# The advantages of interpolating dynamic in solving the electromagnetic field equations.

Group Meeting

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# Outline

- Motivation.
- General Solution of covariant Lorentz force equation
    - -Special Cases.
- Solutions for interpolating Lorentz force equation
    - -Example Problem
- Instant Form dynamic Solutions as function of Euclidian time
- Light-Front dynamic Solutions as function of Light-Front time

# Lorentz Force Equation

$$m\frac{d\,U^{\mu}(\tau)}{d\,\tau} = qF^{\mu\nu}U_{\nu}(\tau)$$

## Electromagnetic Field Strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ Ex & 0 & -Bz & By \\ Ey & Bz & 0 & -Bx \\ Ez & -By & Bx & 0 \end{pmatrix}$$

Correspondence between

Electric Fields and Boost $E \leftrightarrow K$
Magnetic Fields and Rotation $\mathrm{B} \leftrightarrow -J$

## Poincare Matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

- In the Instant form dynamic

  $J^1, J^2, J^3$ -> Kinematic Operators
  $K^1, K^2, K^3$ -> Dynamic Operators

- In the Light-form dynamic

  $J^1, J^2, J^3$ , $K^3$ -> Kinematic Operators
  $K^1, K^2$, -> Dynamic Operators

# Interpolating Poincare Matrix

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$
$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$
$$\mathcal{D}^{\hat{1}} = -K^1 \cos\delta + J^2 \sin\delta,$$
$$\mathcal{D}^{\hat{2}} = -J^1 \sin\delta - K^2 \cos\delta.$$

TABLE I. Kinematic and dynamic generators for different interpolation angles

| | Kinematic | Dynamic |
|---|---|---|
| $\delta = 0$ | $\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$ | $\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$ |
| $0 \le \delta < \pi/4$ | $\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$ | $\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$ |
| $\delta = \pi/4$ | $\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P^+$ | $\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P^-$ |

- Among the ten Poincare generators, the six generators are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformation generated by them. $\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P_1, P_2, P_{\hat{-}}$

- Light-Front dynamics (LFD) has one more kinematic operator than the Instant Form dynamic (IFD).

$K^3$

# Interpolating Electromagnetic Field Strength Tensor

$$F^{\hat{\mu}\,\hat{\nu}} = \begin{pmatrix} 0 & -ExCos[\delta] - BySin[\delta] & -EyCos[\delta] + BxSin[\delta] & Ez \\ ExCos[\delta] + BySin[\delta] & 0 & -Bz & -ByCos[\delta] + ExSin[\delta] \\ EyCos[\delta] - BxSin[\delta] & Bz & 0 & BxCos[\delta] + EySin[\delta] \\ -Ez & ByCos[\delta] - ExSin[\delta] & -BxCos[\delta] - EySin[\delta] & 0 \end{pmatrix}$$

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

Kinematic

Dynamic

- Since kinematic operators leave the time-invariant, their usage is beneficial in describing the characteristics of the motion with a simpler time-variant expression.

Interpolating Lorentz force equation.

$$m\frac{d\,U^{\hat{\mu}}(\tau)}{d\,\tau} = qF^{\hat{\mu}\hat{\nu}}U_{\hat{\nu}}(\tau)$$

# Covariant Lorentz Force Equation

$$m\frac{d\,U^{\mu}(\tau)}{d\,\tau} = qF^{\mu\nu}U_{\nu}(\tau)$$

General solution of motion in a uniform electromagnetic field

$$u^{\mu}(\tau) = \left(e^{\frac{qF\tau}{m}}\right)^{\mu}_{\ \nu} u^{\nu}(0)$$

Method-A

According to the Quantum field theory by Itzkson and Zuber

$\sigma$=Pauli matrices

For the four-vector u , introduce the 2 x 2 matrix (spinorial representation)

$$\underline{u} = u_0 I + \mathbf{u}.\sigma$$

$$\frac{d\underline{u}}{d\tau} = \frac{q}{m}\left(\frac{\mathbf{E}+i\mathbf{B}}{2}.\sigma\underline{u} + \underline{u}\frac{\mathbf{E}-i\mathbf{B}}{2}.\sigma\right)$$

After integrating

$$\underline{u}(\tau) = \exp\left(q\frac{\mathbf{E}+i\mathbf{B}}{2m}.\sigma\tau\right)\underline{u}(0)\exp\left(q\frac{\mathbf{E}-i\mathbf{B}}{2m}.\sigma\tau\right)$$

Using Mathematica

$u^0(\tau) = \frac{Tr[\underline{u}(\tau).I]}{2}$ $u^1(\tau) = \frac{Tr[\underline{u}(\tau).\sigma_1]}{2}$ $u^2(\tau) = \frac{Tr[\underline{u}(\tau).\sigma_2]}{2}$ $u^3(\tau) = \frac{Tr[\underline{u}(\tau).\sigma_3]}{2}$

- We can also calculate the space- time coordinates after integrating above equations

Manually calculating the analytical expression

$\boldsymbol{n} = \boldsymbol{E} + \boldsymbol{iB}$ $a = \frac{q}{2m}(\boldsymbol{n}^2)^{1/2}$ $\boldsymbol{n}^* = \boldsymbol{E} - \boldsymbol{iB}$ $a^* = \frac{e}{2m}(\boldsymbol{n}^{*2})^{1/2}$

$$\exp\left(q\frac{\mathbf{E}+i\mathbf{B}}{2m}.\sigma\tau\right) = \cosh(a\tau)I + \frac{\mathbf{n}.\sigma}{\sqrt{\mathbf{n}^2}}\sinh(a\tau).$$

$$\underline{u}(\tau) \quad = \quad \left(\cosh(a\tau)I + \frac{\mathbf{n}.\sigma}{\sqrt{\mathbf{n}^2}}\sinh(a\tau)\right)(u^0(0)I + \mathbf{u}(0).\sigma)\left(\cosh(a^*\tau)I + \frac{\mathbf{n}^*.\sigma}{\sqrt{\mathbf{n}^{*2}}}\sinh(a^*\tau)\right)$$

## Simplification techniques,

- The complex values inside the square toot , derived by converting the values into polar coordinate

$$x = r\cos\theta \qquad y = r\sin\theta \qquad x + iy = r\,e^{i\theta}$$

$$(x+iy)^{1/2} = r^{1/2}(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2})$$

$$\sqrt{x+iy} = \left(\frac{\sqrt{x+\sqrt{x^2+y^2}}}{\sqrt{2}} + i\,\mathrm{sgn}(y)\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{\sqrt{2}}\right)$$

$$\sqrt{\mathbf{n}^2} = \sqrt{\mathbf{E}^2-\mathbf{B}^2+2i\mathbf{E}.\mathbf{B}} = \left(\frac{\sqrt{\mathbf{E}^2-\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}} + i\frac{\sqrt{-\mathbf{E}^2+\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}}\right)$$

$$\sqrt{\mathbf{n}^{*2}} = \sqrt{\mathbf{E}^2-\mathbf{B}^2-2i\mathbf{E}.\mathbf{B}} = \left(\frac{\sqrt{\mathbf{E}^2-\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}} - i\frac{\sqrt{-\mathbf{E}^2+\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}}\right)$$

$$\sqrt{\mathbf{n}^2\mathbf{n}^{*2}} = \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}$$

- Vector identities

$$\begin{aligned}
\mathbf{n}\times\mathbf{n}^* &= (\mathbf{E}+i\mathbf{B})\times(\mathbf{E}-i\mathbf{B}) = -2i\mathbf{E}\times\mathbf{B}, \\
\mathbf{n}\times\mathbf{u}(0)\times\mathbf{n}^* &= -(\mathbf{n}^*.\mathbf{u})\mathbf{n} + (\mathbf{n}^*.\mathbf{n})\mathbf{u} \\
&= (\mathbf{E}\times\mathbf{u}(0) + i\mathbf{B}\times\mathbf{u}(0))\times(\mathbf{E}-i\mathbf{B}) = 2i\mathbf{u}(0)\times\mathbf{E}\times\mathbf{B}, \\
\mathbf{n}\times\mathbf{u}(0) &= \mathbf{E}\times\mathbf{u}(0) + i\mathbf{B}\times\mathbf{u}(0), \\
\mathbf{u}(0)\times\mathbf{n}^* &= -\mathbf{E}\times\mathbf{u}(0) + i\mathbf{B}\times\mathbf{u}(0), \\
(\mathbf{n}\times\mathbf{u}(0)).\mathbf{n}^* &= (\mathbf{n}^*\times\mathbf{n}).\mathbf{u}(0) = 2i(\mathbf{E}\times\mathbf{B}).\mathbf{u}(0).
\end{aligned}$$

