Interpolating Lorentz force equation and solution between the Instant Form Dynamics and Light-Front Dynamics

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Light Cone

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Lorentz Force Equation

$$m\frac{d\,U^{\mu}(\tau)}{d\,\tau} = qF^{\mu\nu}U_{\nu}(\tau)$$

Electromagnetic Field Strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ Ex & 0 & -Bz & By \\ Ey & Bz & 0 & -Bx \\ Ez & -By & Bx & 0 \end{pmatrix}$$

Correspondence between

Electric Fields and Boost $E \leftrightarrow K$
Magnetic Fields and Rotation $\mathrm{B} \leftrightarrow -J$

Poincare Matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

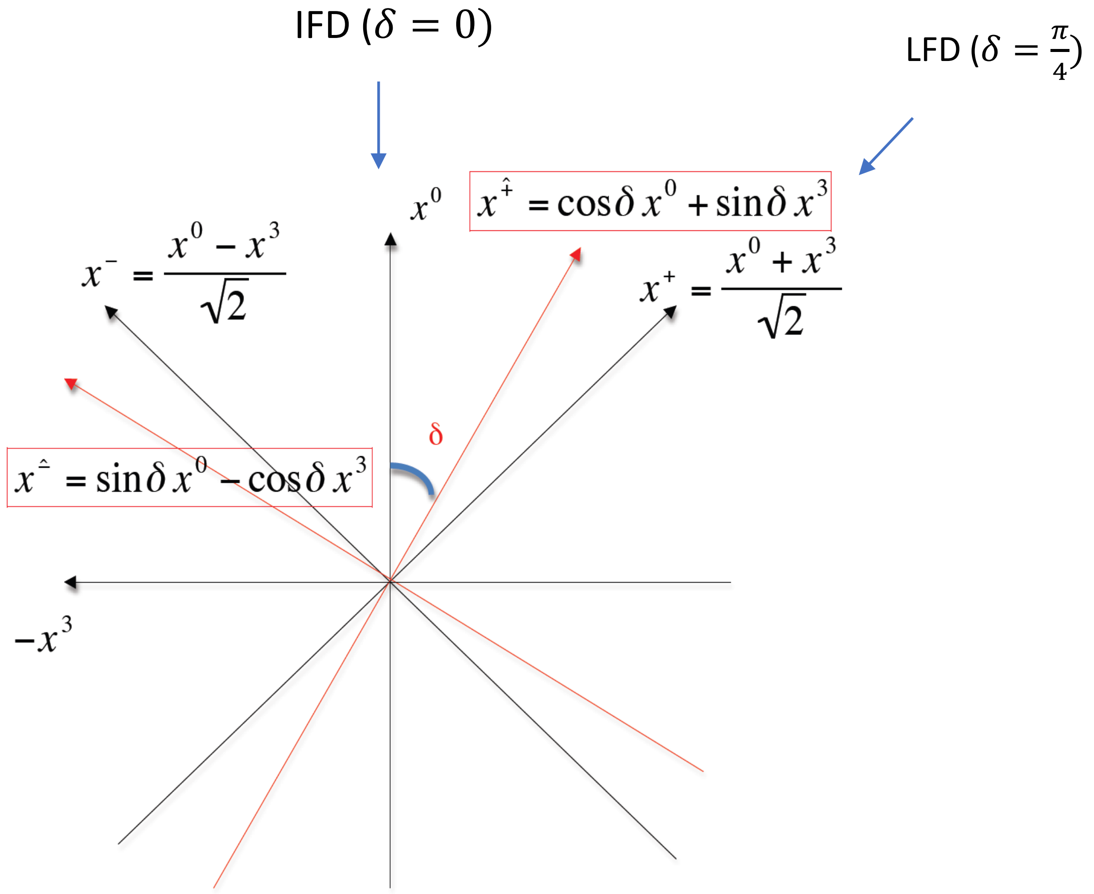
- In the Instant form dynamic

  J^1, J^2, J^3 -> Kinematic Operators
  K^1, K^2, K^3 -> Dynamic Operators

- In the Light-form dynamic

  J^1, J^2, J^3 , K^3 -> Kinematic Operators
  $K^1, K^2,$ -> Dynamic Operators

Interpolating Dynamic



Interpolating transformation matrix

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} cos(\delta) & 0 & 0 & \sin(\delta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\delta) & 0 & 0 & -cos(\delta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolation space time matrix

$$g_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{bmatrix}$$

K. Hornsbostel, Phys. Rev. D **45**, 3781 (1992)

C.-R. Ji, Z. Li, A.T Suzuki and B. Ma Phys. Rev. D **98**, 036017 (2018)

