# Poincare Algebra with Scaled Interpolating Variables.

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# Out Line

- Summary of the last talk
  - Novel basis set of the scaled interpolating variable
  - Key features of Poincare operators
- Spin operator
- Poincare Algebra
- Translation operator
  - Different approaches and references

## Novel Interpolating Basis

The interpolating space-time coordinate between Instant Form Dynamic and Light-Front Vacuum are defined by a transformation from the ordinary space-time coordinates,

$$x^{\bar{\mu}} = H^{\bar{\mu}}_{\nu} x^{\nu}$$

$$\begin{pmatrix} x^{\tilde{+}} \\ x^{\tilde{1}} \\ x^{\tilde{2}} \\ x^{\tilde{-}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\sin\delta}{\sqrt{\mathbb{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\cos\delta}{\sqrt{\mathbb{C}}} \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

The new interpolating basis in the ordinarily interpolating basis

$$x^{\tilde{+}} = \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}, x^{\tilde{1}} = x^{\hat{1}}, x^{\tilde{2}} = x^{\hat{2}}, x^{\tilde{-}} = \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}$$

Note: In ordinarily Interpolation transformation in IFD, invert the direction of z direction in the IFD ( $x^{\hat{-}} \rightarrow -x^3$ ).

: Orthogonal basis set for all interpolation angle

Novel basis

IFD $(\delta \rightarrow 0)$, $x^{\tilde{+}} \rightarrow x^0, x^{\tilde{1}} \rightarrow x^1, x^{\tilde{2}} \rightarrow x^2, x^{\tilde{-}} \rightarrow x^3$

- Skew coordinate system ( all interpolation angle except $\delta \rightarrow 0$)

LFD $(\delta \rightarrow \frac{\pi}{4})$ , $x^{\hat{+}} \rightarrow x^+$ , $x_{\hat{-}} \rightarrow x^+$ and $\sqrt{\mathbb{C}} \rightarrow 0$ , therefore $x^{\tilde{+}}$ and $x^{\tilde{-}}$ become indeterminate unless we consider $x^+ = 0$ .

When $\delta \rightarrow \pi/4 - \epsilon$ ($\epsilon$ is a very small value), The expansion of $x^{\tilde{+}}$ and $x^{\tilde{-}}$ can be written as

$$x^{\tilde{+}}_{\delta \rightarrow \frac{\pi}{4} - \epsilon} = \frac{x^+}{\sqrt{2\epsilon}} + \frac{x^- \epsilon}{\sqrt{2}} - \frac{x^+ \epsilon^{3/2}}{6\sqrt{2}} \ldots\ldots$$

$$x^{\tilde{-}}_{\delta \rightarrow \frac{\pi}{4} - \epsilon} = \frac{x^+}{\sqrt{2\epsilon}} - \frac{x^- \epsilon}{\sqrt{2}} - \frac{x^+ \epsilon^{3/2}}{6\sqrt{2}} \ldots\ldots$$

This clearly shows that $x^{\tilde{+}}$ and $x^{\tilde{-}}$ reach to same value for very small $\epsilon$ values as $\delta \rightarrow \pi/4$