Simplification techniques,

$$\sinh(a\tau)\cosh(a^*\tau) = \frac{1}{2}\left(\sinh((a+a^*)\tau) + \sinh((a-a^*)\tau)\right) = \frac{\sinh(2Re(a)\tau)}{2} + \frac{\sinh(2iIm(a)\tau)}{2}$$

$$= \frac{1}{2}\sinh\left(q\tau\sqrt{\mathbf{E}^2 - \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}/(\sqrt{2}m)\right) +$$

$$\frac{1}{2}\sin\left(q\tau\sqrt{-\mathbf{E}^2 + \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}/(\sqrt{2}m)\right),$$

- Relationship of trigonometric functions

$$\begin{aligned}
\cosh(a\tau)\cosh(a^*\tau) &= \frac{1}{2}\left(\cosh((a+a^*)\tau) + \cosh((a-a^*)\tau)\right) \\
&= \frac{1}{2}\left(\cosh(2Re(a)\tau) + \cosh(2iIm(a)\tau)\right), \\
&= \frac{1}{2}\Bigg\{\cosh\left(\frac{q\tau\sqrt{\mathbf{E}^2 - \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}m}\right) \\
&+ \cos\left(\frac{q\tau\sqrt{-\mathbf{E}^2 + \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}m}\right)\Bigg\}, \\
\sinh(a\tau)\sinh(a^*\tau) &= \frac{1}{2}\left(\cosh((a+a^*)\tau) - \cosh((a-a^*)\tau)\right) \\
&= \frac{1}{2}\left(\cosh(2Re(a)\tau) - \cosh(2iIm(a)\tau)\right), \\
&= \frac{1}{2}\Bigg\{\cosh\left(\frac{q\tau\sqrt{\mathbf{E}^2 - \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}m}\right) \\
&- \cos\left(\frac{q\tau\sqrt{-\mathbf{E}^2 + \mathbf{B}^2 + \sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2 + 4(\mathbf{E}.\mathbf{B})^2}}}{\sqrt{2}m}\right)\Bigg\}.
\end{aligned}$$

$$
\begin{aligned}
u^0(\tau) &= \frac{u^0(0)}{2}\Bigg\{ \frac{\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}+\mathbf{E}^2+\mathbf{B}^2}{\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\cosh\left(\frac{q\tau\mathbb{E}}{\sqrt{2}m}\right) \\
&+ \frac{\sqrt{(\mathbf{E}-\mathbf{B})^2+4(\mathbf{E}.\mathbf{B})^2}-\mathbf{E}^2-\mathbf{B}^2}{\sqrt{(\mathbf{E}-\mathbf{B})^2+4(\mathbf{E}.\mathbf{B})^2}}\cos\left(\frac{q\tau\mathbb{B}}{\sqrt{2}m}\right)\Bigg\} \\
&+ \frac{\mathbf{E}.\mathbf{u}(0)\mathbb{E}+\mathbf{B}.\mathbf{u}(0)\mathbb{B}}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\sinh\left(q\tau\mathbb{E}/(\sqrt{2}m)\right) \\
&+ \frac{\mathbf{E}.\mathbf{u}(0)\mathbb{B}-\mathbf{B}.\mathbf{u}(0)\mathbb{E}}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\sin\left(q\tau\mathbb{B}/(\sqrt{2}m)\right) \\
&- \frac{(\mathbf{E}\times\mathbf{B}).\mathbf{u}(0)}{\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2-4(\mathbf{E}.\mathbf{B})^2}}\left\{\cosh\left(\frac{q\tau\mathbb{E}}{\sqrt{2}m}\right)-\cos\left(\frac{q\tau\mathbb{B}}{\sqrt{2}m}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E} &= \sqrt{\mathbf{E}^2-\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}, \\
\mathbb{B} &= \sqrt{-\mathbf{E}^2+\mathbf{B}^2+\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u}(\tau) &= \frac{(\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}-(\mathbf{E}^2+\mathbf{B}^2))\mathbf{u}(0)}{2\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\cosh\left(\frac{q\tau\mathbb{E}}{\sqrt{2}m}\right)\\
&+ \frac{(\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}+(\mathbf{E}^2+\mathbf{B}^2))\mathbf{u}(0)}{2\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\cos\left(\frac{q\tau\mathbb{B}}{\sqrt{2}m}\right)\\
&+ \frac{\mathbf{E}\times\mathbf{B}u^0(0)}{\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\left\{\cosh\left(\frac{q\tau\mathbb{E}}{\sqrt{2}m}\right)-\cos\left(\frac{q\tau\mathbb{B}}{\sqrt{2}m}\right)\right\}\\
&+ \frac{\mathbf{E}u^0(0)\mathbb{E}+\mathbf{B}u^0(0)\mathbb{B}}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\sinh\left(q\tau\mathbb{E}/(\sqrt{2}m)\right)\\
&+ \frac{\mathbf{E}u^0(0)\mathbb{B}-\mathbf{B}u^0(0)\mathbb{E}}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\sin\left(q\tau\mathbb{B}/(\sqrt{2}m)\right)\\
&- \frac{1}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\left\{(\mathbf{E}\times\mathbf{u}(0))\mathbb{E}+(\mathbf{B}\times\mathbf{u}(0))\mathbb{B}\right\}\sin\left(q\tau\mathbb{B}/(\sqrt{2}m)\right)\\
&+ \frac{1}{\sqrt{2}\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\left\{(\mathbf{E}\times\mathbf{u}(0))\mathbb{B}-(\mathbf{B}\times\mathbf{u}(0))\mathbb{E}\right\}\sinh\left(q\tau\mathbb{E}/(\sqrt{2}m)\right)\\
&+ \frac{(\mathbf{E}.\mathbf{u}(0)\mathbf{E}+(\mathbf{B}.\mathbf{u}(0))\mathbf{B})}{\sqrt{(\mathbf{E}^2-\mathbf{B}^2)^2+4(\mathbf{E}.\mathbf{B})^2}}\left\{\cosh\left(\frac{q\tau\mathbb{E}}{\sqrt{2}m}\right)-\cos\left(\frac{q\tau\mathbb{B}}{\sqrt{2}m}\right)\right\}.
\end{aligned}
$$

We confirmed our general results by,

- Comparing the Mathematica results with the manually calculated results.

- Comparing the results for the special scenario mentioned in the book with the results derived from the most general results.

❑ Result of the simplified scenario shown in Itzkson and Zuber's textbook.

$$\underset{\sim}{u}(0) = I$$

Electric and magnetic field are perpendicular to each other → $\boldsymbol{E}.\boldsymbol{B}$=0

$$\underset{\sim}{u}(\tau) = \left[\cosh^2(a\tau) + \sinh^2(a\tau)\frac{\mathbf{E}^2 + \mathbf{B}^2}{\mathbf{n}^2}\right]I + \left[2\sinh(a\tau)\cosh(a\tau)\frac{\mathbf{E}}{(\mathbf{n}^2)^{1/2}} + 2\sinh^2(a\tau)\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{n}^2}\right]\cdot\boldsymbol{\sigma}$$

Analytical expressions for special cases.

