Poincare algebra of scaled interpolating variables between Instant Form Dynamics and Light-Front Dynamics

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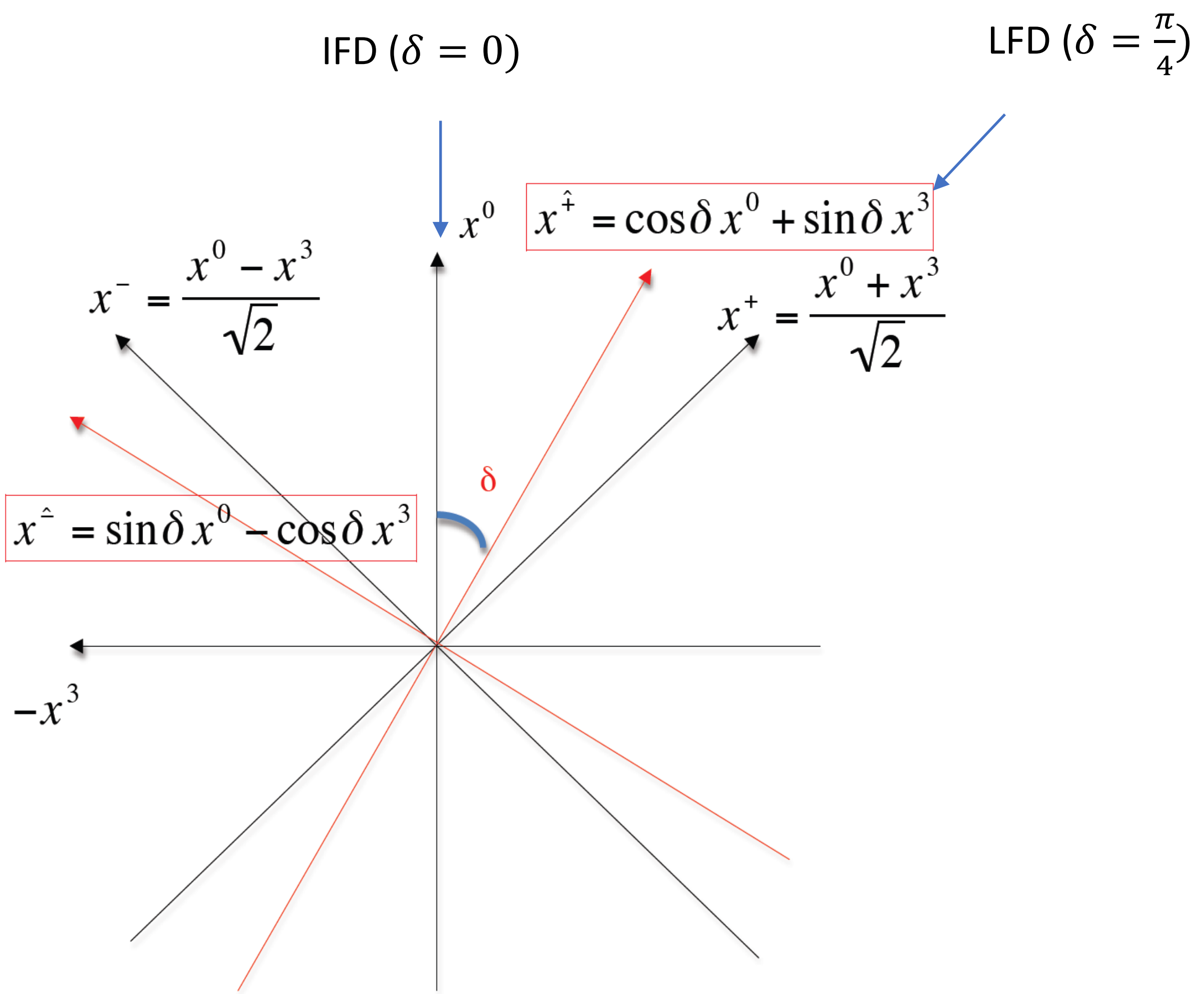
Interpolating Dynamic



$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos(\delta) & 0 & 0 & \sin(\delta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\delta) & 0 & 0 & -\cos(\delta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolation space time matrix

$$g_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{bmatrix}$$

$\mathbb{S} = Sin(2\delta)$
$\mathbb{C} = Cos(2\delta$

- Relate IFD and LFD, and show whole landscape in between
- Investigate the zero-mode by varying the δ parameter.
- Clarify any conceivable confusion between Infinite momentum frame and LFD

