A pedagogical introduction to Solitons in Classical field theory - I

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Outline

- Simple Topological Soliton - ‘The Kink’
- Vortices
- Dirac’s Monopole
- Wigner Distribution of Classical Solitons

Simple Topological Soliton - ‘The Kink’ [1]

[1] David Tong. TASI lectures on solitons: Instantons, monopoles, vortices and kinks. 2005.

Classical Solitons

1. Solitons are the solutions of classical field equations. Precisely, they are the solutions of the finite energy configuration.
2. Topological solitons is a solution of a system of partial differential equations (or) of quantum field theory *homotopically distinct* from the vacuum solution[2].
3. Two continuous function from one topological space is homo-topically distinct, if one can be continuously deformed into each other. The deformation is defined as *homotopy*.
4. The simplest topological solitons are namely, the Kink solitons and the Sine gordan soliton.

[2] N. S. Manton and P. Sutcliffe, *"Topological solitons*,", Cambridge University Press.

Kink Soliton - An overview

The simple topological soliton is the *kink soliton* which occurs in 1+1 dimensional space time.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) \tag{1}$$

$$U(\phi) = \frac{-\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\mu^2}{4\lambda}$$

which can be simplified as,

$$U(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2 \tag{2}$$

where, $a = \frac{\mu}{\sqrt{\lambda}}$. The minima of the potential occurs when $\frac{dU}{d\phi} = 0$.

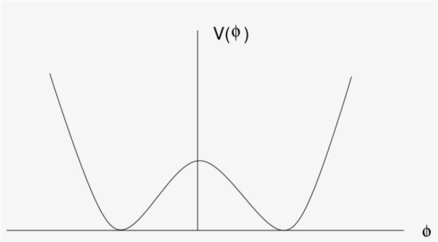

Figure: Mexican hat potential

Classical Vacua and Domain walls

- The classical vacua occurs at the minima of the potential. Thus the classical vacua solution occurs at $\phi = \pm a$.
- ϕ is continous and between two vacua there should be a transition region. This is called *Domain wall*.
- This domain wall is theoretically the spontaneous breaking of discrete symmetry.
- The simplest soliton namely the kink soliton has the form,

$$\phi = atanh \frac{\mu}{\sqrt{2}} x$$

.

Derricks Theorem and its consequence

Derrick's theorem[3] is an argument due to the physicist G.H.Derrick which shows that stationary localised solutions to the non linear wave equation or non linear klein gordan equation in spatial dimensions three and higher are unstable. Derricks theorem which was considered an obstacle to interpreting soliton-like solutions as particles, contained the following physical argument about non-existence of stable localized stationary solutions to the nonlinear wave equation. We look for solutions which :

- Localised
- Finite energy configuration
- Static (Time independent)

[3] V. Robokov, *Classical theory of gauge fields*, Princeton University Press

Energy of Kink soliton

Calculating the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}(\partial_x \phi)^2 + U(\phi) \tag{3}$$

The Energy can be given by,

$$E = \int_{-\infty}^{\infty} dx \mathcal{H} \tag{4}$$

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_x \phi)^2 + \frac{\lambda}{4}(\phi^2 - a^2)^2 \right] \tag{5}$$

By substituting the known values we get, *(briefly discussed in 1+1 Instantons)*.

$$E = \frac{\mu^2}{4\lambda} * \frac{\lambda}{3} * 16a^2 \tag{6}$$

$$E = \frac{4}{3}\mu a^2 < \infty \tag{7}$$

This gives the finite energy solution.

Parity:
Under parity transformation $\phi \rightarrow -\phi$, as $x \rightarrow -x$.
Thus, for kink soliton transforms as , $\phi = -atanh\frac{\mu}{\sqrt{2}}x$. The parity transformed kink soliton is called as the *Anti-kink soliton*.
Time Translation:
Since the kink soliton is invariant in the time translation as they are independent of the time co-ordinates. The kink soliton remains invariant under time translation.
Spatial transformation:
Under the spatial transformation, $x \rightarrow x - \alpha$ the kink soliton transforms as, $\phi = atanh\frac{\mu}{\sqrt{2}}(x - \alpha)$. The spatial transformation doesnot leave the kink to be invariant, but changes the origin of the kink soliton.
Lorentz Symmetry:
The lorentz transformation is as follows,

$$x = \gamma(x' - ut')$$

$$t = \gamma(t' - ux')$$

The kink under the lorentz transformation leaves the solution to satisfy the equations of motion which can be described as follows, $\phi = atanh\frac{\mu}{\sqrt{2}}\gamma(x' - ut')$