B. Ma and C.-R. Ji , Phys. Rev. D **104**, 036004 (2021)

- Connect two relativistic dynamics, proposed by Dirac

P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949)

Interpolating Poincare Matrix

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^{\hat{1}} \sin\delta - J^{\hat{2}} \cos\delta, \\ \mathcal{K}^{\hat{2}} &= J^{\hat{1}} \cos\delta - K^{\hat{2}} \sin\delta, \\ \mathcal{D}^{\hat{1}} &= -K^{\hat{1}} \cos\delta + J^{\hat{2}} \sin\delta, \\ \mathcal{D}^{\hat{2}} &= -J^{\hat{1}} \sin\delta - K^{\hat{2}} \cos\delta. \end{aligned}$$

TABLE I. Kinematic and dynamic generators for different interpolation angles

	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \le \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P^+$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P^-$

- Among the ten Poincare generators, the six generators are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformation generated by them. $\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}}$

- Light-Front dynamics (LFD) has one more kinematic operator than the Instant Form dynamic (IFD).

K^3

Interpolating Electromagnetic Field Strength Tensor

$$F^{\hat{\mu}\,\hat{\nu}} = \begin{pmatrix} 0 & -ExCos[\delta] - BySin[\delta] & -EyCos[\delta] + BxSin[\delta] & Ez \\ ExCos[\delta] + BySin[\delta] & 0 & -Bz & -ByCos[\delta] + ExSin[\delta] \\ EyCos[\delta] - BxSin[\delta] & Bz & 0 & BxCos[\delta] + EySin[\delta] \\ -Ez & ByCos[\delta] - ExSin[\delta] & -BxCos[\delta] - EySin[\delta] & 0 \end{pmatrix}$$

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^{3} \\ -\mathcal{D}^{\hat{1}} & 0 & J^{3} & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^{3} & 0 & -\mathcal{K}^{\hat{2}} \\ -K^{3} & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

- Since kinematic operators leave the time-invariant, their usage is beneficial in describing the characteristics of the motion with a simpler time-variant expression.

Kinematic

Dynamic

Interpolating Lorentz force equation.

$$m\frac{d\,U^{\hat{\mu}}(\tau)}{d\,\tau} = qF^{\hat{\mu}\hat{\nu}}U_{\hat{\nu}}(\tau)$$

$$\begin{pmatrix} \dot{u}^{\hat{+}} \\ \dot{u}^{\hat{1}} \\ \dot{u}^{\hat{2}} \\ \dot{u}^{\hat{-}} \end{pmatrix} = \frac{q}{m}\begin{pmatrix} 0 & -ExCos[\delta]-BySin[\delta] & -EyCos[\delta]+BxSin[\delta] & Ez \\ ExCos[\delta]+BySin[\delta] & 0 & -Bz & -ByCos[\delta]+ExSin[\delta] \\ EyCos[\delta]-BxSin[\delta] & Bz & 0 & BxCos[\delta]+EySin[\delta] \\ -Ez & ByCos[\delta]-ExSin[\delta] & -BxCos[\delta]-EySin[\delta] & 0 \end{pmatrix}\begin{pmatrix} u_{\hat{+}} \\ u_{\hat{1}} \\ u_{\hat{2}} \\ u_{\hat{-}} \end{pmatrix}$$