$\mathbf{E} = 0$

$$u^0(\tau) = u^0(0),$$

$$\mathbf{u}(\tau) = \mathbf{u}(0)\cos\left(\frac{q|\mathbf{B}\tau}{m}\right) - \frac{(\mathbf{B}.\mathbf{u}(0))\mathbf{B}}{\mathbf{B}^2}\left(-1+\cos\left(\frac{q|\mathbf{B}\tau}{m}\right)\right) - \frac{\mathbf{B}\times\mathbf{u}(0)}{|\mathbf{B}|}\sin\left(\frac{q|\mathbf{B}|\tau}{m}\right)$$

$\mathbf{B} = 0$

$$u^0(\tau) = u^0(0)\cosh\left(\frac{q|\mathbf{E}|\tau}{m}\right) + \frac{\mathbf{E}.\mathbf{u}(0)}{|\mathbf{E}|}\sinh\left(\frac{q|\mathbf{E}|\tau}{m}\right),$$

$$\mathbf{u}(\tau) = \mathbf{u}(0) + \frac{(\mathbf{E}.\mathbf{u}(0))\mathbf{E}}{\mathbf{E}^2}\left(-1+\cosh\left(\frac{q|\mathbf{E}|\tau}{m}\right)\right) + \frac{\mathbf{E}u^0(0)}{|\mathbf{E}|}\sinh\left(\frac{q|\mathbf{E}|\tau}{m}\right)$$

It seems we will have a singularity ,when $\mathbf{E}^2 - \mathbf{B}^2 = 0$ $\mathbf{E}.\mathbf{B} = 0$

$$
\begin{aligned}
u^0(\tau) &= u^0(0)\left(1 + \frac{q^2(\mathbf{E}^2 + \mathbf{B}^2)\tau^2}{4m^2}\right) + \frac{q\mathbf{E}.\mathbf{u}(0)\tau}{m} - \frac{q^2(\mathbf{E}\times\mathbf{B}).\mathbf{u}(0)\tau^2}{2m^2}, \\
\mathbf{u}(\tau) &= \mathbf{u}(0)\left(1 - \frac{q^2(\mathbf{E}^2 + \mathbf{B}^2)\tau^2}{4m^2}\right) + \frac{q^2\mathbf{E}\times\mathbf{B}u^0(0)\tau^2}{2m^2} + \frac{q\mathbf{E}u^0(0)\tau}{m} - \frac{q\mathbf{B}\times\mathbf{u}(0)\tau}{m} \\
&+ \frac{q^2((\mathbf{E}.\mathbf{u}(0))\mathbf{E} + (\mathbf{B}.\mathbf{u}(0))\mathbf{B})\tau^2}{2m^2}
\end{aligned}
$$

After integrating, We also can easily find out the trajectories of this situation with specific initial conditions.

# Interpolating Lorentz force equation.

$$m\frac{d\,U^{\hat{\mu}}(\tau)}{d\,\tau} = qF^{\hat{\mu}\hat{\nu}}U_{\hat{\nu}}(\tau)$$

$$\begin{pmatrix} \dot{u}^{\hat{+}} \\ \dot{u}^{\hat{1}} \\ \dot{u}^{\hat{2}} \\ \dot{u}^{\hat{-}} \end{pmatrix} = \frac{q}{m}\begin{pmatrix} 0 & -ExCos[\delta] - BySin[\delta] & -EyCos[\delta] + BxSin[\delta] & Ez \\ ExCos[\delta] + BySin[\delta] & 0 & -Bz & -ByCos[\delta] + ExSin[\delta] \\ EyCos[\delta] - BxSin[\delta] & Bz & 0 & BxCos[\delta] + EySin[\delta] \\ -Ez & ByCos[\delta] - ExSin[\delta] & -BxCos[\delta] - EySin[\delta] & 0 \end{pmatrix}\begin{pmatrix} u_{\hat{+}} \\ u_{\hat{1}} \\ u_{\hat{2}} \\ u_{\hat{-}} \end{pmatrix}$$

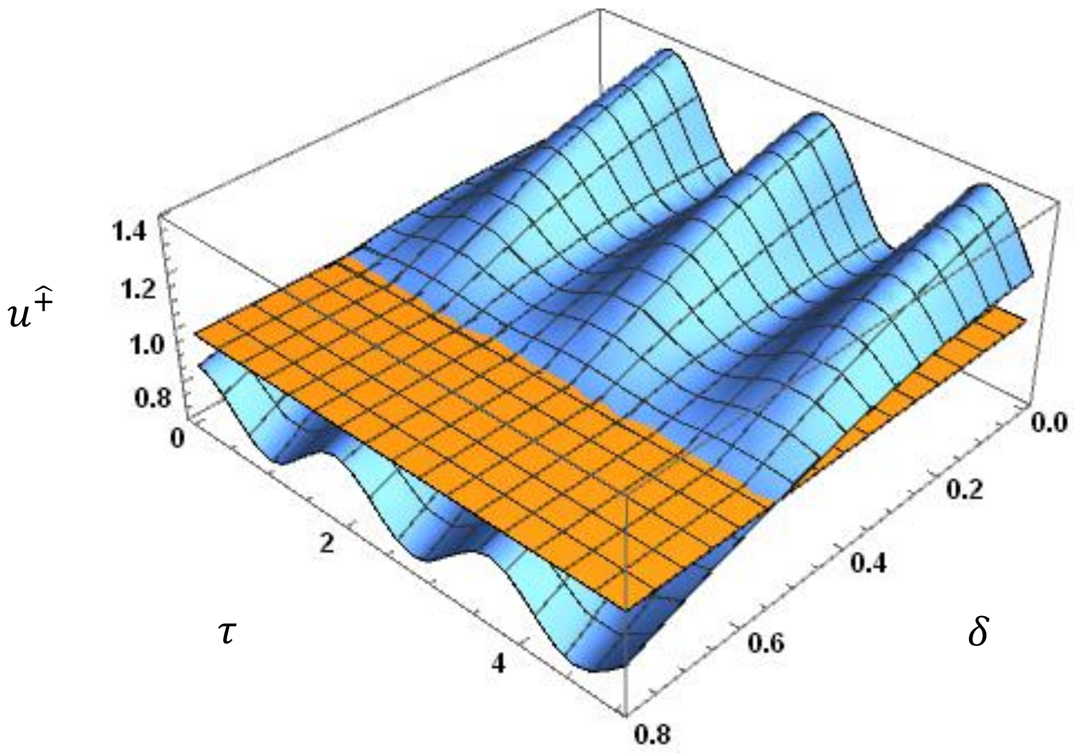
- We convert the field analogous to the dynamic generators into kinematic, using the special choice of interpolation angle between the fields.

$$Tan[\delta] = -\frac{Ex}{By} = \frac{Ey}{Bx}, \qquad Ez = 0$$

$$\dot{u}^{\hat{+}}(\tau) = 0 \qquad x^{\hat{+}}(\tau) = U^{\hat{+}}(0)\tau \quad \longleftarrow$$

Direct connection between interpolating time and the proper time

$$U^{\hat{\mp}}(\tau) = u^0(\tau)\, \mathrm{Cos}[\delta] + u^3(\tau)\, \mathrm{Sin}[\delta]$$




$u^{\hat{\mp}}(0)$=0.9859

$x^{\hat{\mp}}(\tau) = 0.9859\tau$

Time-invariant plane.

$Ex = -2\, Tan[\pi/6]\, ,\, Ey = Tan[\pi/6]\, ,\, Ez = 0$

B$x = 1,\, By = 2,\, Bz = 3,\, vz = 0.2\, ,\, q = 1,\, m = 1$

Example problem

Electric field --> x, Magnetic Field --> y , Velocity --> z

$$F^{\hat{\mu}\,\hat{\nu}} = \begin{pmatrix} 0 & -\mathrm{ExCos}[\delta] - \mathrm{BySin}[\delta] & 0 & 0 \\ \mathrm{ExCos}[\delta] + \mathrm{BySin}[\delta] & 0 & 0 & -\mathrm{ByCos}[\delta] + \mathrm{ExSin}[\delta] \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{ByCos}[\delta] - \mathrm{ExSin}[\delta] & 0 & 0 \end{pmatrix}$$