Anti-Kink soliton

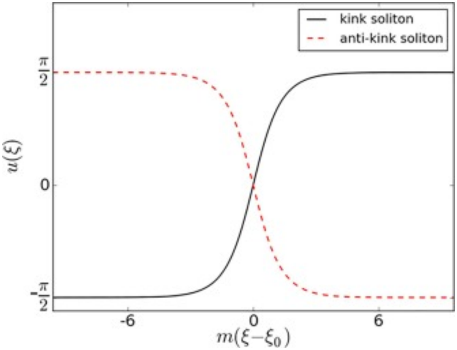

Figure: Kink and Anti-Kink soliton

Magnetic Vortices

Magnetic Vortices - A glimpse

- The solitons of curly type are called the vortices. Vortices can occur at field theory with spontaneously broken continuous symmetry.
- Topological vortices in the $2+1$ dimensions are called *vortices* and in $3+1$ dimensions are called *flux tubes* (or) *strings*.
- During the transformation of vortices in $2+1$ to $3+1$ the solutions donot change because in $3+1$ the flux tube is observed projecting out in the third coordinate.

Abrikosov-Nielson-Oleson Vortex [4]

Consider a system in the Abelian Higgs model, $(2+1)$,

$$\mathcal{L} = \frac{-1}{4e^2} F_{\mu\nu} F^{\mu\nu} + D^{\mu}\phi D_{\mu}\phi - U(\phi) \tag{8}$$

$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\nu}$
$D_{\mu}\phi = (\partial_{\mu} - iqA_{\mu})\phi$
$U(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2$ where $a = \frac{\mu}{\sqrt{\lambda}}$
The model is abelian, So it is invariant under the $U(1)$ Gauge transormation.

$$\phi \to e^{i\beta(x)}\phi$$

$$A_{\mu} \to A_{\mu} + \frac{1}{q}\partial_{\mu}\beta$$

Here we consider the pure gauge situation, So, $A_{\mu} = 0.\phi = a$ corresponds to the classical vacua. The Higgs mechanism of the vector field acquires a mass,

$$m_v = \sqrt{2e}qa \tag{9}$$

The real field which is the Higgs field has the mass such that,

$$m_H = 2\sqrt{\lambda}a \tag{10}$$

[4] H.B. Nielsen and P. Otesen, Niels Bohr Institute preprint, Copenhagen,(May 1973)

Energy of ABO vortices [5]

We find the vortices when $r \to \infty$. The energy should be **Time independent and Finite**.

$$\mathcal{H} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left[D_\mu \phi \right]^2 + U(\phi) \tag{11}$$

Since it is time independent the time derivatives vanish. Thus,

$$E = \int d^2 x \mathcal{H} \tag{12}$$

$$E = \int d^2 x \left[\frac{1}{4e^2} F_{ij} F^{ij} + [D_i \phi]^2 + U(\phi) \right] < \infty \tag{13}$$

We find the vortices when $r \to \infty$. The energy should be **Time independent and Finite**.

$$\mathcal{H} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left[D_\mu \phi \right]^2 + U(\phi) \tag{14}$$

Since it is time independent the time derivatives vanish. Thus,

$$E = \int d^2 x \mathcal{H} \tag{15}$$

$$E = \int d^2 x \left[\frac{1}{4e^2} F_{ij} F^{ij} + [D_i \phi]^2 + U(\phi) \right] < \infty \tag{16}$$

[5] B. Zumino, Lectures given at the 1973 Nato Summer Institute in Capri, CERN preprint TH. 1779 (1973)

Bogomolnyi's trick [6]

By the process of completing the squares,

$$E \mp \frac{1}{2}\int (a^3 F_{12}) dx^1 dx^2 \geqslant 0 \tag{17}$$

By simplifying further we get,

$$E \mp a^3 \frac{2n\pi}{e} \geqslant 0 \tag{18}$$

Thus when $n > 0$ we have $E \geqslant \frac{2n\pi a^3}{e}$
When $n < 0$ we have $E \geqslant \frac{-2n\pi a^3}{e}$
Equating the similar terms of first order to 0, we get the first order BPS solution,

$$D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - ieA_{\bar{z}}\phi = 0 \tag{19}$$

and

$$F_{12} = \frac{-e}{2}\left[\phi^2 - a^2\right] \tag{20}$$

This condition can be termed as the *Saturation of bounds.*

[6] E.B. Bogomol'nyi. Sov. J. Nucl. Phys. 24, 449 (1976).