- We connect two relativistic dynamics, proposed by Dirac

P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949)

K. Hornsbostel, Phys. Rev. D **45**, 3781 (1992)

C.-R. Ji, Z. Li, and B. Ma Phys. Rev. D **98**, 036017 (2018)- QED_{3+1}

B. Ma, C.-R. Ji Phys. Rev. D **104**, 036004 (2021)-QCD_{1+1}

Interpolating Poincare Matrix

$$M_{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$\mathcal{K}^{\hat{1}} = -K^{\hat{1}} \sin\delta - J^{\hat{2}} \cos\delta,$$
$$\mathcal{K}^{\hat{2}} = J^{\hat{1}} \cos\delta - K^{\hat{2}} \sin\delta,$$
$$\mathcal{D}^{\hat{1}} = -K^{\hat{1}} \cos\delta + J^{\hat{2}} \sin\delta,$$
$$\mathcal{D}^{\hat{2}} = -J^{\hat{1}} \sin\delta - K^{\hat{2}} \cos\delta.$$

TABLE I. Kinematic and dynamic generators for different interpolation angles

	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3$
$0 \le \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2$

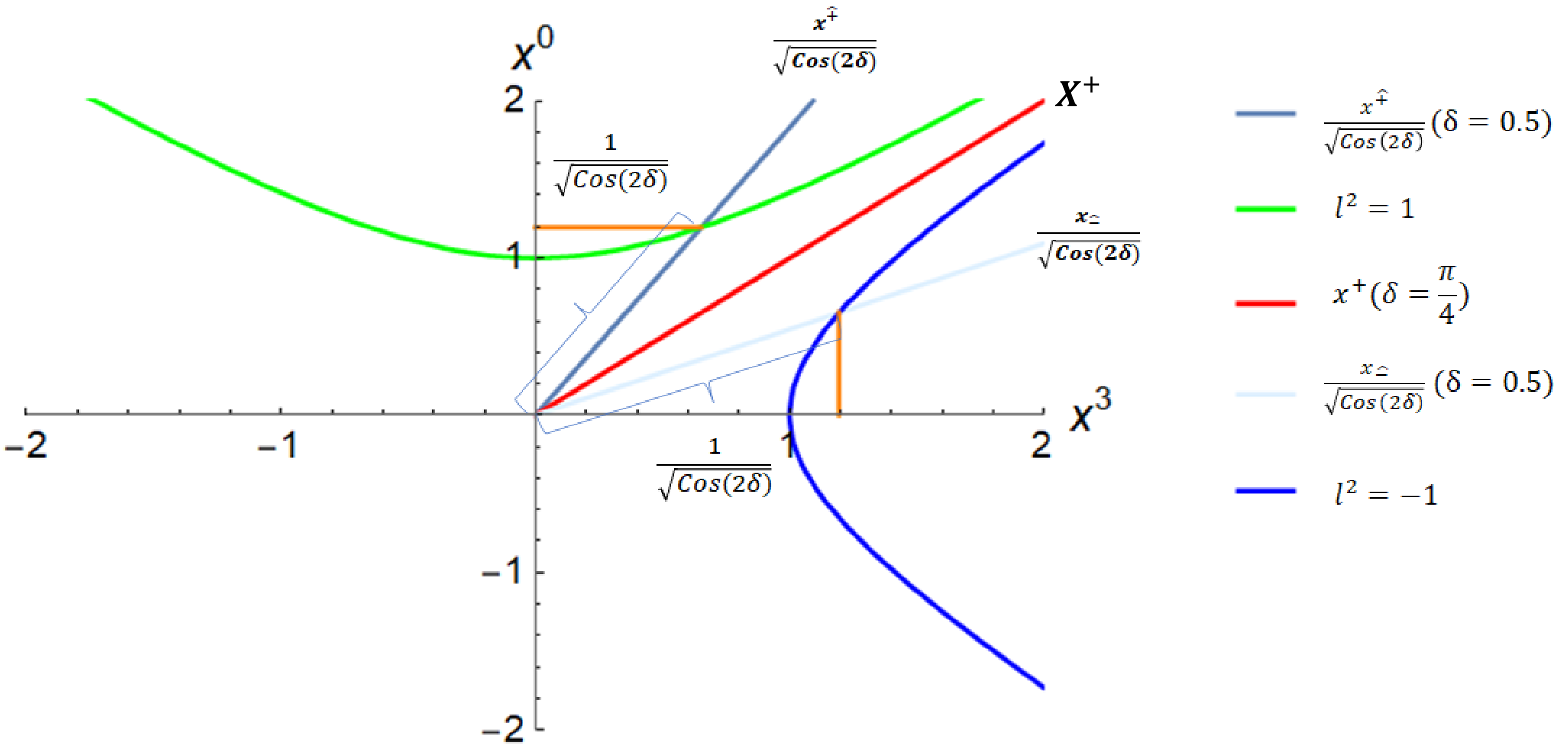
- Among these Poincare generators, these three generators are always kinematic in the sense that the $x^{\hat{+}} = 0$ plane is intact under the transformation generated by them.