Field analogous to dynamic operators

$$\mathrm{Ex} = -\mathrm{By}\, Tan[\delta]$$

$$\tau = \frac{x^{\hat{+}}}{U^{\hat{+}}(0)}$$

Interpolating space - coordinates as functions of interpolating time

$$x^{\hat{1}}\left(x^{\hat{+}}\right) = -\left(\frac{m\,Cos[\delta]}{q\,ByCos[2\delta]}\right)\left(Cos[2\delta]\,U^{\hat{-}}(0) - Sin[2\delta]U^{\hat{+}}(0)\right)\left(Cos\left[\frac{q\,By\,\sqrt{Cos[2\delta]}x^{\hat{+}}}{m\,Cos[\delta]U^{\hat{+}}(0)}\right] - 1\right)$$

$$x^{\hat{2}}\left(x^{\hat{+}}\right) = 0$$

$$x^{\hat{-}}\left(x^{\hat{+}}\right) = \frac{Sin[2\delta]}{Cos[2\delta]}\,x^{\hat{+}} + \left[\frac{Cos[2\delta]\,U^{\hat{-}}(0) - Sin[2\delta]U^{\hat{+}}(0)}{Cos[2\delta]}\right]\left(\frac{m\,Cos[\delta]}{q\,By\sqrt{Cos[2\delta]}}\right)Sin\left[\frac{q\,By\,\sqrt{Cos[2\delta]}x^{\hat{+}}}{m\,Cos[\delta]U^{\hat{+}}(0)}\right]$$

- We also can find the general space-time coordinates as functions of proper time

$$t(\tau) = x^{\hat{+}}(\tau)Cos\delta + x^{\hat{-}}(\tau)Sin\delta\,, \qquad x(\tau) = x^{\hat{1}}(\tau), \qquad y(\tau) = x^{\hat{2}}(\tau), \qquad z(\tau) = x^{\hat{+}}(\tau)Sin\delta - x^{\hat{-}}(\tau)Cos\delta$$

$$Sin(\delta) = \frac{-Ex}{\sqrt{E_x^2 + B_y^2}} \qquad Cos(\delta) = \frac{By}{\sqrt{E_x^2 + B_y^2}}$$

$$t(\tau) = \frac{By\big(By\, u_t\,(0) - Ex\, u_z\,(0)\big)\tau}{B_y^2 - E_x^2} - \frac{Ex\, m\,\big(Ex\, u_t\,(0) - By\, u_z\,(0)\big)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;\, q\tau}{m}\right]}{q\big(B_y^2 - E_x^2\big)^{3/2}}$$

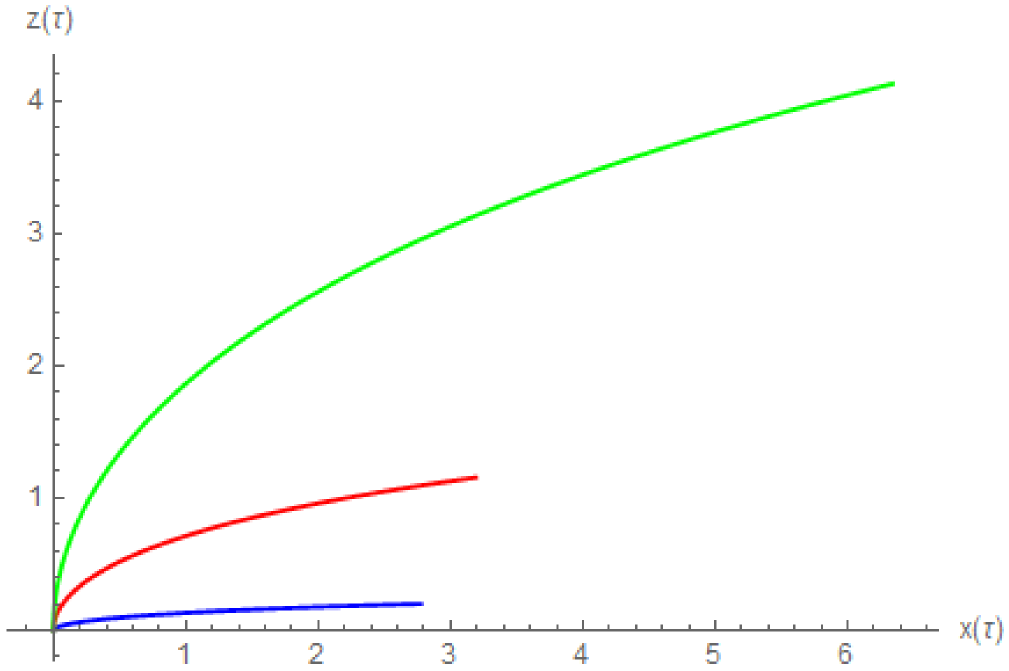
$$x(\tau) = \frac{2m\,\big(Ex\, u_t\,(0) - By u_z\,(0)\big)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;\, q\tau}{2m}\right]^2}{\big(B_y^2 - E_x^2\big)q} \qquad y(\tau) = 0$$

$$z(\tau) = \frac{qEx\big(By\, u_t\,(0) - Ex\, u_z\,(0)\big)\tau}{B_y^2 - E_x^2} + \frac{By\, m\big(-Ex\, u_t\,(0) + By\, u_z\,(0)\big)Sin\left[\frac{\sqrt{B_y^2 - E_x^2}\;\, q\tau}{m}\right]}{q\big(B_y^2 - E_x^2\big)^{3/2}}$$

- This results also can be derived using the general expressions we find previously .

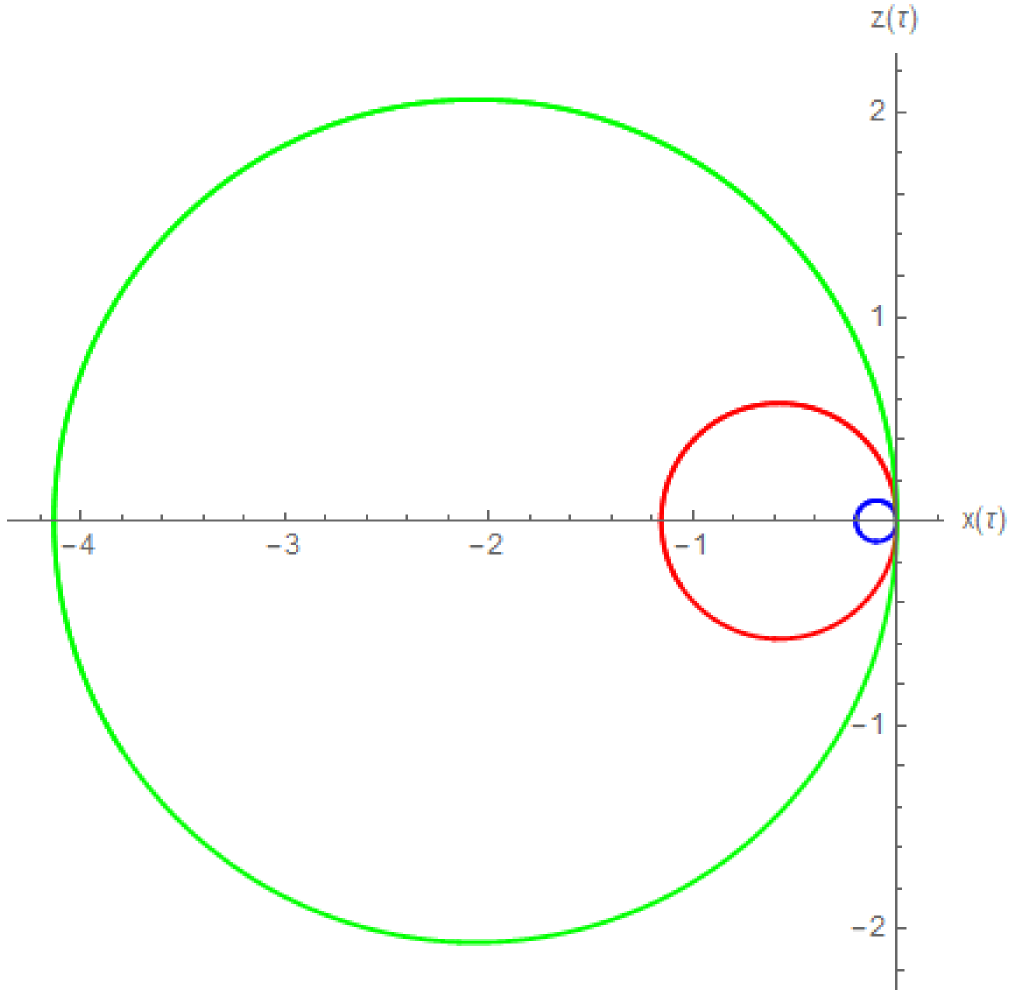
Special Cases.