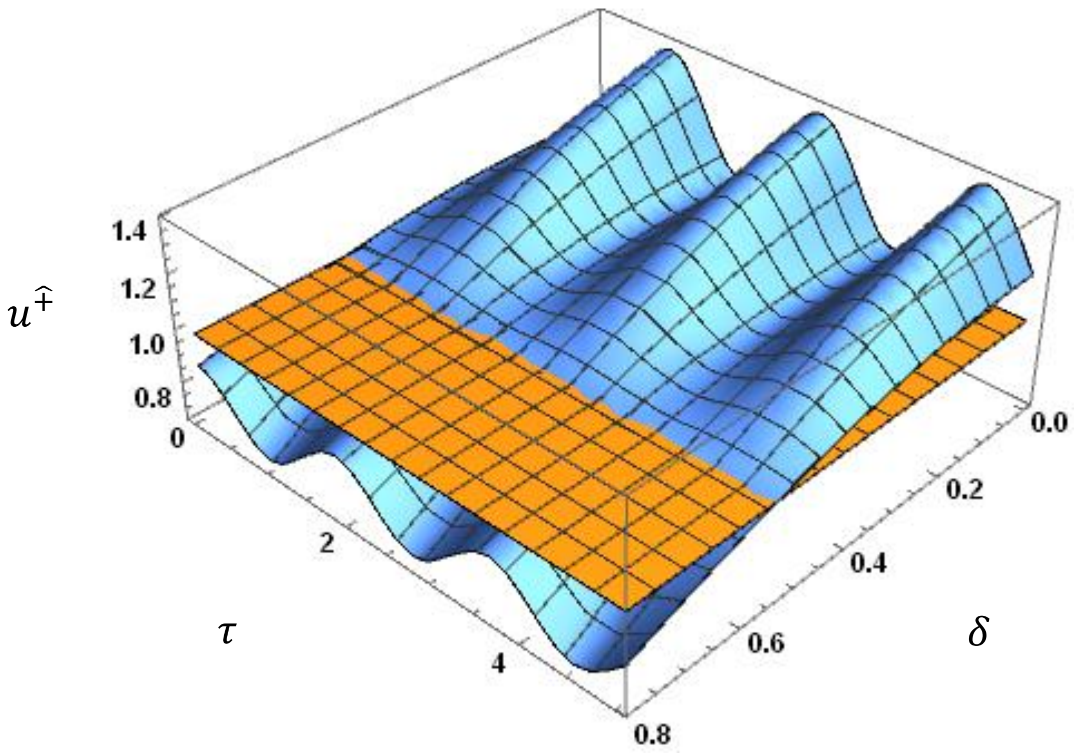
- We convert the field analogous to the dynamic generators into kinematic, using the special choice of interpolation angle between the fields.

$$Tan[\delta] = -\frac{Ex}{By} = \frac{Ey}{Bx}, \qquad Ez = 0$$

$$\dot{u}^{\hat{+}}(\tau) = 0 \qquad x^{\hat{+}}(\tau) = U^{\hat{+}}(0)\tau$$

← Direct connection between interpolating time and the proper time

$U^{\hat{\mp}}(\tau) = u^0(\tau)\,\mathrm{Cos}[\delta] + u^3(\tau)\,\mathrm{Sin}[\delta]$



$u^{\hat{\mp}}(0)$=0.9859

$x^{\hat{\mp}}(\tau) = 0.9859\tau$

Time-invariant plane.

$Ex = -2\,Tan[\pi/6]\,,\ Ey = Tan[\pi/6]\,,\ Ez = 0$

$Bx = 1,\ By = 2,\ Bz = 3,\ vz = 0.2\,,\ q = 1,\ m = 1$

Example problem

Electric field --> x, Magnetic Field --> y , Velocity --> z

$$F^{\hat{\mu}\,\hat{\nu}} = \begin{pmatrix} 0 & -\mathrm{ExCos}[\delta] - \mathrm{BySin}[\delta] & 0 & 0 \\ \mathrm{ExCos}[\delta] + \mathrm{BySin}[\delta] & 0 & 0 & -\mathrm{ByCos}[\delta] + \mathrm{ExSin}[\delta] \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{ByCos}[\delta] - \mathrm{ExSin}[\delta] & 0 & 0 \end{pmatrix}$$