  $\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3$

- Light-Front dynamics (LFD) has one more kinematic operator than the Instant Form dynamic (IFD).

  K^3

Novel Scaled Interpolating Variables



$$x^N = H.x$$

$$\begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\sin\delta}{\sqrt{\mathbb{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\cos\delta}{\sqrt{\mathbb{C}}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}$$

$$x^- = \frac{x^0 - x^3}{\sqrt{2}}$$

IFD　　　　LFD

$$l^2 = \underbrace{(x^0)^2 - (x^3)^2}_{\text{IFD}} = \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 = \underbrace{2\,x^+ x^-}_{\text{LFD}}$$

Boost operators

$$(x^0, x^1, x^2, x^3)$$

$$(x^{\hat{+}}/\sqrt{\mathbb{C}}, x^{\hat{1}}, x^{\hat{2}}, x_{\hat{-}}/\sqrt{\mathbb{C}})$$

$$e^{(-i\beta_x K_1)}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_x) & \sinh(\beta_x) & 0 & 0 \\ \sinh(\beta_x) & \cosh(\beta_x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cosh(\beta_x)+1)\mathbb{C}+(\cosh(\beta_x)-1)}{2\mathbb{C}} & \frac{\sinh(\beta_x)\cos\delta}{\sqrt{\mathbb{C}}} & 0 & -\frac{(\cosh(\beta_x)-1)\mathbb{S}}{2\mathbb{C}} \\ \frac{\sinh(\beta_x)\cos\delta}{\sqrt{\mathbb{C}}} & \cosh(\beta_x) & 0 & -\frac{\sinh(\beta_x)\sin\delta}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{(\cosh(\beta_x)-1)\mathbb{S}}{2\mathbb{C}} & \frac{\sinh(\beta_x)\sin\delta}{\sqrt{\mathbb{C}}} & 0 & \frac{(\cosh(\beta_x)+1)\mathbb{C}-(\cosh(\beta_x)-1)}{2\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{(-i\beta_y K_2)}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_y) & 0 & \sinh(\beta_y) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(\beta_y) & 0 & \cosh(\beta_y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cosh(\beta_y)+1)\mathbb{C}+(\cosh(\beta_y)-1)}{2\mathbb{C}} & 0 & \frac{\sinh(\beta_y)\cos\delta}{\sqrt{\mathbb{C}}} & -\frac{(\cosh(\beta_y)-1)\mathbb{S}}{2\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sinh(\beta_y)\cos\delta}{\sqrt{\mathbb{C}}} & 0 & \cosh(\beta_y) & -\frac{\sinh(\beta_y)\sin\delta}{\sqrt{\mathbb{C}}} \\ \frac{(\cosh(\beta_y)-1)\mathbb{S}}{2\mathbb{C}} & 0 & \frac{\sinh(\beta_y)\sin\delta}{\sqrt{\mathbb{C}}} & \frac{(\cosh(\beta_y)+1)\mathbb{C}-(\cosh(\beta_y)-1)}{2\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{(-i\beta_z K_3)}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_z) & 0 & 0 & \sinh(\beta_z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta_z) & 0 & 0 & \cosh(\beta_z) \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- Form Invariant and independent of interpolation angle