Number of Vortices- A mathematical prelude [7]

1. Let X, Y be two topological spaces. Suppose $f : X \to Y$ is a mapping which is continuous, if it takes a neighbourhood of X with that of Y.
2. If $f : X \to Y$ and $g : X \to Y$ are said to *homotopic* if one can be continuously deformed to the other. A mapping $f : X \to Y$ is said to be homotopic to zero, if it is homotopic to a mapping taking the whole space X to a single point of Y.
3. If Y is connected then all such mappings are homotopic to each other and the equivalence class is called the *zero homotopy class*.
4. **Fundamental group:** Let us consider the mapping of S^1 to X,, where X is some topological space. Let $f : X \to Y$ is a mapping in the interval $[0, 1]$ such that,

   $$f(0) = f(1) = c, say$$

   . The set of homotopy mapping from S^1 to X in that interval is denoted by $\Pi_1(X, c)$.

[7] R. Piccinni, *Lectures of Homotopy theory*, Springer

Homotopy groups

Let f and g be two mappings from S^1 to X such that,

$$f(0) = f(1) = c$$

$$g(0) = g(1) = c$$

in the interval $[0, 1]$. Then $f * g$ is defined as the path that first runs along g and then along f. $f * g$ satisfies all the properties of the group and $\Pi_1(X, c)$ is called as the *fundamental group*. A fundmental group is the simplest group which can generate other elements of that group. For $S^1 \to X$, where $X = S^1$, then

$$\Pi_1(S^1) = \mathcal{Z}$$

where, $\mathcal{Z}$ is the set of all integers. The group $\Pi_1(S^1)$ is isomorphic to the set of all integers under addition.

Some results from the Homotopy groups [8]

- If $f : S^n \to S^m; for n < m$ then ,

$$\pi_n(S^m) = 0$$

  This mapping is called the *trivial map*. Because it has only the zero element.
- The homotopy group of $\pi_n(S^n)$ is isomorphic to all the group of all integers under addition $\mathcal{Z}$.
- The degree of mapping is called the topological number.
- The degree of mapping is given by the winding number. The number of times the function goes around the given function.
- In the 3 dimensional space, $\pi_n(M)$ is trivial which arises by a consquence of *Poincar'e Conjecture*.
- For, $\pi_n(S^1) = \pi_n(S^2) = \pi_n(S^3) = \mathcal{Z}$, which is the set of integers under addition.

[8] R. Piccinni, *Lectures of Homotopy theory*, Springer

Number of Vortices [9]

Let us take a system in the $2+1$ dimension, which possess the $U(1)$ symmetry. The lagrangian of that system is,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) \tag{21}$$

The vacuum solution occurs when $\phi = \pm a.\phi$ travels along the contour in the given interval. This on co-ordinate transformation to the polar coordinates would result in ,

$$\phi(r, \alpha) = ae^{in\alpha} \tag{22}$$

where, α is the polar angle and n is a integer. This leads us to the application of the topological formula for first homotopy group.

$$\pi_1(U_1) = \mathcal{Z} \tag{23}$$

Here,$\mathcal{Z}$ is vortex number, (i.e)., Number of counts around the vacuum manifold circle.

[9] R. Rajaraman. *An Introduction to Solitons and Instantons in Quantum Field Theory*. Amsterdam: North- Holland, 1982.

Vortices in Superconductivity [10]

1. The vortex in Abelian Higgs model is similar to the *Abrikosov vortices* in the Landau-Ginzberg model of superconductivity.
2. An Abrikosov vortex (also called a fluxon) is a vortex of supercurrent in a type-II superconductor. Abrikosov vortices can be explicitly demonstrated as solutions to that theory in a general mathematical setting, viz. as vortices in complex line bundles on Riemannian manifolds.
3. Landau–Ginzburg theory is a mathematical physical theory used to describe superconductivity. It was postulated as a phenomenological model which could describe type-I superconductors without examining their microscopic properties. It further extends to quantum field theory and string theory because they are closely resembled.