Field analogous to dynamic operators

$$\mathrm{Ex} = -\mathrm{By}\,Tan[\delta]$$

$\tau = \frac{x^{\hat{+}}}{U^{\hat{+}}(0)}$

Interpolating space - coordinates as functions of interpolating time

$$x^{\hat{1}}\left(x^{\hat{+}}\right) = -\left(\frac{m\ Cos[\delta]}{q\ ByCos[2\delta]}\right)\left(Cos[2\delta]\ U^{\hat{-}}(0) - Sin[2\delta]U^{\hat{+}}(0)\right)\left(Cos\left[\frac{q\ By\ \sqrt{Cos[2\delta]}x^{\hat{+}}}{m\ Cos[\delta]U^{\hat{+}}(0)}\right] - 1\right)$$

$$x^{\hat{2}}\left(x^{\hat{+}}\right) = 0$$

$$x^{\hat{-}}\left(x^{\hat{+}}\right) = \frac{Sin[2\delta]}{Cos[2\delta]}x^{\hat{+}} + \left[\frac{Cos[2\delta]\ U^{\hat{-}}(0) - Sin[2\delta]U^{\hat{+}}(0)}{Cos[2\delta]}\right]\left(\frac{m\ Cos[\delta]}{q\ By\sqrt{Cos[2\delta]}}\right)Sin\left[\frac{q\ By\ \sqrt{Cos[2\delta]}x^{\hat{+}}}{m\ Cos[\delta]U^{\hat{+}}(0)}\right]$$

- We also can find the general space-time coordinates as functions of proper time

$$t(\tau) = x^{\hat{+}}(\tau)Cos\delta + x^{\hat{-}}(\tau)Sin\delta\,, \qquad x(\tau) = x^{\hat{1}}(\tau), \qquad y(\tau) = x^{\hat{2}}(\tau), \qquad z(\tau) = x^{\hat{+}}(\tau)Sin\delta - x^{\hat{-}}(\tau)Cos\delta$$

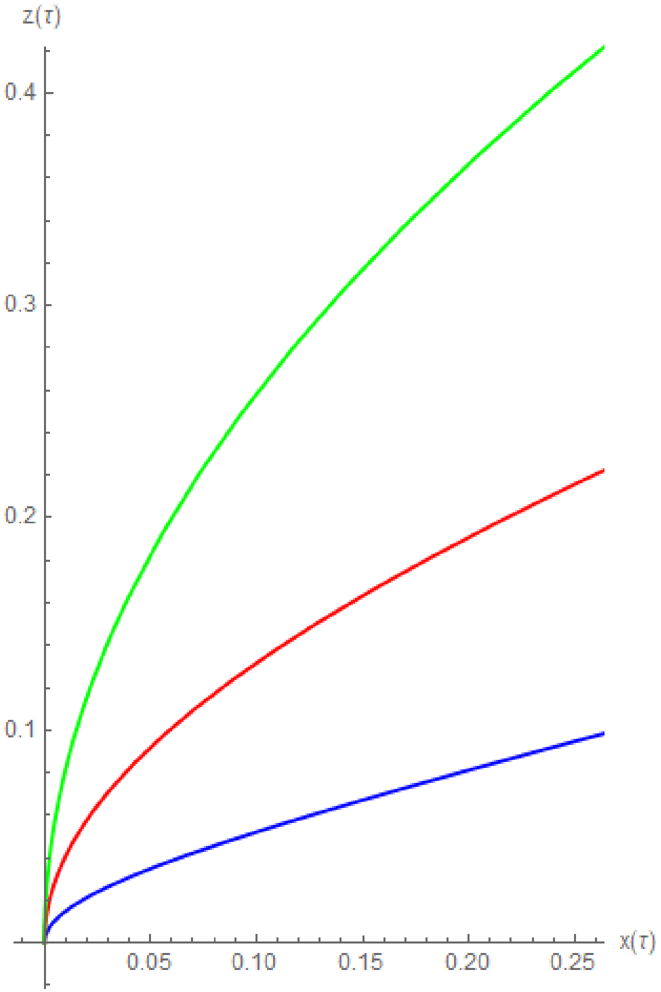
$$Sin(\delta) = \frac{-Ex}{\sqrt{E_x^2 + B_y^2}} \qquad Cos(\delta) = \frac{By}{\sqrt{E_x^2 + B_y^2}}$$

$$t(\tau) = \frac{qBy\sqrt{B_y^2 - E_x^2}\,(By\,u_t\,(0) - Ex\,u_z\,(0))\tau - Ex\,m\,\left(Ex\,u_t\,(0) - By\,u_z\,(0)\right)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;q\tau}{m}\right]}{q\left(B_y^2 - E_x^2\right)^{3/2}}$$