Rotation operators

$$(x^{\hat{+}}/\sqrt{\mathbb{C}}, x^{\hat{1}}, x^{\hat{2}}, x_{\hat{-}}/\sqrt{\mathbb{C}})$$

$$(x^0, x^1, x^2, x^3)$$

$$e^{-iJ_1\theta_x}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cos\delta^2 - \cos\theta_x \sin\delta^2)}{\mathbb{C}} & 0 & \frac{\sin\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & -\frac{(\sin(\theta_x/2)^2\mathbb{S})}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sin\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & 0 & \cos\theta_x & -\frac{\cos\delta\sin\theta_x}{\sqrt{\mathbb{C}}} \\ \frac{(\sin(\theta_x/2)^2\mathbb{S})}{\mathbb{C}} & 0 & \frac{\cos\delta\sin\theta_x}{\sqrt{\mathbb{C}}} & \frac{(\cos\delta^2\cos\theta_x - \sin\delta^2)}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{-iJ_2\theta_y}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{(\cos\delta^2 - \cos\theta_y \sin\delta^2)}{\mathbb{C}} & -\frac{\sin\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & 0 & -\frac{(\sin(\theta_y/2)^2\mathbb{S})}{\mathbb{C}} \\ -\frac{\sin\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & \cos\theta & 0 & \frac{\cos\delta\sin\theta_y}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{(\sin(\theta_y/2)^2\mathbb{S})}{\mathbb{C}} & -\frac{\cos\delta\sin\theta_y}{\sqrt{\mathbb{C}}} & 0 & \frac{(\cos\delta^2\cos\theta_y - \sin\delta^2)}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{-iJ_3\theta_z}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- Form Invariant and independent of interpolation angle

Kinematic Generator $\mathcal{K}^{\hat{1}}$

$$\mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta$$

$$e^{i\beta_1 \mathcal{K}^{\hat{1}}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_1 \sin\delta^2}{\mathbb{C}} & \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ \frac{\sin\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & \cos\alpha_1 & 0 & \frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ -\frac{\sin(\alpha_1/2)^2 \mathbb{S}}{\mathbb{C}} & -\frac{\cos\delta \sin\alpha_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2 \cos\alpha_1 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\alpha_1 = \beta_1 \sqrt{\mathbb{C}}$$

$\mathcal{K}^{\hat{1}} \to -J^2$

IFD

$$e^{-i\beta_1 J^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{Cos}[\beta1] & 0 & \text{Sin}[\beta1] \\ 0 & 0 & 1 & 0 \\ 0 & -\text{Sin}[\beta1] & 0 & \text{Cos}[\beta1] \end{pmatrix}$$

β_1 is an angle

$\mathcal{K}^{\hat{1}} \to -E_1$

LFD

$$e^{-i\beta_1 E_1} = \begin{pmatrix} \frac{1}{4}(4+\beta1^2) & \frac{\beta1}{\sqrt{2}} & 0 & \frac{\beta1^2}{4} \\ \frac{\beta1}{\sqrt{2}} & 1 & 0 & \frac{\beta1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ -\frac{\beta1^2}{4} & -\frac{\beta1}{\sqrt{2}} & 0 & 1-\frac{\beta1^2}{4} \end{pmatrix}$$

β_1 is the rapidity

$e^{i\beta_1 \mathcal{K}^{\hat{1}}}$ $(x^{\hat{+}}/\sqrt{\mathbb{C}}, x^{\hat{1}}, x^{\hat{2}}, x_{\hat{-}}/\sqrt{\mathbb{C}})$

$$\alpha_1 = \beta_1\sqrt{\mathbb{C}}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha_1 & 0 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- $\mathcal{K}^{\hat{1}}$ plays the role of rotation around y axis for all interpolation angles in this new basis

- $\mathcal{K}^{\hat{2}}$ plays the role of rotation around x axis for all interpolation angles in this new basis

- Kinematic operators exclusively independent of interpolation angles in the new basis

Kinematic Generator $\mathcal{K}^{\hat{2}}$

$$\mathcal{K}^{\hat{2}} = J^1\cos\delta - K^2\sin\delta$$

$e^{i\beta_2 \mathcal{K}^{\hat{2}}}$

$$\alpha_2 = \beta_2\sqrt{\mathbb{C}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 - \cos\alpha_2\sin\delta^2}{\mathbb{C}} & 0 & \frac{\sin\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} & \frac{\sin(\alpha_1/2)^2\mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\sin\delta\sin\alpha_2}{\sqrt{\mathbb{C}}} & 0 & \cos\alpha_2 & \frac{\cos\delta\sin\alpha_1}{\sqrt{\mathbb{C}}} \\ -\frac{\sin(\alpha_2/2)^2\mathbb{S}}{\mathbb{C}} & 0 & -\frac{\cos\delta\sin\alpha_2}{\sqrt{\mathbb{C}}} & \frac{\cos\delta^2\cos\alpha_2 - \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & \sin\alpha_2 \\ 0 & 0 & -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