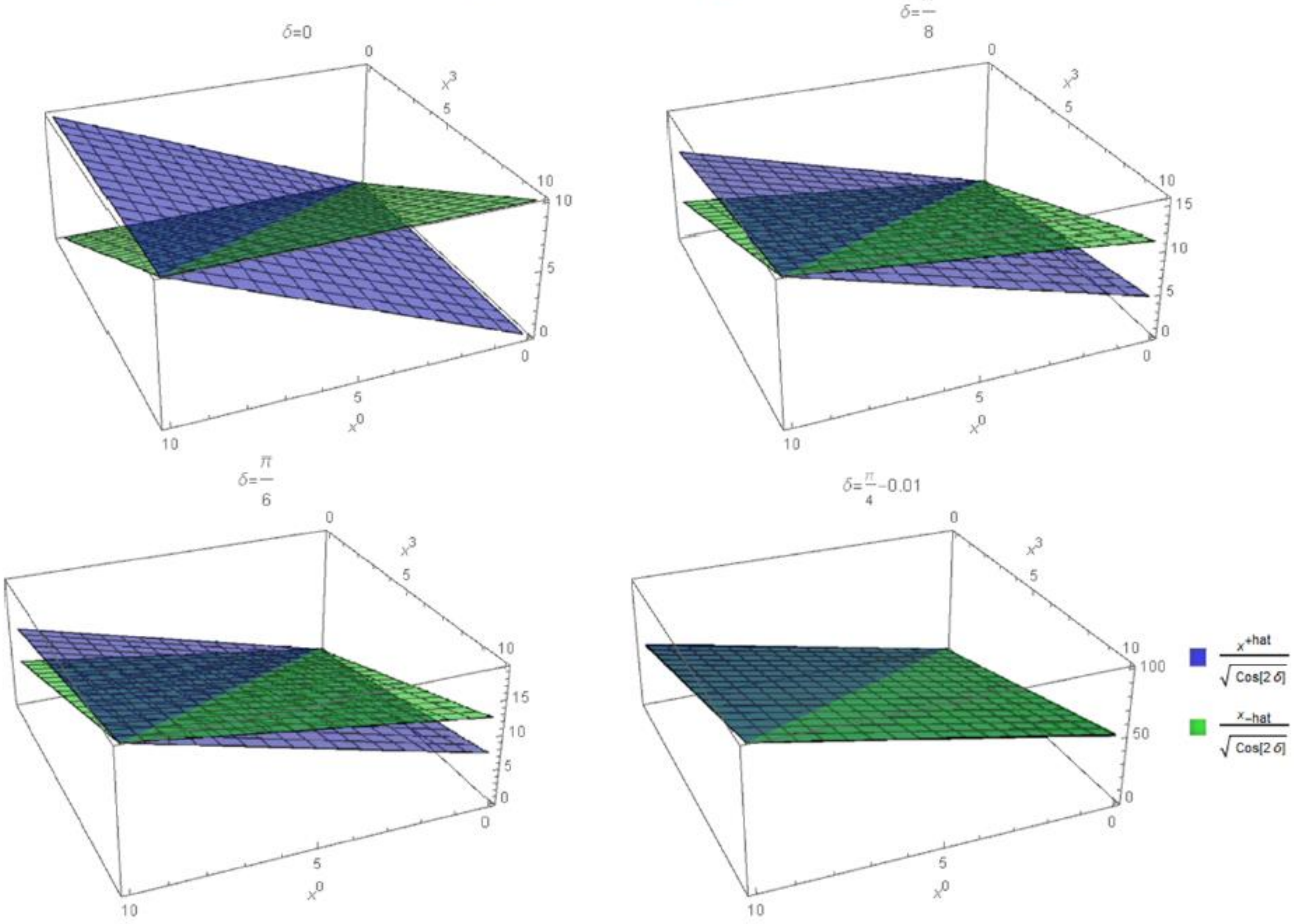


Fig 01: $x^{\hat{+}}$ and $x^{\hat{-}}$ vary with $x^0$ and $x^3$ for different $\delta$ values

## Space-Time Interval

$$s^2 = x^{\mu} x_{\mu} = x^{\hat{\mu}} x_{\hat{\mu}} = (x^{\tilde{+}})^2 - (x^{\tilde{1}})^2 - (x^{\tilde{2}})^2 - (x^{\tilde{-}})^2$$

Momentum space for the particle mass M, $P^{\mu} P_{\mu}$ on the mass shell is equal to $M^2$

$$M^2 = P^{\mu} P_{\mu} = P^{\hat{\mu}} P_{\hat{\mu}} = (P^{\tilde{+}})^2 - (P^{\tilde{1}})^2 - (P^{\tilde{2}})^2 - (P^{\tilde{-}})^2$$

Lorentz transformation related to the new interpolating basis can be written as

$$x'^{\tilde{\mu}} = H^{\tilde{\mu}}_{\nu} x'^{\nu} = H^{\tilde{\mu}}_{\nu} \Lambda^{\nu}_{\alpha} x^{\alpha} = H^{\tilde{\mu}}_{\nu} \Lambda^{\nu}_{\alpha} (H^{-1})^{\alpha}_{\tilde{\nu}} x^{\tilde{\nu}} = \Lambda^{\tilde{\mu}}_{\tilde{\nu}} x^{\tilde{\nu}}$$

After transforming bases using the Lorentz transformation

$$s^2 = x'^{\mu} x'_{\mu} = x'^{\hat{\mu}} x'_{\hat{\mu}} = (x^{\tilde{+}})^2 - (x'^{\tilde{1}})^2 - (x'^{\tilde{2}})^2 - (x'^{\tilde{-}})^2$$

$$M^2 = P'^{\mu} P'_{\mu} = P'^{\hat{\mu}} P'_{\hat{\mu}} = (P'^{\tilde{+}})^2 - (P'^{\tilde{1}})^2 - (P'^{\tilde{2}})^2 - (P'^{\tilde{-}})^2$$

Boost and Rotation operators in 4-vector representation

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Poincare Matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

Interpolating Poincare Matrix

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$

$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$

$$\mathcal{D}^{\hat{1}} = -K^1 \cos\delta + J^2 \sin\delta,$$

$$\mathcal{D}^{\hat{2}} = -J^1 \sin\delta - K^2 \cos\delta.$$

$$e^{(-i\beta_z K_3)}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\tilde{+}} \\ x'^{\tilde{1}} \\ x'^{\tilde{2}} \\ x'^{\tilde{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} x^{\tilde{+}} \\ x^{\tilde{1}} \\ x^{\tilde{2}} \\ x^{\tilde{-}} \end{pmatrix}$$

$$e^{-iJ_3\theta_z}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\tilde{+}} \\ x'^{\tilde{1}} \\ x'^{\tilde{2}} \\ x'^{\tilde{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\tilde{+}} \\ x^{\tilde{1}} \\ x^{\tilde{2}} \\ x^{\tilde{-}} \end{pmatrix}$$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta \qquad e^{i\beta_1 \mathcal{K}^{\hat{1}}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_1 \sin\delta^2}{\mathbb{C}} & \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & \cos\alpha_1 & 0 & \frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ -\frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} & -\frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2 \cos\alpha_1 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\tilde{+}} \\ x'^{\tilde{1}} \\ x'^{\tilde{2}} \\ x'^{\tilde{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x^{\tilde{+}} \\ x^{\tilde{1}} \\ x^{\tilde{2}} \\ x^{\tilde{-}} \end{pmatrix}$$