2D Trajectories




Ex =1 , By= 0 $q = 1, m = 1, \tau = 2$

$$\frac{z(\tau)}{x(\tau)} = -\frac{Ex\, q\, vz\, \tau}{m * \left(1 - \text{Cosh}\left[\frac{\sqrt{Ex^2}\, q\, \tau}{m}\right]\right)}$$




Ex =0 , By= 1 $q = 1, m = 1, \tau = 10$

Blue → vz=0.1 , Red → vz= 0.5, Green → vz=0.9

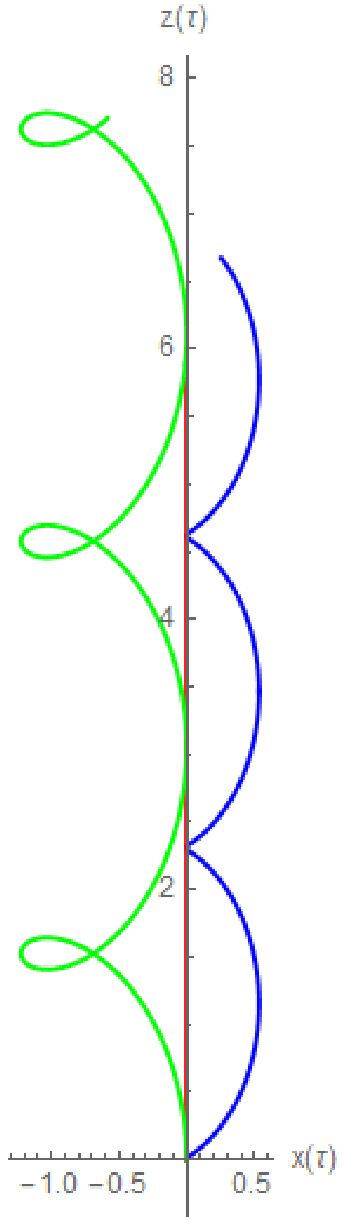
$$\frac{z(\tau)}{x(\tau)} = \frac{2\,m\,vz}{(Ex\,q - By\,q\,vz)\,\tau} - \frac{\left(q\left(-2\,By\,Ex + \left(By^2 + Ex^2\right)vz\right)\right)\tau}{6\,(m\,(Ex - By\,vz))} - \frac{\left((By - Ex)\,(By + Ex)\,q^3\left(-4\,By\,Ex + By^2\,vz + 3\,Ex^2\,vz\right)\right)\tau^3}{360\left(m^3\,(Ex - By\,vz)\right)} + O[\tau]^5$$

When $\tau \ll 1$, $\frac{z(\tau)}{x(\tau)} \approx \frac{2\,m_v}{q(E_x - B_y V_z)\,\tau}$

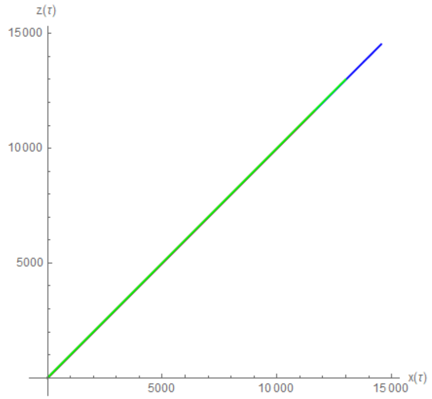
When $V_z = \frac{E_x}{B_y}$ $z(\tau)$ motion more dominant

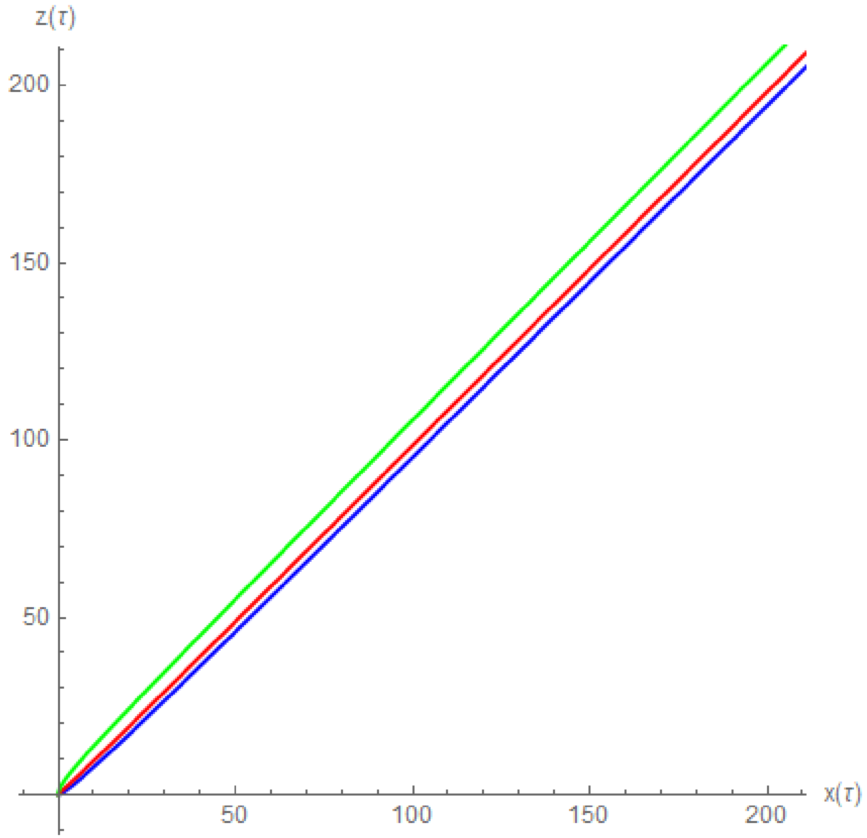
Ex =1 , By= 2 $q = 1, m = 1, \tau = 10$

Blue → vz=0.1 , Red → vz= 0.5, Green → vz=0.9



# 2D Trajectories




When $E_x = \sqrt{2}\, B_y$

$$\frac{z(\tau)}{x(\tau)} = -\frac{\text{Byq}\left(\sqrt{2} - 2\text{vz}\right)\tau}{m\left(\sqrt{2} - \text{vz}\right)\left(-1 + \text{Cosh}\left[\frac{\text{Byq}\tau}{m}\right]\right)} + \text{Coth}\left[\frac{\text{Byq}\tau}{2m}\right]$$

$$\frac{z(\tau)}{x(\tau)} |\tau \to \infty = 1$$

By= 1 $q = 1, m = 1$

Blue → vz=0.2 , Red → vz= 0.5, Green → vz=0.8

Special interpolating angles.

# Instant Form dynamic

$$(\dot{u}^{\mu}) = \frac{q}{m}\begin{pmatrix} 0 & -Ex & -Ey & Ez \\ Ex & 0 & -Bz & -By \\ Ey & Bz & 0 & Bx \\ -Ez & By & -Bx & 0 \end{pmatrix}(u_{\nu})$$