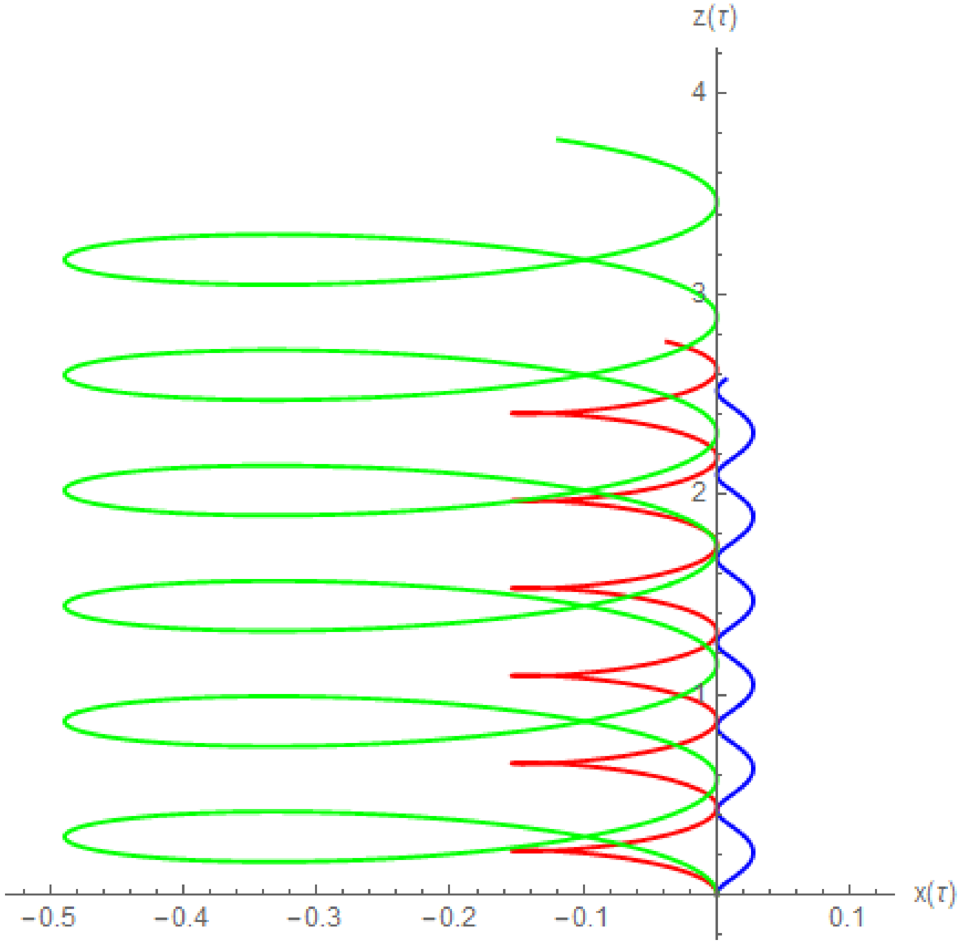
$$x(\tau) = \frac{2m\,\left(Ex\,u_t\,(0) - Byu_z\,(0)\right)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;q\tau}{2m}\right]^2}{\left(B_y^2 - E_x^2\right)q} \qquad y(\tau) = 0$$

$$z(\tau) = \frac{qEx\sqrt{B_y^4 - E_x^4}\,(By\,u_t\,(0) - Ex\,u_z\,(0))\tau + By\sqrt{B_y^2 - E_x^2}\,m\,\left(-Ex\,u_t\,(0) + By\,u_z\,(0)\right)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;q\tau}{m}\right]}{q\left(B_y^2 - E_x^2\right)^{3/2}\sqrt{B_y^2 - E_x^2}}$$

2D Trajectories



Ex =4 , By= 1 $q = 1, m = 1$



Ex =1 , By= 4 $q = 1, m = 1$

Blue → vz=0.2 , Red → vz= 0.5, Green → vz=0.8

Light-Front dynamic solution.

$$F^{\mu\nu}{}_{LF} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -Ex-By & -Ey+Bx & \sqrt{2}\,Ez \\ Ex+By & 0 & -\sqrt{2}Bz & -By+Ex \\ Ey-Bx & \sqrt{2}\,Bz & 0 & Bx+Ey \\ -\sqrt{2}\,Ez & By-Ex & -Bx-Ey & 0 \end{pmatrix}$$