$$\alpha_1 = \sqrt{\beta_1^2 \cos 2\delta} = \sqrt{\beta_1^2 \mathbb{C}}$$

- Similar to the $e^{-i\theta_y J^2}$ , but here we have $e^{-i\alpha_1 J^2}$
- It seems $\mathcal{K}^{\hat{1}}$ play the role of rotation around y axis starting from the IFD to LF vacuum with this new basis
- Kinematic operator $\mathcal{K}^{\hat{1}}$ exclusively independent of interpolation angle in the new basis

$$e^{-iJ_2\theta_y}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta$$

$$e^{i\beta_2 \mathcal{K}^{\hat{2}}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_2 \sin\delta^2}{\mathbb{C}} & 0 & \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sin\delta \sin\alpha_2}{\sqrt{\mathbb{C}}} & 0 & \cos\alpha_2 & \frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} \\ -\frac{\sin(\alpha_2/2)^2 \mathbb{S}}{\mathbb{C}} & 0 & -\frac{\cos\delta \sin\alpha_2}{\sqrt{\mathbb{C}}} & \frac{\cos\delta^2 \cos\alpha_2 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & \sin\alpha_2 \\ 0 & 0 & -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$$e^{-iJ_1\theta_x}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

- Similar to the $e^{i\theta_x J^1}$ , but here we have $e^{i\alpha_2 J^1}$
- It seems $\mathcal{K}^{\hat{2}}$ play the role of rotation around x- axis starting from the IFD to LF vaccume with this new basis
- Kinematic operator $\mathcal{K}^{\hat{2}}$ exclusively independent of interpolation angle in the new basis

$$T = T_{12}T_3 = e^{i\beta_1 \mathcal{K}^{\hat{1}} + i\beta_2 \mathcal{K}^{\hat{2}}} e^{-i\beta_3 K^3}$$

IFD

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta B_1 - \sin\delta\cos\alpha B_2}{\mathbb{C}} & \frac{\beta_1 \sin\delta \sin\alpha}{\alpha} & \frac{\beta_2 \sin\delta\sin\alpha}{\alpha} & \frac{\cos\delta B_2 - \sin\delta\cos\alpha B_1}{\mathbb{C}} \\ \frac{\beta_1 \sin\alpha B_2}{\alpha} & \frac{\beta_2^2 + \beta_1^2 \cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha - 1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1\sin\alpha B_1}{\alpha} \\ \frac{\beta_2 \sin\alpha B_2}{\alpha} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_2\sin\alpha B_1}{\alpha} \\ \frac{\cos\delta\cos\alpha B_2 - \sin\delta B_1}{\mathbb{C}} & -\frac{\beta_1\cos\delta\sin\alpha}{\alpha} & -\frac{\beta_2\cos\delta\sin\alpha}{\alpha} & \frac{\cos\delta\cos\alpha B_1 - \sin\delta B_2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \text{Cosh}[\beta 3] & 0 & 0 & \text{Sinh}[\beta 3] \\ \frac{\beta 1 \text{Sin}[\alpha i]\text{Sinh}[\beta 3]}{\sqrt{\beta 1^2 + \beta 2^2}} & \frac{\beta 2^2 + \beta 1^2 \text{Cos}[\alpha i]}{\beta 1^2 + \beta 2^2} & \frac{\beta 1 \beta 2(-1 + \text{Cos}[\alpha i])}{\beta 1^2 + \beta 2^2} & \frac{\beta 1 \text{Cosh}[\beta 3]\text{Sin}[\alpha i]}{\sqrt{\beta 1^2 + \beta 2^2}} \\ \frac{\beta 2 \text{Sin}[\alpha i]\text{Sinh}[\beta 3]}{\sqrt{\beta 1^2 + \beta 2^2}} & \frac{\beta 1 \beta 2(-1 + \text{Cos}[\alpha i])}{\beta 1^2 + \beta 2^2} & \frac{\beta 1^2 + \beta 2^2 \text{Cos}[\alpha i]}{\beta 1^2 + \beta 2^2} & \frac{\beta 2 \text{Cosh}[\beta 3]\text{Sin}[\alpha i]}{\sqrt{\beta 1^2 + \beta 2^2}} \\ \text{Cos}[\alpha i]\text{Sinh}[\beta 3] & -\frac{\beta 1 \text{Sin}[\alpha i]}{\sqrt{\beta 1^2 + \beta 2^2}} & -\frac{\beta 2 \text{Sin}[\alpha i]}{\sqrt{\beta 1^2 + \beta 2^2}} & \text{Cos}[\alpha i]\text{Cosh}[\beta 3] \end{pmatrix}$$