$J^1, J^2, J^3$ -> Kinematic Operators , Analogues field $Bx, By$ , $Bz$
$K^1, K^2, K^3$ -> Dynamic Operators , Analogues field E$x$, $Ey$ , $Ez$

$$\mathrm{E}x = 0, Ey = 0\,, Ez = 0$$

$$\dot{u}^0(\tau) = 0 \qquad \mathrm{t}(\tau) = U^0(0)\tau$$

Direct connection between Euclidean time and the proper time

$$U^0(0) = \frac{1}{\sqrt{1-\boldsymbol{V}^2}} = \gamma$$

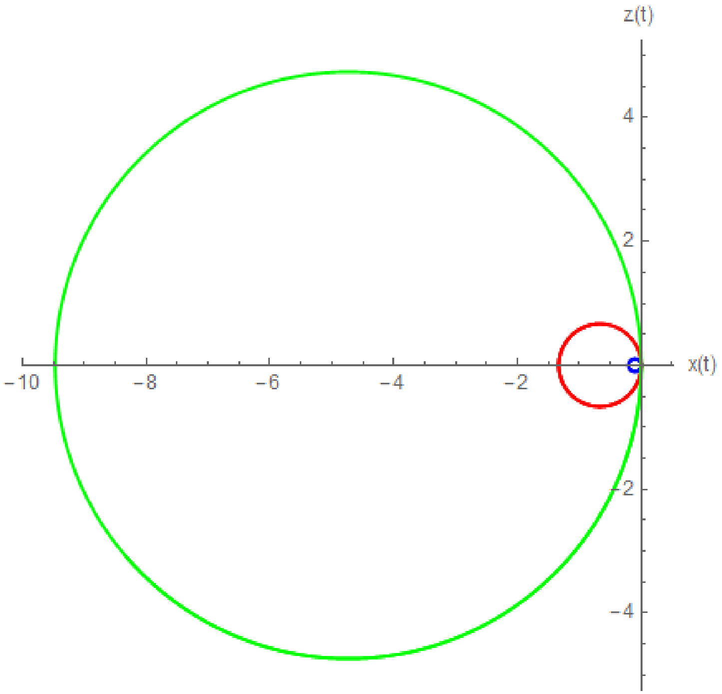
Instant form dynamic space coordinates as functions of Euclidean time

$\boldsymbol{B} = Bx^2+By^2+Bz^2$

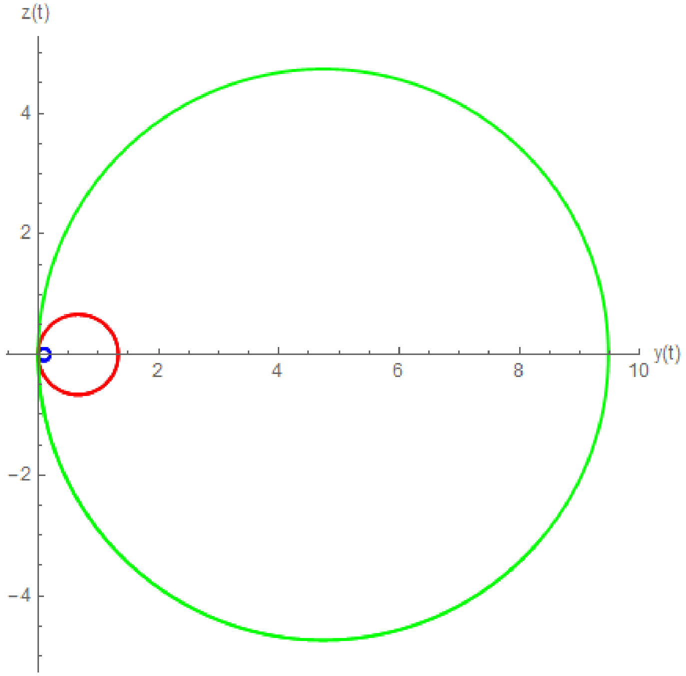
$$x(t) = \frac{\mathrm{vz}\left(-\sqrt{\boldsymbol{B}}\mathrm{BxBz}qt\sqrt{1-\mathrm{vz}^2} - \sqrt{\boldsymbol{B}}\mathrm{By}m\left(-1+\mathrm{Cos}\left[\frac{\sqrt{\boldsymbol{B}}qt\sqrt{1-\mathrm{vz}^2}}{m}\right]\right) + \mathrm{BxBz}m\mathrm{Sin}\left[\frac{\sqrt{\boldsymbol{B}}qt\sqrt{1-\mathrm{vz}^2}}{m}\right]\right)}{\boldsymbol{B}^{3/2}q(-1+\mathrm{vz}^2)}$$

$$y(t) = \frac{\mathrm{vz}\left(-\sqrt{\boldsymbol{B}}\left(\mathrm{Bx}m + \mathrm{ByBz}qt\sqrt{1-\mathrm{vz}^2}\right) + \sqrt{\boldsymbol{B}}\mathrm{Bx}m\mathrm{Cos}\left[\frac{\sqrt{\boldsymbol{B}}qt\sqrt{1-\mathrm{vz}^2}}{m}\right] + \mathrm{ByBz}m\mathrm{Sin}\left[\frac{\sqrt{\boldsymbol{B}}qt\sqrt{1-\mathrm{vz}^2}}{m}\right]\right)}{\boldsymbol{B}^{3/2}q(-1+\mathrm{vz}^2)}$$

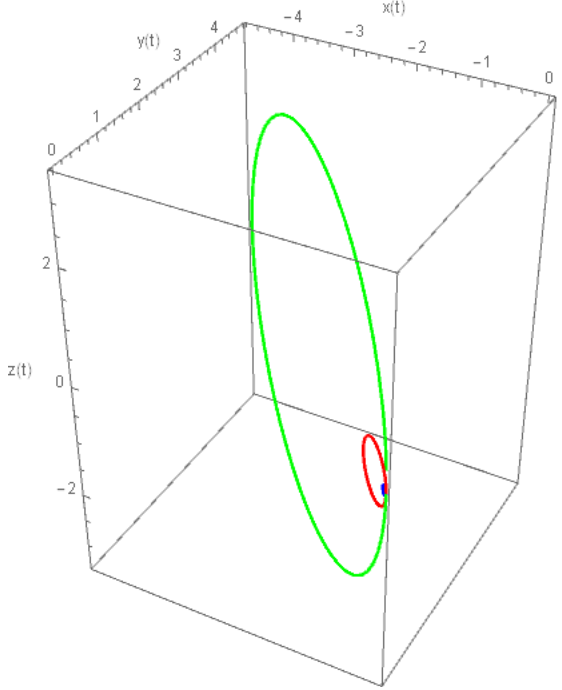
$$z(t) = \frac{\mathrm{vz}\left(\sqrt{\boldsymbol{B}}\mathrm{Bz}^2qt\sqrt{1-\mathrm{vz}^2} + (\mathrm{Bx}^2+\mathrm{By}^2)m\mathrm{Sin}\left[\frac{\sqrt{\boldsymbol{B}}qt\sqrt{1-\mathrm{vz}^2}}{m}\right]\right)}{\boldsymbol{B}^{3/2}q(-1+\mathrm{vz}^2)}$$



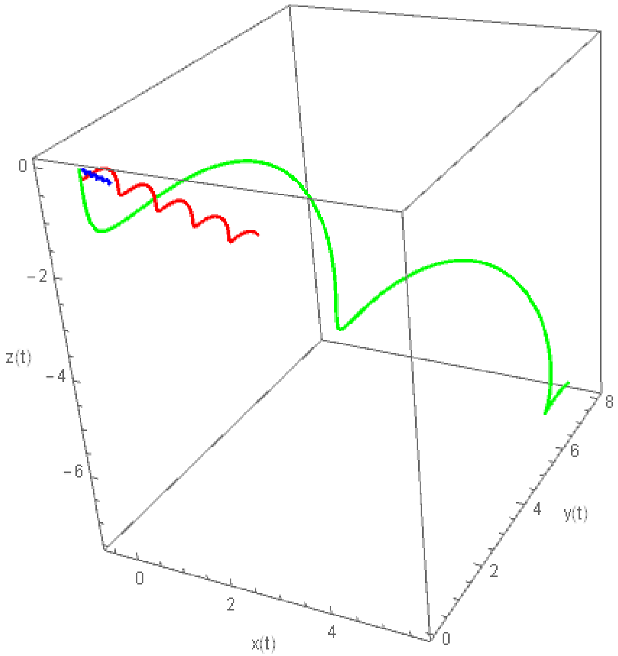

By= 1 $q = 1, m = 1, t = 20$




Bx= 1 $q = 1, m = 1, t = 20$




Bx= By =1
$q = 1, m = 1$
$, \tau = 20$




Bx= By =Bz=2
$q = 1, m = 1$
$, \tau = 20$

Blue → vz=0.1 , Red → vz= 0.5, Green → vz=0.9

When all Electric and magnetic fields are arbitrary in the IFD.