[10] R. Rajaraman. *An Introduction to Solitons and Instantons in Quantum Field Theory*. Amsterdam: North- Holland, 1982.

The Vortices

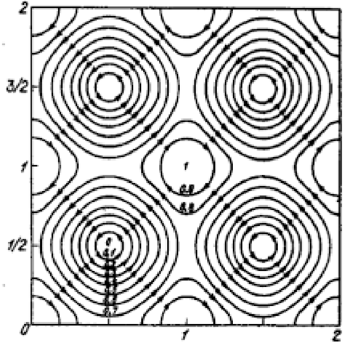

Figure: Abrikosov Vortices

Monopoles

Monopoles

- The magnetically charged solutions are much of interest inorder to study the grand unified theories by unifying the Weak, strong and electromagnetic interactions.
- The initial approach of monopoles were given by *Dirac*, *t'Hooft* and *Polyakov*.

Dirac's Monopole [11]

By Maxwell's equation there is no source free magnetic charges unlike electric charges. If there is a magnetic charges then , $\nabla . B$ is not equal to zero.

$$B = \nabla \times A$$

is false if magnetic charges exist. Also, if electric charges are present $\nabla . E = 4\pi\rho_e$ where,

$$\rho e = e\delta^3(x) \tag{24}$$

$$\vec{E} = \frac{e}{r^2}\hat{e_r} \tag{25}$$

Dirac in 1931, found the effect of a presence of magnetic pole at origin. So assuming the presence of magnetic charge we have ,

$$\vec{B} = \frac{g}{r^2}\hat{e_r} \tag{26}$$

such that,

$$\nabla . B = 4\pi g\delta^3(x) \tag{27}$$

[11] P.A.M. Dirac, Proc. Roy. Soc. A133 (1934) 60; Phys. Rev. 74 (1948) 817

Dirac's Monopole - An illustration [12]

At origin of $\mathcal{R}^{\ni}$, $\nabla.B = 0$ where A exists everywhere apart from origin. But this cannot be true. By Gauss law,

$$\oint_{S^3_R} E.dS = 4\pi e \tag{28}$$

Similarly,

$$\oint_{S^3_R} B.dS = 4\pi g \tag{29}$$

Illustration:
Let's assume, $\vec{A^s} = \frac{-qy}{r(r+z)}\hat{e_x} + \frac{gx}{r(r+z)}\hat{e_y}$
$\vec{A^s}$ is defined except at $r = 0$ and $r = -z$. Therefore,

$$\vec{\nabla}.\vec{A^s} = \vec{B_m} \tag{30}$$

To make Gauss law work, we need to have a flux tube of zero thickness along *'-z axis'*. This is termed as *Dirac String*. Consider a free electron in a time independent magnetic field,

$$H = \frac{(p - eA)^2}{2m} \tag{31}$$

$$H\psi = E\psi \tag{32}$$

[12] P.A.M. Dirac, Proc. Roy. Soc. A133 (1934) 60; Phys. Rev. 74 (1948) 817

Illustration continued [13]

$$\psi(x) = exp\left[\frac{ie}{h}\int_{x_0}^{x} \vec{dl}.\vec{A}\right]\tilde{\psi(\tilde{x})}$$

Substituting the value of ψ in the previous equation we get,

$$(p - eA)\psi = exp\left[i\Phi\right]P\psi \tag{33}$$

where, $exp\left[i\Phi\right]$ is the phase. In Quantum mechanics, phase is not an observable but the difference of phase is an observable. Also, phase is path dependent.

$$\Delta\Phi = \Phi(r_1) - \Phi(r_2) \tag{34}$$

Therefore,

$$\Delta\Phi = \frac{e}{\bar{h}}\Phi_m \tag{35}$$

where, Φ_m is the flux through the surface. This can be termed as *Ahronkov Bohm effect*.

[13] P.A.M. Dirac, Proc. Roy. Soc. A133 (1934) 60; Phys. Rev. 74 (1948) 817

Ahronkov Bohm effect

1. The Aharonov–Bohm effect is a quantum mechanical phenomenon in which an electrically charged particle is affected by an electromagnetic potential, despite being confined to a region in which both the magnetic field B and electric field E are zero.
2. The underlying mechanism is the coupling of the electromagnetic potential with the complex phase of a charged particle's wave function, and the Aharonov–Bohm effect is accordingly illustrated by interference experiments.