$\alpha = \sqrt{(\beta_1^2 + \beta_2^2)\mathbb{C}}, \quad B_1 = \sin\delta\sinh\beta_3 + \cos\delta\cosh\beta_3, \; B_2 = \sin\delta\cosh\beta_3 + \cos\delta\sinh\beta_3$

$$\begin{pmatrix} x'^{\tilde{+}} \\ x'^{\tilde{1}} \\ x'^{\tilde{2}} \\ x'^{\tilde{-}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_3 & 0 & 0 & \sinh\beta_3 \\ \frac{\beta_1 \sin\alpha\sinh\beta_3}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_2^2+\beta_1^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1\cosh\beta_3\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} \\ \frac{\beta_2\sin\alpha\sinh\beta_3}{\sqrt{\beta_1^2+\beta_2^2}} & \frac{\beta_1\beta_2(\cos\alpha-1)}{\beta_1^2+\beta_2^2} & \frac{\beta_1^2+\beta_2^2\cos\alpha}{\beta_1^2+\beta_2^2} & \frac{\beta_2\cosh\beta_3\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} \\ \cos\alpha\sinh\beta_3 & -\frac{\beta_1\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} & -\frac{\beta_2\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}} & \cos\alpha\cosh\beta_3 \end{pmatrix} \begin{pmatrix} x^{\tilde{+}} \\ x^{\tilde{1}} \\ x^{\tilde{2}} \\ x^{\tilde{-}} \end{pmatrix}$$

# Spin Operator

$$\mathcal{J}_i = T J_i T^{-1}$$

$$\cos\alpha = \frac{P_{\hat{-}}}{\mathbb{P}}$$

$$\sin\alpha = \frac{\sqrt{\mathbf{P}_\perp^2 \mathbb{C}}}{\mathbb{P}}$$

Generalized Helicity operator $\mathcal{J}_3 = T J_3 T^{-1}$

$$\frac{\beta_j}{\alpha} = \frac{P^j}{\sqrt{\mathbf{P}_\perp^2 \mathbb{C}}}$$

$(x^0, x^1, x^2, x^3)$

$$\mathcal{J}_3 = J_3 \cos\alpha + (\beta_1 \mathcal{K}^{\hat{2}} - \beta_2 \mathcal{K}^{\hat{1}}) \frac{\sin\alpha}{\alpha}.$$

$$\alpha = \sqrt{\mathbb{C}(\beta_1^2 + \beta_2^2)}$$

$$\mathcal{J}_3 = \frac{1}{\mathbb{P}} (P_{\hat{-}} J_3 + P^1 \mathcal{K}^{\hat{2}} - P^2 \mathcal{K}^{\hat{1}}),$$

$$\mathbb{P} \equiv \sqrt{(P^{\hat{+}})^2 - M^2 \mathbb{C}} = \sqrt{P_{\hat{-}}^2 + \mathbf{P}_\perp^2 \mathbb{C}}.$$

$(\delta \to 0)$, $\mathcal{K}^{\hat{1}} \to -J^2$, $\mathcal{K}^{\hat{2}} \to J^1$, $P_{\hat{-}} \to P^3$ $\mathbb{P} \to \sqrt{(P^0)^2 - M^2} = |\mathbf{P}|$ $\mathcal{J}_3 = \mathbf{P} \cdot \mathbf{J} / |\mathbf{P}|$

Jackob-Wick Helicity

$(\delta \to \pi/4)$, $\mathcal{K}^{\hat{1}} \to -E_1$, $\mathcal{K}^{\hat{2}} \to -E_2$, $P_{\hat{-}} \to P^+$ $\mathbb{P} \to \sqrt{(P^+)^2} = P^+$

Light-front helicity $\mathcal{J}_3 = J_3 + \frac{1}{P^+}(P^2 E_1 - P^1 E_2)$

# Generalized Helicity operator

$$\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}, x^{\hat{1}}, x^{\hat{2}}, \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} = (x^{\tilde{+}}, x^{\tilde{1}}, x^{\tilde{2}}, x^{\tilde{-}})$$

$$\mathcal{J}_3 = T J_3 T^{-1}$$

$$\mathcal{J}_3 = J_3 \cos\alpha + (\beta_1 J_1 + \beta_2 J_2)\frac{\sin\alpha}{\sqrt{\beta_1^2 + \beta_2^2}}$$

$$\mathcal{J}_3 = \frac{\sqrt{\mathbb{C}}}{\mathbb{P}}\left(P^1 J_1 + P^2 J_2 + \frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}} J_3\right) \longrightarrow \mathcal{J}_3 = \frac{1}{\sqrt{\mathbf{P}_\perp^2 + \frac{P_{\hat{-}}^2}{\mathbb{C}}}}\left(P^1 J_1 + P^2 J_2 + \frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}} J_3\right)$$

Magnitude of the total momentum in the new basis $|\tilde{\boldsymbol{P}}|$

$$\mathcal{J}_3 = \frac{\tilde{\boldsymbol{P}}.\boldsymbol{J}}{|\tilde{\boldsymbol{P}}|}$$

This valid for any interpolation angle in the basis, It seems the structure is invariant throughout

$$\mathcal{J}_3 = \frac{\sqrt{\mathbb{C}}}{\mathbb{P}}\left(P^1 J_1 + P^2 J_2 + \frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}} J_3\right)$$

$$\boxed{\mathcal{J}_3 = \frac{\tilde{\boldsymbol{P}}.\boldsymbol{J}}{|\tilde{\boldsymbol{P}}|}}$$