$$\begin{aligned}
\frac{d^2t(\tau)}{d\tau^2} &= \frac{q}{m}\left(E_x\frac{dx(\tau)}{d\tau}+E_y\frac{dy(\tau)}{d\tau}+E_z\frac{dz(\tau)}{d\tau}\right),\\
\frac{d^2x(\tau)}{d\tau^2} &= \frac{q}{m}\left(E_x\frac{dt(\tau)}{d\tau}+B_z\frac{dy(\tau)}{d\tau}-B_y\frac{dz(\tau)}{d\tau}\right),\\
\frac{d^2y(\tau)}{d\tau^2} &= \frac{q}{m}\left(E_y\frac{dt(\tau)}{d\tau}-B_z\frac{dx(\tau)}{d\tau}+B_x\frac{dz(\tau)}{d\tau}\right),\\
\frac{d^2z(\tau)}{d\tau^2} &= -\frac{q}{m}\left(E_z\frac{dt(\tau)}{d\tau}+B_y\frac{dx(\tau)}{d\tau}-B_x\frac{dy(\tau)}{d\tau}\right).
\end{aligned}$$

After integrating and applying the initial condition.

$$\begin{aligned}
\frac{dt(\tau)}{d\tau} &= \frac{q}{m}\left(E_x x(\tau)+E_y y(\tau)+E_z z(\tau)\right)+u_t(0),\\
\frac{dx(\tau)}{d\tau} &= \frac{q}{m}\left(E_x t(\tau)+B_z y(\tau)-B_y z(\tau)\right),\\
\frac{dy(\tau)}{d\tau} &= \frac{q}{m}\left(E_y t(\tau)-B_z x(\tau)+B_x z(\tau)\right),\\
\frac{dz(\tau)}{d\tau} &= -\frac{q}{m}\left(E_z t(\tau)+B_y x(\tau)-B_x y(\tau)\right)+u_z(0).
\end{aligned}$$

$$\frac{d^2t(\tau)}{d\tau^2} = \frac{q^2}{m^2}\left((E_x^2+E_y^2-E_z^2)t(\tau)+(E_zB_y-E_yB_z)x(\tau)+(E_xB_z+E_zB_x)y(\tau)+(E_yB_x-E_xB_y)z(\tau)\right)+\frac{qE_z}{m}u_z$$

$$\frac{d^2x(\tau)}{d\tau^2} = \frac{q^2}{m^2}\left((E_yB_z+E_zB_y)t(\tau)+(E_x^2-B_z^2-B_y^2)x(\tau)+(E_xE_y-B_xB_y)y(\tau)+(E_xE_z+B_xB_z)z(\tau)\right)$$
$$+ \frac{q}{m}(E_xu_t(0)-B_yu_z(0)),$$

$$\frac{d^2y(\tau)}{d\tau^2} = \frac{q^2}{m^2}\left((E_zB_x-E_xB_z)t(\tau)+(E_xE_y-B_xB_y)x(\tau)+(E_y^2+B_x^2-B_z^2)y(\tau)+(E_yE_z+B_yB_z)z(\tau)\right)$$
$$+ \frac{q}{m}(E_yu_t(0)+B_xu_z(0)),$$

$$\frac{d^2z(\tau)}{d\tau^2} = \frac{q^2}{m^2}\left((E_yB_x-E_xB_y)t(\tau)-(E_xE_z+B_xB_z)x(\tau)-(E_yE_z+B_yB_z)y(\tau)-(E_z^2+B_x^2+B_y^2)z(\tau)\right)-\frac{qE_z}{m}u_t$$

It is not easy to solve complicated cases like this and find out the IFD space coordinate as a function of Euclidean time.

- we could reduce some equations of motion dynamic into kinematic by using different interpolation angle values.
- We can discuss more different scenarios.

Example problem

# Light-Front dynamic solution.

$$F^{\mu\nu}{}_{LF} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -Ex-By & -Ey+Bx & \sqrt{2}\,Ez \\ Ex+By & 0 & -\sqrt{2}Bz & -By+Ex \\ Ey-Bx & \sqrt{2}\,Bz & 0 & Bx+Ey \\ -\sqrt{2}\,Ez & By-Ex & -Bx-Ey & 0 \end{pmatrix}$$

We remove field analogous to dynamic operators

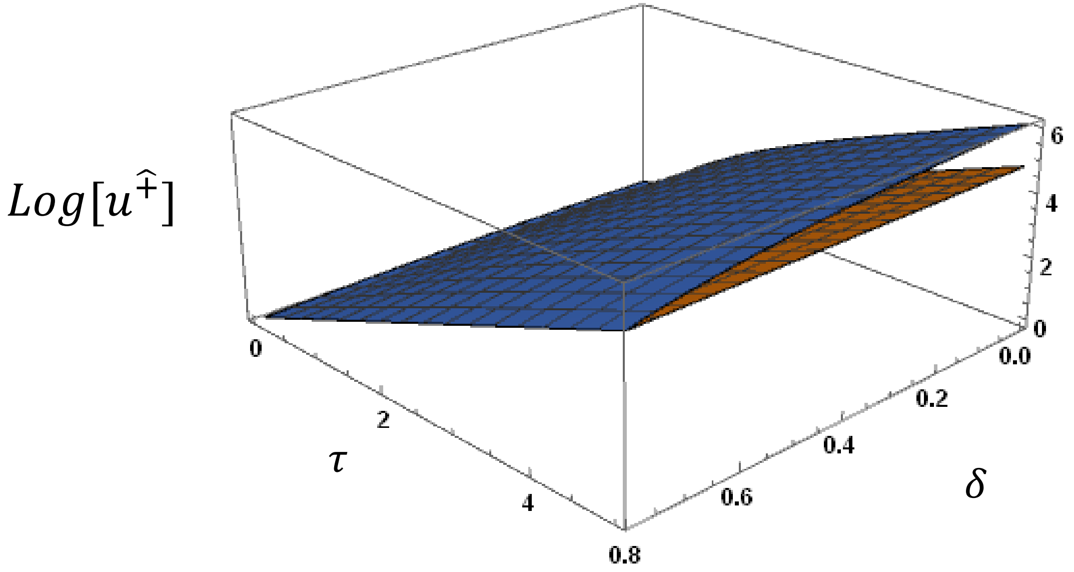
$$Ex = -By, \qquad Ey = Bx$$

- At the Light-Front $\delta = \frac{\pi}{4}$, $K^3$ becomes kinematic,

Field analogous to $K^3$ is $Ez$

$$U^+(\tau) = U^+(0)e^{\frac{Ez\,q}{m}\tau}$$

$$x^+(\tau) = \frac{m}{Ez\,q}U^+(0)(e^{\frac{Ez\,q}{m}\tau}-1)$$




$Bx = 0.5, By = 2, Bz = 0.2, vz = 0.2, q = 1, m = 1,\quad Ex = -2, Ey = 0.5, Ez = 1$

## Light-Front Space coordinates as functions of Light-Front time

$$\tau = \frac{m}{Ez\, q}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]$$

$$x^1(x^+) = -\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(By\, Bz + Bx\, Ez)mU^+(0)\left(1 - Cos\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]\right]\right)\right]$$
$$+\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(Bx\, Bz - By\, Ez)\left(Bz\, qx^+ - mU^+(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]\right]\right)\right]$$

$$x^2(x^+) = \frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(Bx\, Bz - By\, Ez)mU^+(0)\left(1 - Cos\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]\right]\right)\right]$$
$$+\frac{1}{Bz(B_z^2 + E_z^2)q}\left[\sqrt{2}(By\, Bz + Bx\, Ez)\left(Bz\, qx^+ - mU^+(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]\right]\right)\right]$$

$$x^-(x^+) = \frac{mU^-(0)x^+}{(mU^+(0) + Ezqx^+)} + \frac{(Bx^2 + By^2)}{(Bz^2+Ez^2)}\left(\frac{(2mU^+(0) + Ezqx^+)x^+}{(mU^+(0) + Ezqx^+)} - \frac{2mU^+(0)Sin\left[\frac{Bz}{Ez}\ Log[1 + \frac{Ez\, qx^+}{U^+(0)m}]\right]}{Bzq}\right)$$