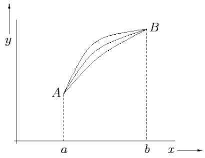

Figure: Phase difference

AB effect

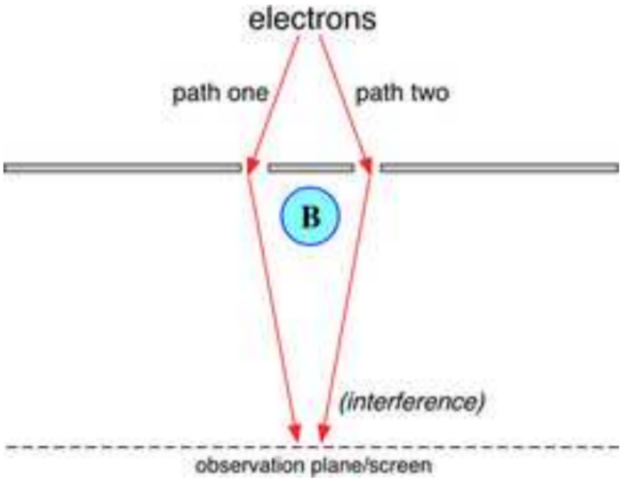

Figure: Ahrankov - Bohm effect

Dirac's string

Dirac's string is unobservable if $\Delta\Phi$ is a multiple of 2π.

$$\Delta\Phi = 2\pi n \tag{36}$$

$$2\pi n = \frac{e}{\bar{h}}\Phi_m \tag{37}$$

where, $\Phi_m = 4\pi g$ is the flux through the surface. Therefore,

$$eg = \frac{n\bar{h}}{2} \tag{38}$$

This is called the *Dirac quantisation condition*. "*Thus a mere existence of one pole of strength g would require all electric charges to be quantised by* $\frac{\bar{h}c}{2g}$ *and similarly the existence of one electric charge would require all poles to be quantised*".

Dirac string [14]

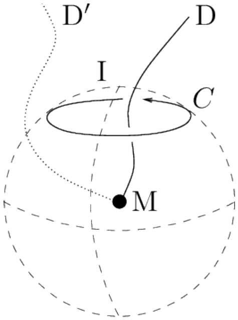

Figure: Dirac's String

[14] P.A.M. Dirac, Proc. Roy. Soc. A133 (1934) 60; Phys. Rev. 74 (1948) 817

Wigner Distributions of Classical Solitons

Based on the work with V.K.Ojha [arxiv:2205.02531]

Wigner Distributions [15]

1. One of the seminal works in semi-classical physics was carried out by Wigner, who combined the distribution of a particle's position (coordinate) and momentum in terms of a wave function.
2. This function known as *Wigner function* or *Wigner distribution* shows the phase space formulation of quantum mechanics. The function defined by Wigner is not unique, and there are more such functions generally known as the quasi-probability distribution function.
3. The quasi probability distributions are similar to the probability distribution function but do not satisfies all the axioms required to call them probability distribution functions. For example, they are not always positive definite and normalized to 1. However, it acts as a standard tool to study the quantum-classical interface.

[15] Wigner, E. P. (1932). *On the quantum correction for thermodynamic equilibrium.* Physical Review 40(5), 749-759.

- The motivation of us to calculate the Wigner distribution of these solitons is due to the behaviour of these solitons, such *that soliton forms the solution of semi-classical approximation in the second quantized relativistic field theory*.[16]
- Moreover, the Wigner distribution also emphasises a similar idea as a quasiprobability distribution.
- We have given a short glimpse of how the Wigner distribution helps us to find the Classical speed limit time , semi-classical speed limit time , and Quantum speed limit time in the present context of solitons towards the end, as the quantum speed limit time forms the foundations of quantum information computing.

[16] R. Rajaraman. *An Introduction to Solitons and Instantons in Quantum Field Theory*. Amsterdam: North-Holland, 1982.