$(\delta \to 0)$

$P_{\hat{-}} \to P^3 \quad \mathbb{P} \to |\mathbf{P}|$

$$\mathcal{J}_3 = \frac{\boldsymbol{P}.\boldsymbol{J}}{|\boldsymbol{P}|}$$

Jackob-Wick Helicity

$(\delta \to \pi/4)$

$P_{\hat{-}} \to P^+ \quad \mathbb{P} \to \sqrt{(P^+)^2} = P^+$

$$\mathcal{J}_3 = J_3$$

LF-zero-mode

$\sqrt{\mathbb{C}} \to 0$ and $P^+ \to 0$

$$\mathcal{J}_3 = J_3 + a\left(P^1 J_1 + P^2 J_2\right)$$

$a$ = finite value

# Poincare Algebra in the new basis ( 45 commutation relations)

| | $\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}$ | $P^{\hat{1}}$ | $P^{\hat{2}}$ | $\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}$ | $\mathcal{D}^{\hat{1}}$ | $\mathcal{D}^{\hat{2}}$ | $J^3$ | $\mathcal{K}^{\hat{1}}$ | $\mathcal{K}^{\hat{1}}$ | $K^3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}$ | 0 | 0 | 0 | 0 | $\frac{iP^{\hat{1}}}{\sqrt{\mathbb{C}}}$ | $\frac{iP^{\hat{2}}}{\sqrt{\mathbb{C}}}$ | 0 | 0 | 0 | $\frac{iP_{\hat{-}}}{\sqrt{\mathbb{C}}}$ |
| $P^{\hat{1}}$ | 0 | 0 | 0 | 0 | $\frac{-iP^{\hat{+}}}{\mathbb{C}}+\frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$ | 0 | $-iP^{\hat{2}}$ | $-iP_{\hat{-}}$ | 0 | 0 |
| $P^{\hat{2}}$ | 0 | 0 | 0 | 0 | 0 | $\frac{-iP^{\hat{+}}}{\mathbb{C}}+\frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$ | $iP^{\hat{1}}$ | 0 | $-iP_{\hat{-}}$ | 0 |
| $\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}$ | 0 | 0 | 0 | 0 | $\frac{-i\mathbb{S}P^{\hat{1}}}{\sqrt{\mathbb{C}}}$ | $\frac{-i\mathbb{S}P^{\hat{2}}}{\sqrt{\mathbb{C}}}$ | 0 | $i\sqrt{\mathbb{C}}P^{\hat{1}}$ | $i\sqrt{\mathbb{C}}P^{\hat{2}}$ | $\frac{iP^{\hat{+}}}{\sqrt{\mathbb{C}}}$ |
| $\mathcal{D}^{\hat{1}}$ | $-\frac{iP^{\hat{1}}}{\sqrt{\mathbb{C}}}$ | $\frac{iP^{\hat{+}}}{\mathbb{C}}-\frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}})$ | 0 | $\frac{i\mathbb{S}P^{\hat{1}}}{\sqrt{\mathbb{C}}}$ | 0 | $-i\mathbb{C}J^3$ | $-i\mathcal{D}^{\hat{2}}$ | $-iK^3$ | $-i\mathbb{S}J^3$ | $-i\mathbb{S}\mathcal{D}^{\hat{1}}+i\mathbb{C}\mathcal{K}^{\hat{1}}$ |
| $\mathcal{D}^{\hat{2}}$ | $-\frac{iP^{\hat{2}}}{\sqrt{\mathbb{C}}}$ | 0 | $\frac{iP^{\hat{+}}}{\mathbb{C}}-\frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$ | $\frac{i\mathbb{S}P^{\hat{2}}}{\sqrt{\mathbb{C}}}$ | $i\mathbb{C}J^3$ | 0 | $i\mathcal{D}^{\hat{1}}$ | $i\mathbb{S}J^3$ | $-iK^3$ | $-i\mathbb{S}\mathcal{D}^{\hat{2}}+i\mathbb{C}\mathcal{K}^{\hat{2}}$ |
| $J^3$ | 0 | $iP^{\hat{2}}$ | $-iP^{\hat{1}}$ | 0 | $i\mathcal{D}^{\hat{2}}$ | $-i\mathcal{D}^{\hat{1}}$ | 0 | $i\mathcal{K}^{\hat{2}}$ | $-i\mathcal{K}^{\hat{1}}$ | 0 |
| $\mathcal{K}^{\hat{1}}$ | 0 | $iP_{\hat{-}}$ | 0 | $-i\sqrt{\mathbb{C}}P^{\hat{1}}$ | $iK^3$ | $-i\mathbb{S}J^3$ | $-i\mathcal{K}^{\hat{2}}$ | 0 | $i\mathbb{C}J^3$ | $i\mathbb{S}\mathcal{K}^{\hat{1}}+i\mathbb{C}\mathcal{D}^{\hat{1}}$ |
| $\mathcal{K}^{\hat{2}}$ | 0 | 0 | $iP_{\hat{-}}$ | $-i\sqrt{\mathbb{C}}P^{\hat{2}}$ | $i\mathbb{S}J^3$ | $iK^3$ | $i\mathcal{K}^{\hat{1}}$ | $-i\mathbb{C}J^3$ | 0 | $i\mathbb{S}\mathcal{K}^{\hat{2}}+i\mathbb{C}\mathcal{D}^{\hat{2}}$ |
| $K^3$ | $\frac{-iP_{\hat{-}}}{\sqrt{\mathbb{C}}}$ | 0 | 0 | $\frac{-iP^{\hat{+}}}{\sqrt{\mathbb{C}}}$ | $\mathbb{S}\mathcal{D}^{\hat{1}}+i\mathbb{C}\mathcal{K}^{\hat{1}}$ | $i\mathbb{S}\mathcal{D}^{\hat{2}}-i\mathbb{C}\mathcal{K}^{\hat{2}}$ | 0 | $-i\mathbb{S}\mathcal{K}^{\hat{1}}-i\mathbb{C}\mathcal{D}^{\hat{1}}$ | $-i\mathbb{S}\mathcal{K}^{\hat{2}}-i\mathbb{C}\mathcal{D}^{\hat{2}}$ | 0 |

