac{\beta_1\sin\alpha(\cosh\beta_3+\mathbb{S}\sinh\beta_3)}{\alpha} \\ -\frac{\beta_2\sin\alpha\sinh\beta_3\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & -\frac{\beta_2\sin\alpha(\cosh\beta_3+\mathbb{S}\sinh\beta_3)}{\alpha} \\ \cos\alpha\sinh\beta_3\mathbb{C} & \frac{\beta_1\sin\alpha\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_2\sin\alpha\sqrt{\mathbb{C}}}{\sqrt{\beta_1^2+\beta_2^2}} & \cos\alpha(\cosh\beta_3 + \mathbb{S}\sinh\beta_3) \end{pmatrix} \begin{pmatrix} x_{\hat{+}} \\ x_{\hat{1}} \\ x_{\hat{2}} \\ x_{\hat{-}} \end{pmatrix}$$

$$
\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}
=
\begin{pmatrix}
\cosh\beta_3 & 0 & 0 & \sinh\beta_3 \\
\frac{\beta_1 \sin\alpha \sinh\beta_3}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_2^2+\beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1\cosh\beta_3\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} \\
\frac{\beta_2 \sin\alpha \sinh\beta_3}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_2\cosh\beta_3\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} \\
\cos\alpha\sinh\beta_3 & -\frac{\beta_1\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} & -\frac{\beta_2\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} & \cos\alpha\cosh\beta_3
\end{pmatrix}
\begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}
$$

$$
\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}
=
\begin{pmatrix}
\frac{\cos\delta B_1-\sin\delta\cos\alpha B_2}{\mathbb{C}} & \frac{\beta_1\sin\delta\sin\alpha}{\alpha} & \frac{\beta_2\sin\delta\sin\alpha}{\alpha} & \frac{\cos\delta B_2-\sin\delta\cos\alpha B_1}{\mathbb{C}} \\
\frac{\beta_1\sin\alpha B_2}{\alpha} & \frac{\beta_2^2+\beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1\sin\alpha B_1}{\alpha} \\
\frac{\beta_2\sin\alpha B_2}{\alpha} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_2\sin\alpha B_1}{\alpha} \\
\frac{\cos\delta\cos\alpha B_2-\sin\delta B_1}{\mathbb{C}} & -\frac{\beta_1\cos\delta\sin\alpha}{\alpha} & -\frac{\beta_2\cos\delta\sin\alpha}{\alpha} & \frac{\cos\delta\cos\alpha B_1-\sin\delta B_2}{\mathbb{C}}
\end{pmatrix}
\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}
$$

$$
\alpha = \sqrt{(\beta_1^2+\beta_2^2)\mathbb{C}}, \quad B_1 = \sin\delta\sinh\beta_3 + \cos\delta\cosh\beta_3, \; B_2 = \sin\delta\cosh\beta_3 + \cos\delta\sinh\beta_3
$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_3 & 0 & 0 & \sinh\beta_3 \\ \frac{\alpha_1 \sin\sqrt{\alpha_1^2+\alpha_2^2}\sinh\beta_3}{\sqrt{\alpha_1^2+\alpha_2^2}} & \frac{\alpha_2^2+\alpha_1^2\cos\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1^2+\alpha_2^2} & \frac{\alpha_1\alpha_2(\cos\sqrt{\alpha_1^2+\alpha_2^2}-1)}{\alpha_1^2+\alpha_2^2} & \frac{\alpha_1\cosh\beta_3\sin\sqrt{\alpha_1^2+\alpha_2^2}}{\sqrt{\alpha_1^2+\alpha_2^2}} \\ \frac{\alpha_2 \sin\sqrt{\alpha_1^2+\alpha_2^2}\sinh\beta_3}{\sqrt{\alpha_1^2+\alpha_2^2}} & \frac{\alpha_1\alpha_2(\cos\sqrt{\alpha_1^2+\alpha_2^2}-1)}{\alpha_1^2+\alpha_2^2} & \frac{\alpha_1^2+\alpha_2^2\cos\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1^2+\alpha_2^2} & \frac{\alpha_2\cosh\beta_3\sin\sqrt{\alpha_1^2+\alpha_2^2}}{\sqrt{\alpha_1^2+\alpha_2^2}} \\ \cos\sqrt{\alpha_1^2+\alpha_2^2}\sinh\beta_3 & -\frac{\alpha_1\sin\sqrt{\alpha_1^2+\alpha_2^2}}{\sqrt{\alpha_1^2+\alpha_2^2}} & -\frac{\alpha_2\sin\sqrt{\alpha_1^2+\alpha_2^2}}{\sqrt{\alpha_1^2+\alpha_2^2}} & \cos\sqrt{\alpha_1^2+\alpha_2^2}\cosh\beta_3 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$\alpha_1 = \sqrt{\beta_1^2\cos 2\delta} = \sqrt{\beta_1^2\mathbb{C}} \qquad \alpha_2 = \sqrt{\beta_1^2\cos 2\delta} = \sqrt{\beta_2^2\mathbb{C}}$$

Conclusion

- We can prove the invariant of space time interval in all transformed (using rotation and boost or combination of them) four vectors in space-time in all the considered basis independently interpolation angle.

$$s^2 = x^{\mu}x_{\mu} = x'^{\mu}x'_{\mu} = x^{\hat{\mu}}x_{\hat{\mu}} = x'^{\hat{\mu}}x'_{\hat{\mu}}$$

$$= \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (x^1)^2 - (x^2)^2 - \left(\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 = \left(\frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (x'^1)^2 - (x'^2)^2 - \left(\frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2$$

- Momentum space, For the particle of mass M, $P^{\mu}P_{\mu}$ on the mass shell is equal to M^2

$$M^2 = P^{\mu}P_{\mu} = P'^{\mu}P'_{\mu} = P^{\hat{\mu}}P_{\hat{\mu}} = P'^{\hat{\mu}}P'_{\hat{\mu}}$$

$$= \left(\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (P^1)^2 - (P^2)^2 - \left(\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 = \left(\frac{P'^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (P'^1)^2 - (P'^2)^2 - \left(\frac{P'_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2$$

- We prove that the new basis gives the unique structure for some generators.

Future work

- Try to find more application in the new basis and understand physical interpretation behind all the generators in the new basis

Thank you