Wigner distributions for the kink soliton

$$W(x,p) = \frac{1}{h}\int_a^b dy\psi(x+y)\psi^*(x-y)e^{\frac{iPy}{\hbar}} \tag{39}$$

The Lagrangian density for the simple kink soliton is written by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\psi\partial_\mu\psi + V(\psi)$$

where $V(\psi)$ represents the potential energy which can be given by

$$V(\psi) = \frac{-\mu^2}{2}\psi^2 + \frac{\lambda}{4}\psi^4 + \frac{\mu^2}{4\lambda}$$

The simplest kink soliton can be given by

$$\psi(x) = a\tanh\left(\sqrt{\frac{\lambda}{2}}ax\right) \tag{40}$$

WD of Kink Solitons [17]

Substituting the value of the wave function

$$W(x,p) = \frac{1}{h}\int_{-a}^{a} dy \left[a \tanh\left(\sqrt{\frac{\lambda}{2}}a(x+y)\right) a \tanh\left(\sqrt{\frac{\lambda}{2}}a(x-y)\right) e^{\frac{iPy}{\hbar}} \right] \tag{41}$$

Therefore, upon substituting the boundary values we obtain the Wigner distribution of the kink soliton as

$$W(x,p) = \frac{2a^3}{h}\left[\tanh^2\left(\sqrt{\frac{\lambda}{2}}ax\right) - \frac{2p^2a^2}{9\hbar}\tanh^2\left(\sqrt{\frac{\lambda}{2}}ax\right)\right] \tag{42}$$

[17] Radhakrishnan, R., and Ojha, V. K. (2022). *Quasiprobability distribution of Classical solitons*. arXiv preprint arXiv:2205.02531.

WD of Kink [18]

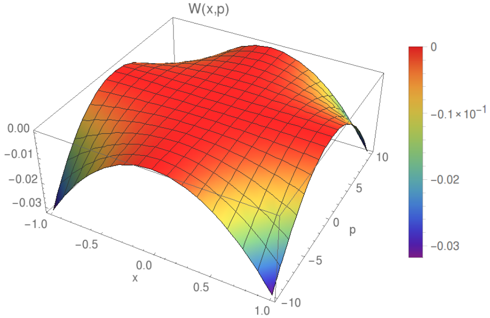

Figure: Wigner distribution for Kink soliton; For $-a < x < a$ and $a = 10^{-10}m$

[18] Radhakrishnan, R., and Ojha, V. K. (2022). *Quasiprobability distribution of Classical solitons*. arXiv preprint arXiv:2205.02531.

WD of Sine gordan solitons [19]

The function of the Sine-Gordan soliton can be given by

$$\psi_{SG} = \frac{4}{\beta}\tan^{-1}[e^{\sqrt{\alpha}\beta x}] \tag{43}$$

With suitable boundary conditions we obtain the Wigner distribution of the Sine-Gordan soliton as

$$W(x,p) = \frac{2b}{h}\left[[\tan^{-1}(e^{\sqrt{\alpha}}\beta x)]^2 - b^2\left[\frac{p^2}{3\hbar^2}[\tan^{-1}(e^{\sqrt{\alpha}}\beta x)]^2 + \frac{\alpha\beta^2 e^{2\sqrt{\alpha}\beta x}}{3(e^{2\sqrt{\alpha}\beta x}+1)^2}\right] + \mathcal{O}(b^3)\right] \tag{44}$$

[19] Radhakrishnan, R., and Ojha, V. K. (2022). *Quasiprobability distribution of Classical solitons*. arXiv preprint arXiv:2205.02531.

WD of SG soliton [20]

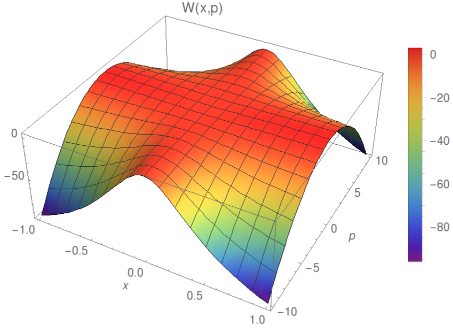

Figure: Wigner distribution for Sine-Gordan soliton; For $-b < x < b$ and $b = 10^{-10}m$

[20] Radhakrishnan, R., and Ojha, V. K. (2022). *Quasiprobability distribution of Classical solitons*. arXiv preprint arXiv:2205.02531.

Concluding remarks

1. We used the Wigner distribution to calculate the current and charge distributions.
2. The Wigner distributions help us to find the quantum speed limit time and other quantum information entropies.
3. Next week, I will introduce with those calculation and subsequently will discuss the following:
    - Polykov, t'Hooft monopoles
    - Julia -zee dyons
    - Euclidean instantons
    - Instantons in Yang mills theory
    - A naive introduction towards dualities

Thank you very much for your attention