All the commutation relation has been calculated previously , are considered ordinarily covariant and contravariant form.

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, K^{\hat{3}}\right] = i\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}} \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, K^{\hat{3}}\right] = i\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}$$

$\delta \to 0,\ [P^0, K^3] = iP^3 \qquad [P^3, K^3] = iP^0$

$\delta \to \frac{\pi}{4}, P^+ \to 0,$ Kinematic behavior of the longitudinal boost

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, J^{\hat{3}}\right] = 0 \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, J^{\hat{3}}\right] = 0$$

$J^3$ is kinematic operator for all interpolation angle

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, \mathcal{K}^{\hat{1}}\right] = 0 \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, \mathcal{K}^{\hat{1}}\right] = i\sqrt{\mathbb{C}}P^{\hat{1}}$$

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, \mathcal{K}^{\hat{2}}\right] = 0 \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, \mathcal{K}^{\hat{2}}\right] = i\sqrt{\mathbb{C}}P^{\hat{2}}$$

$\mathcal{K}^{\hat{1}}$ $\mathcal{K}^{\hat{2}}$ Kinematic operators in all range of interpolation angle

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, \mathcal{D}^{\hat{1}}\right] = -i\frac{P^{\hat{1}}}{\sqrt{\mathbb{C}}} \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, \mathcal{D}^{\hat{1}}\right] = -i\frac{P^{\hat{1}}\mathbb{S}}{\sqrt{\mathbb{C}}}$$

$$\left[\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}, \mathcal{D}^{\hat{2}}\right] = -i\frac{P^{\hat{2}}}{\sqrt{\mathbb{C}}} \qquad \left[\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}, \mathcal{D}^{\hat{2}}\right] = -i\frac{P^{\hat{2}}\mathbb{S}}{\sqrt{\mathbb{C}}}$$

$\delta \to 0,\ [P^0, K^1] = iP^1 \qquad [P^3, K^1] = 0$

$[P^0, K^2] = iP^2 \qquad [P^3, K^2] = 0$

$\delta \to \frac{\pi}{4}, P^+ \to 0,$

Infinite longitudinal momentum -> negligible $P^1$ and $P^2$

Commutation relations still give finite values

# Poincare Generators in spinor representation



TABLE I. – *4-dimensional representation of the Poincaré group.*

| | |
|---|---|
| rotations | $M^{ij}=\frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$, |
| pure Lorentz transformations | $M^{0k}=\frac{1}{2}\begin{pmatrix}i\sigma^k & 0\\ 0 & -i\sigma^k\end{pmatrix}$, |
| translations | $P^{\mu}=\frac{1}{2}\begin{pmatrix}0 & 0\\ \sigma^{\mu} & 0\end{pmatrix}$, |

where $\sigma^{\mu}=(1,\boldsymbol{\sigma})$ are the $2\times 2$ unit matrix and the usual Pauli matrices.

$$[M^{\mu\nu}, M^{\varrho\lambda}] = i(g^{\nu\varrho}M^{\mu\lambda}-g^{\nu\lambda}M^{\mu\varrho}-g^{\mu\varrho}M^{\nu\lambda}+g^{\mu\lambda}M^{\nu\varrho}),$$

$$[M^{\mu\nu}, P^{\varrho}] = i(g^{\nu\varrho}P^{\mu}-g^{\mu\varrho}P^{\nu}),$$

$$[P^{\mu}, P^{\nu}] = 0,$$

Approach- 1

One dimensional non-relativistic

- Time translation

$t' = t + a_t \qquad z' = z$

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{bmatrix} 1 + \frac{a_t}{t} & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

$$I + \frac{a_t}{t}\left(\frac{I + \sigma_3}{2}\right)$$

$$H = \left(\frac{I+\sigma_3}{2}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Space translation in z-direction