We remove field analogous to dynamic operators

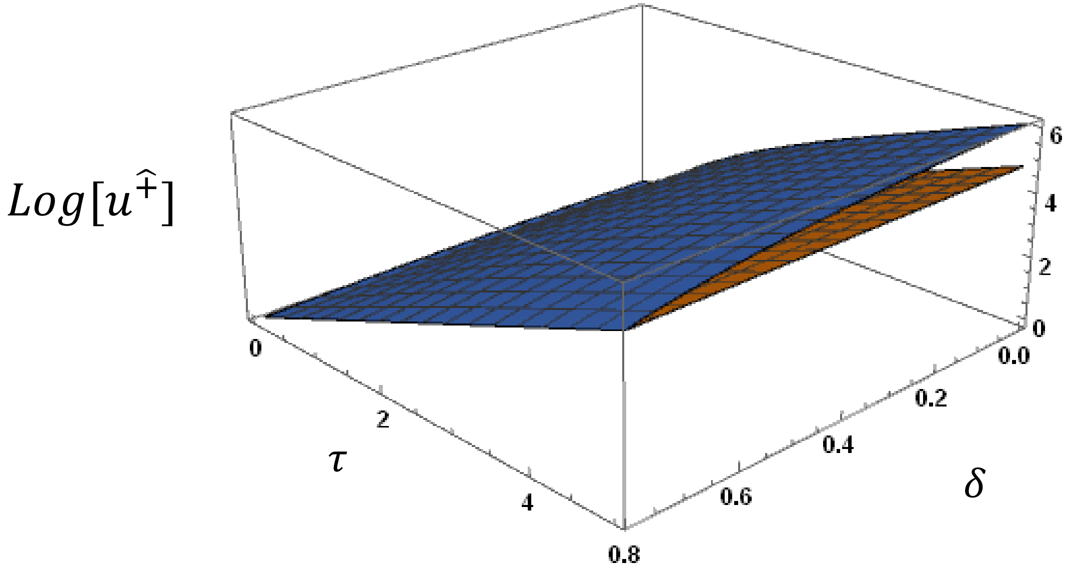
$$Ex = -By, \qquad Ey = Bx$$

- At the Light-Front $\delta = \frac{\pi}{4}$, K^3 becomes kinematic,

Field analogous to K^3 is Ez

$$U^+(\tau) = U^+(0)e^{\frac{Ez\,q}{m}\tau}$$

$$x^+(\tau) = \frac{m}{Ez\,q}U^+(0)(e^{\frac{Ez\,q}{m}\tau}-1)$$



$Bx = 0.5, By = 2, Bz = 0.2, vz = 0.2, q = 1, m = 1, \qquad Ex = -2, Ey = 0.5, Ez = 1$

Light-Front Space coordinates as functions of Light-Front time

$$\tau = \frac{m}{Ez\, q}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]$$

$$x^{1}(x^{+}) = -\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(By\, Bz + Bx\, Ez)mU^{+}(0)\left(1 - Cos\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]\right]\right)\right]$$
$$+\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(Bx\, Bz - By\, Ez)\left(Bz\, qx^{+} - mU^{+}(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]\right]\right)\right]$$

$$x^{2}(x^{+}) = \frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(Bx\, Bz - By\, Ez)mU^{+}(0)\left(1 - Cos\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]\right]\right)\right]$$
$$+\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(By\, Bz + Bx\, Ez)\left(Bz\, qx^{+} - mU^{+}(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]\right]\right)\right]$$

$$x^{-}(x^{+}) = \frac{mU^{-}(0)x^{+}}{(mU^{+}(0) + Ezqx^{+})} + \frac{(Bx^2 + By^2)}{(Bz^2 + Ez^2)}\left(\frac{(2mU^{+}(0) + Ezqx^{+})x^{+}}{(mU^{+}(0) + Ezqx^{+})} - \frac{2mU^{+}(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^{+}}{U^{+}(0)m}]\right]}{Bzq}\right)$$