- We also can find the general space-time coordinates as functions of proper time

$$t(\tau) = \frac{x^{+}(\tau)+x^{-}(\tau)}{\sqrt{2}}, x(\tau) = x^{1}(\tau), y(\tau) = x^{2}(\tau), z(\tau) = \frac{x^{+}(\tau)-x^{-}(\tau)}{\sqrt{2}}$$

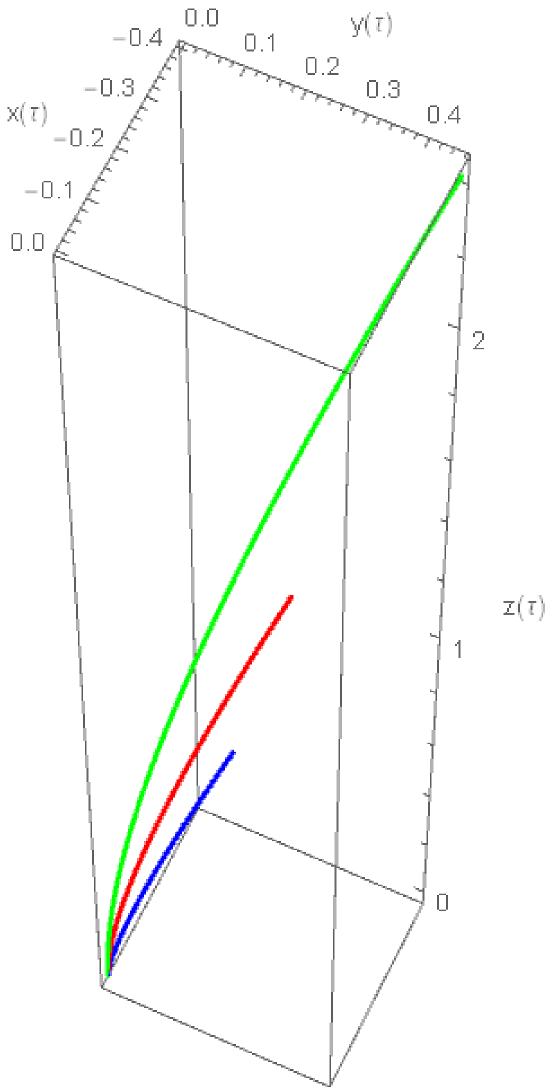
$$t(\tau) = \frac{m}{Ez\,q}\left( u_t(0) - u_z(0)\left[1 - Cosh\left(\frac{Ez\,q\,\tau}{m}\right)\right]\right) + \frac{m(u_t(0) + u_z(0)))}{Bz\,Ez\,q\,(B_z^2 + E_z^2)}\left(Bz\left[B_x^2 + B_y^2 Sinh\left(\frac{Ez\,q\,\tau}{m}\right)\right)\right] - Ez(B_x^2 + B_y^2)\,Sin\left(\frac{Ez\,q\,\tau}{m}\right)\right)$$

$$x(\tau) = -\frac{Bx\,m(u_t(0)+u_z(0))}{BzEzq} + \frac{e^{\frac{Ezq\tau}{m}}(BxBz-ByEz)m(u_t(0)+u_z(0))}{Ez(Bz^2+Ez^2)q} + \frac{(ByBz+BxEz)m(u_t(0)+u_z(0))Cos\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q} - \frac{(BxBz-ByEz)m(u_t(0)+u_z(0))Sin\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q}$$

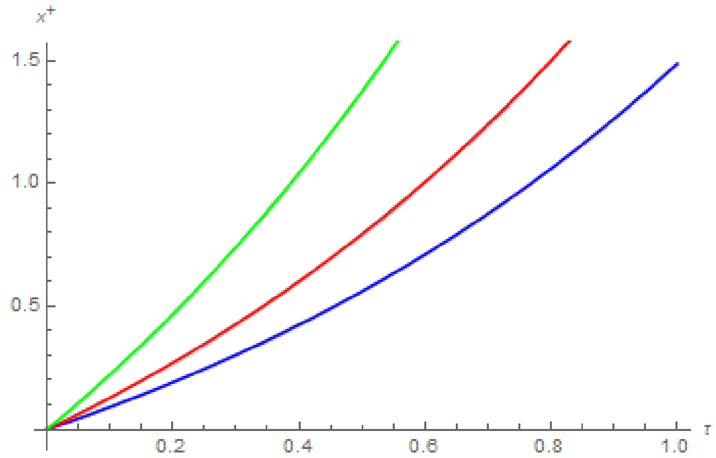
$$y(\tau) = -\frac{Bym(u_t(0)+u_z(0))}{BzEzq} + \frac{e^{\frac{Ezq\tau}{m}}(ByBz+BxEz)m(u_t(0)+u_z(0))}{Ez(Bz^2+Ez^2)q} - \frac{(BxBz-ByEz)m(u_t(0)+u_z(0))Cos\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q} - \frac{(ByBz+BxEz)m(u_t(0)+u_z(0))Sin\left[\frac{Bzq\tau}{m}\right]}{Bz(Bz^2+Ez^2)q}$$

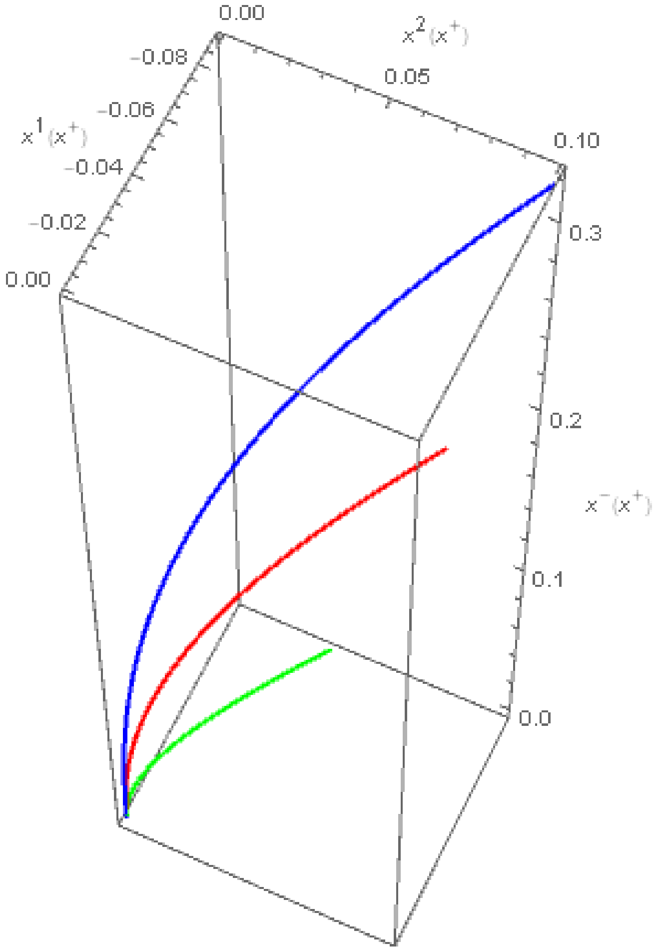
$$z(\tau) = \frac{m}{Ez\,q}\left( u_z(0) - u_t(0)\left[1 - Cosh\left(\frac{Ez\,q\,\tau}{m}\right)\right]\right) - \frac{m(u_t(0) + u_z(0)))}{Bz\,Ez\,q\,(B_z^2 + E_z^2)}\left(Bz\left[B_x^2 + B_y^2 Sinh\left(\frac{Ez\,q\,\tau}{m}\right)\right)\right] - Ez(B_x^2 + B_y^2)\,Sin\left(\frac{Ez\,q\,\tau}{m}\right)\right)$$

# 3D Trajectories




$\tau = 1$




$x^+ = 1$

$$Ez = 1\,, Ex = -0.2\,, Ey = 0.2$$

$$Bx = By = Bz = 0.2$$

$$q = 1, m = 1,$$

Blue → vz=0.2 , Red → vz= 0.5, Green → vz=0.8

# Conclusion

- An alternative method to solve the equation of motion of a charged particle in a relativistic electromagnetic field by using an interpolating angle.
- When we put the constraint to the fields, we limits number of scenarios we can consider.
- This method can effectively gauge the effect of the kinematic generators saving dynamical efforts in  solving the Lorentz force equation for different interpolating time.