$t' = t \qquad z' = z + a_z$

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{a_z}{z} \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

$$I + \frac{a_z}{z}\left(\frac{I - \sigma_3}{2}\right)$$

$$P^3 = \left(\frac{I-\sigma_3}{2}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- Boost in z-direction

$t' = t \qquad z' = vt + z$

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix}$$

$$I + v\left(\frac{\sigma_1 - i\sigma_2}{2}\right)$$

$$K^3 = \left(\frac{\sigma_1 - i\sigma_2}{2}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Commutation Relations

$[H, P^3] = 0$

$[H, K^3] = -K^3$

$[P^3, K^3] = K^3$

## 2. The Lie Algebra of the 2 + 1 Dimensional Galilean Group and its Central Extension

Let $G$ denote the Galilean group in $(2+1)$ space-times and Lie$(G)$ its Lie algebra. We choose a basis for Lie$(G)$ in which the infinitesimal generators of rotation, the boosts along the two spatial directions, that of time translation and those of spatial translation are denoted respectively as $M, N_i, H$ and $P_i$ $(i = 1, 2)$. The commutation relations for these operators are

$[K^3, P^3]$=0

$[P^3, H]$=0

$$[M, N_i] = \varepsilon_{ij} \, N_j, \qquad [N_i, N_j] = 0 \,,$$
$$[P_i, H] = 0, \qquad [P_i, P_j] = 0 \,,$$
$$[M, P_i] = \varepsilon_{ij} \, P_j, \qquad [N_i, P_j] = 0 \,,$$
$$[M, H] = 0, \qquad [N_i, H] = P_i \,. \tag{2.1}$$

$[K^3, H]$= $P^3$

In the above, $\varepsilon_{ij}$ is the antisymmetric symbol with $\varepsilon_{12} = -\varepsilon_{21} = 1$. Summation convention for a repeated index is implied. The physical significance of the generators are well-known. $M$ corresponds to the angular momentum in the plane, $H$ the Hamiltonian and $P_i$ the components of linear momentum.

**The Galilean Group in 2 + 1 Space-Times and its Central Extension**

Commun. Math. Phys. 169, 385 – 395 (1995)

## 2. Dynamical realization of the *l*-conformal Galilei algebra

The $l$-conformal Galilei algebra includes the generators of time translations, dilatations, special conformal transformations, spatial rotations, spatial translations, Galilei boosts and accelerations. Denoting the generators by $(H, D, K, M_{ij}, C_i^{(n)})$, respectively, where $i = 1, \ldots, d$ is a spatial index and $n = 0, 1, \ldots, 2l$, one has the structure relations [3]

$[H, P^3]$=0

$[H, K^3]$= i $P^3$

$$
\begin{aligned}
&[H, D] = iH, \qquad \left[H, C_i^{(n)}\right] = inC_i^{(n-1)}, \\
&[H, K] = 2iD, \qquad [D, K] = iK, \\
&\left[D, C_i^{(n)}\right] = i(n-l)C_i^{(n)}, \qquad \left[K, C_i^{(n)}\right] = i(n-2l)C_i^{(n+1)}, \\
&\left[M_{ij}, C_k^{(n)}\right] = -i\left(\delta_{ik}C_j^{(n)} - \delta_{jk}C_i^{(n)}\right), \\
&[M_{ij}, M_{kl}] = -i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}).
\end{aligned}
\tag{1}
$$

Note that $(H, D, K)$ form $so(2, 1)$ subalgebra, which is the conformal algebra in one dimension. The instances of $n = 0$ and $n = 1$ in $C_i^{(n)}$ correspond to the spatial translations and Galilei boosts. Higher values of $n$ are linked to the accelerations.

Dynamical realization of *l*-conformal Galilei algebra and oscillators

Nuclear Physics B 866 (2013) 212–227

Approach-02

To include translation operator in the four-vector representation, we have to increase one dimension .

$$K^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad K^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix} \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Satisfy usual Poincare algebra

Translation operator

$$\mathbb{P} = e^{-i\, a^{\mu} P_{\mu}} = e^{-i\, g_{\mu\nu} a^{\mu} P^{\nu}}$$

$$e^{-i\, a^{0} P^{0}}$$

$P^{0}$ Operator in the {x0,x1,x2,x3} basis

$$\begin{pmatrix} 1 \\ x^0 + a0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \text{a0} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$e^{i\, a^{1} P^{1}}$$

$P^{1}$ Operator in the {x0,x1,x2,x3} basis

$$\begin{pmatrix} 1 \\ x^0 \\ x^1 + a1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \text{a1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$e^{i\, a^{2} P^{2}}$$

$P^{2}$ Operator in the {x0,x1,x2,x3} basis

$$\begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 + a2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \text{a2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$e^{i\, a^{3} P^{1}}$$

$P^{3}$ Operator in the {x0,x1,x2,x3} basis

$$\begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 \\ x^3 + a3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a3 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