- We also can find the general space-time coordinates as functions of proper time

$$t(\tau) = \frac{x^{+}(\tau)+x^{-}(\tau)}{\sqrt{2}}, x(\tau) = x^{1}(\tau), y(\tau) = x^{2}(\tau), z(\tau) = \frac{x^{+}(\tau)-x^{-}(\tau)}{\sqrt{2}}$$

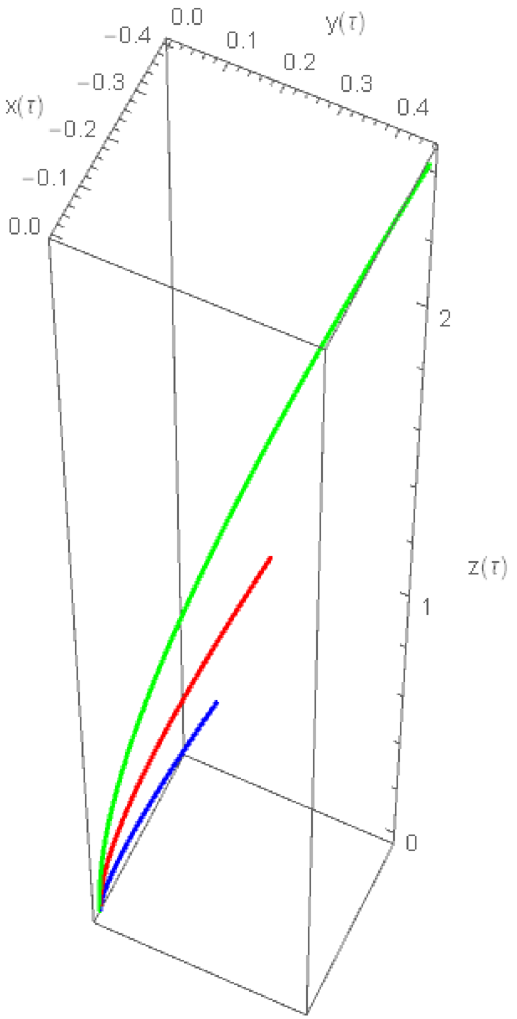
$$t(\tau) = \frac{m}{Ez\,q}\left(u_t(0) - u_z(0)\left[1 - Cosh\left(\frac{Ez\,q\,\tau}{m}\right)\right]\right) + \frac{m(u_t(0) + u_z(0)))}{Bz\,Ez\,q\,(B_z^2 + E_z^2)}\left(Bz\left[B_x^2 + B_y^2 Sinh\left(\frac{Ez\,q\,\tau}{m}\right))\right] - Ez(B_x^2 + B_y^2)\,Sin\left(\frac{Ez\,q\,\tau}{m}\right)\right)$$

$$x(\tau) = -\frac{Bx\,m(u_t(0)+u_z(0))}{BzEzq} + \frac{e^{\frac{Ezq\tau}{m}}(BxBz-ByEz)m(u_t(0)+u_z(0))}{Ez(Bz^2+Ez^2)q} + \frac{(ByBz+BxEz)m(u_t(0) + u_z(0))Cos\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q} - \frac{(BxBz-ByEz)m(u_t(0) + u_z(0))Sin\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q}$$

$$y(\tau) = -\frac{Bym(u_t(0) + u_z(0))}{BzEzq} + \frac{e^{\frac{Ezq\tau}{m}}(ByBz+BxEz)m(u_t(0) + u_z(0))}{Ez(Bz^2+Ez^2)q} - \frac{(BxBz-ByEz)m(u_t(0) + u_z(0))Cos\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q} - \frac{(ByBz+BxEz)m(u_t(0) + u_z(0))Sin\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q}$$

$$z(\tau) = \frac{m}{Ez\,q}\left(u_z(0) - u_t(0)\left[1 - Cosh\left(\frac{Ez\,q\,\tau}{m}\right)\right]\right) - \frac{m(u_t(0) + u_z(0)))}{Bz\,Ez\,q\,(B_z^2 + E_z^2)}\left(Bz\left[B_x^2 + B_y^2 Sinh\left(\frac{Ez\,q\,\tau}{m}\right))\right] - Ez(B_x^2 + B_y^2)\,Sin\left(\frac{Ez\,q\,\tau}{m}\right)\right)$$