## III. Contraction of Lorentz Groups

Let us consider, first, the inhomogeneous Lorentz group with one space-like, one time-like dimension. It is given by the transformations

$$\begin{aligned} x' &= x \cosh \lambda + t \sinh \lambda + a_x \\ t' &= x \sinh \lambda + t \cosh \lambda + a_t. \end{aligned} \tag{26}$$

$$\left\| \begin{matrix} \cosh \lambda & \sinh \lambda & a_x \\ \sinh \lambda & \cosh \lambda & a_t \\ 0 & 0 & 1 \end{matrix} \right\|$$

form a natural, though not unitary, representation of the group of transformations (26). We can carry out the contraction by setting $a_t = b_t$, $\lambda = \epsilon v$, $a_x = \epsilon b_x$ or $\lambda = v/c$, $a_x = b_x/c$ and letting $\epsilon$ converge to 0, or $c$ converge to infinity. If we do this directly in (26a), the representation will not remain faithful for the contracted group. We shall transform therefore (26a) with a suitable $\epsilon$ (or $c$) dependent matrix: multiply the first row with $c$, the first column with $1/c$. If $c$ goes to infinity in the matrix obtained in this way, one obtains the transformations of the contracted group

$$\begin{aligned} x' &= x + vt + b_x \\ t' &= \qquad t + b_t. \end{aligned} \tag{27a}$$

Inönü, E.; Wigner, E.P. On the Contraction of Groups and their Representations. *Proc. Natl. Acad. Sci. (U.S.)* **1953**, *39*, 510–524.

## 2 Two-dimensional Euclidean Grouip and Cylindrical group

The two-dimensional Euclidean group, often called $E(2)$, consists of rotations and translations on a two-dimensional Euclidian plane. The coordinate transformation takes the form

$$x' = x\cos\alpha - y\sin\alpha + u, \qquad y' = x\sin\alpha + y\cos\alpha + v. \tag{1}$$

This transformation can be written in matrix form as

$$\begin{pmatrix} u' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & u \\ \sin\alpha & \cos\alpha & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ y \\ 1 \end{pmatrix} \tag{2}$$

The three-by-three matrix in the above expression can be exponentiated as

$$E(u, v, \alpha) = \exp\left(-i(uP_1 + vP_2)\right)\exp\left(-i\alpha L_3\right), \tag{3}$$

where $L_3$ is the generator of rotations, and $P_1$ and $P_2$ generate translations. These generators take the form

$$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

and satisfy the commutation relations:

$$[P_1, P_2] = 0, \qquad [L_3, P_1] = iP_2, \qquad [L_3, P_2] = -iP_1, \tag{5}$$

which form the Lie algebra for $E(2)$.

Kim, Y. S.; Wigner, E.P. Cylindrical group and massless particles. *J. Math. Phys.* **1987**, *28*, 1175–1179.

## 4. Contraction of *SO*(3, 2) to *ISO*(3, 1)

Let us next go back to the $SO(3,2)$ contents of this two-oscillator system [4]. There are three space-like coordinates $(x, y, z)$ and two time-like coordinates $s$ and $t$. It is thus possible to construct the five-dimensional space of $(x, y, z, t, s)$, and to consider four-dimensional Minkowskian subspaces consisting of $(x, y, z, t)$ and $(x, y, z, s)$.

As for the $s$ variable, we can make it longer or shorter, according to procedure of group contractions introduced first by Inönü and Wigner [17]. In this five-dimensional space, the boosts along the $x$ direction with respect to the $t$ and $s$ variables are generated by

$$A_x = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_x = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{28}$$



Sibel Başkal [1], Young S. Kim [2,*] and Marilyn E. Noz [3]

Let us then introduce the five-by-five contraction matrix [18,19]

$$C(\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}. \tag{29}$$

This matrix leaves the first four columns and rows invariant, and the four-dimensional Minkowskian sub-space of $(x, y, z, t)$ stays invariant.

As for the boost with respect to the $s$ variable, according to the procedure spelled out in Refs. [18,19], the contracted boost generator becomes

$$B_x^c = \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \left[ C^{-1}(\epsilon)\; B_x\; C(\epsilon) \right] = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Likewise, $B_y^c$ and $B_z^c$ become

$$B_y^c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_z^c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

respectively.

As for the $t$ direction, the transformation applicable to the $s$ and $t$ variables is a rotation, generated by

$$B_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}. \tag{32}$$

This matrix also becomes contracted to

$$B_t^c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

These four contracted generators lead to the five-by-five transformation matrix

$$\exp\left\{-i\left(aB_x^c + bB_y^c + cB_z^c + dB_t^c\right)\right\} = \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{34}$$

performing translations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ z+c \\ t+d \\ 1 \end{pmatrix}. \tag{35}$$

This matrix leaves the first four rows and columns invariant. They are for the Lorentz transformation applicable to the Minkowskian space of $(x, y, z, t)$.

## Conclusion

- Kinematic Poincare generators in the basis of scaled interpolating variables produce similar matrix structures of them in the Euclidean basis.

- The kinematic operators $\mathcal{K}^{\hat{1}}$ , $\mathcal{K}^{\hat{2}}$ plays the role of rotation around y and x-axes in the new basis for all interpolation angle

- Translation operator matrix structure is more non-trivial in the Four vector representation

## Future work

- Try to find the four- vector representation of the translation operators in the four dimensions.

- Understanding the extra dimension in the (5x5) matrix in the four- vector representation of translation operators

Thank you