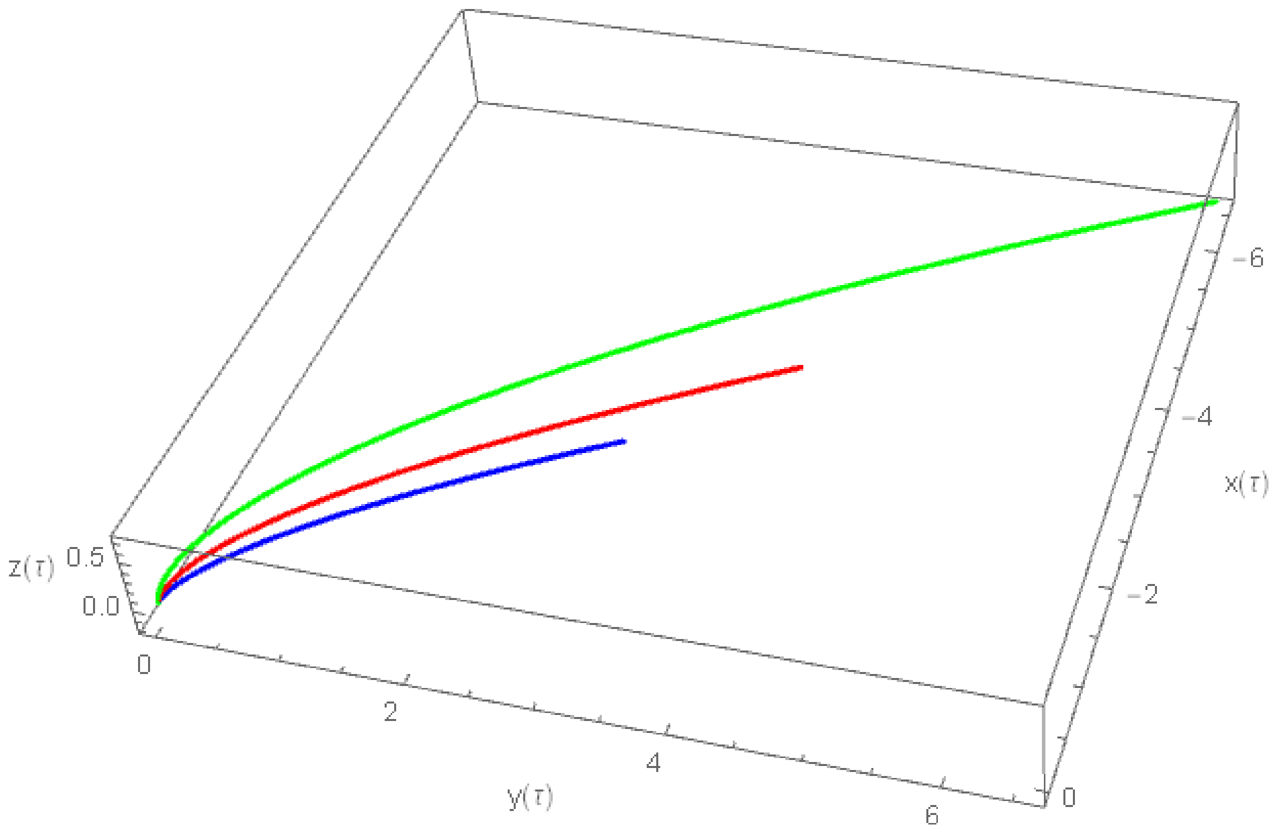
3D Trajectories



$Ez = 1\,, Ex = -0.2\,, Ey = 0.2$

$Bx = By = Bz = 0.2$

$q = 1, m = 1$



$Ez = 2\,, Ex = -2\,, Ey = 2$

$Bx = 2, By = 2\,, Bz = 0$

$q = 1, m = 1$

Blue → vz=0.2 , Red → vz= 0.5, Green → vz=0.8

Conclusion

- An alternative method to solve the equation of motion of a charged particle in a relativistic electromagnetic field by using an interpolating angle.
- This method can effectively gauge the effect of the kinematic generators saving dynamical efforts in solving the Lorentz force equation.

Future work.

- We are looking forward to going beyond the constant electromagnetic field cases.

Thank You.