Dynamic Generator $\mathcal{D}^{\hat{1}}$

$$e^{i\eta_1 \mathcal{D}^{\hat{1}}} \qquad \mathcal{D}^{\hat{1}} = -K^1 \cos\delta + J^2 \sin\delta$$

$$\rho_1 = \sqrt{\eta_1^2 \cos 2\delta} = \sqrt{\eta_1^2 \mathbb{C}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 \cosh\rho_1 - \sin\delta^2}{\mathbb{C}} & \frac{\cos\delta \sinh\rho_1}{\sqrt{\mathbb{C}}} & 0 & -\frac{\sinh(\rho_1/2)^2 \mathbb{S}}{\mathbb{C}} \\ \frac{\cos\delta \sinh\rho_1}{\sqrt{\mathbb{C}}} & \cosh\rho_1 & 0 & -\frac{\sin\delta \sinh\rho_1}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{\sinh(\rho_1/2)^2 \mathbb{S}}{\mathbb{C}} & \frac{\sin\delta \sinh\rho_1}{\sqrt{\mathbb{C}}} & 0 & \frac{\cos\delta^2 - \cosh\rho_1 \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{(-1+\cosh\rho_1)}{\mathbb{C}^2} & \frac{\sinh\rho_1}{\mathbb{C}} & 0 & -\frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} \\ \frac{\sinh\rho_1}{\mathbb{C}} & \cosh\rho_1 & 0 & -\frac{\sinh\rho_1 \mathbb{S}}{\mathbb{C}} \\ 0 & 0 & 1 & 0 \\ \frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} & \frac{\sinh\rho_1 \mathbb{S}}{\mathbb{C}} & 0 & 1 - \frac{(-1+\cosh\rho_1)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

Dynamic Generator $\mathcal{D}^{\hat{2}}$

$$e^{i\eta_2 \mathcal{D}^{\hat{2}}} \qquad \mathcal{D}^{\hat{2}} = -K^2 \cos\delta - J^1 \sin\delta$$

$$\rho_2 = \sqrt{\eta_2^2 \cos 2\delta} = \sqrt{\eta_2^2 \mathbb{C}}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta^2 \cosh\rho_2 - \sin\delta^2}{\mathbb{C}} & 0 & \frac{\cos\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & -\frac{\sinh(\rho_2/2)^2 \mathbb{S}}{\mathbb{C}} \\ 0 & 1 & 0 & 0 \\ \frac{\cos\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & 0 & \cosh\rho_2 & -\frac{\sin\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} \\ \frac{\sinh(\rho_2/2)^2 \mathbb{S}}{\mathbb{C}} & 0 & \frac{\sin\delta \sinh\rho_2}{\sqrt{\mathbb{C}}} & \frac{\cos\delta^2 - \cosh\rho_2 \sin\delta^2}{\mathbb{C}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{(-1+\cosh\rho_2)}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2}{\mathbb{C}} & -\frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} \\ 0 & 1 & 0 & 0 \\ \frac{\sinh\rho_2}{\mathbb{C}} & 0 & \cosh\rho_2 & -\frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} \\ \frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} & 1 - \frac{(-1+\cosh\rho_2)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

Generalized Helicity operator

$$M_{\hat\mu\hat\nu} = \begin{pmatrix} 0 & \mathcal{D}^{\hat 1} & \mathcal{D}^{\hat 2} & K^3 \\ -\mathcal{D}^{\hat 1} & 0 & J^3 & -\mathcal{K}^{\hat 1} \\ -\mathcal{D}^{\hat 2} & -J^3 & 0 & -\mathcal{K}^{\hat 2} \\ -K^3 & \mathcal{K}^{\hat 1} & \mathcal{K}^{\hat 2} & 0 \end{pmatrix}$$

- $\mathcal{D}^{\hat 1}, \mathcal{D}^{\hat 2}$ are dynamic operators in all interpolation angle
- $\mathcal{K}^{\hat 1}, \mathcal{K}^{\hat 2}$ and J_3 are kinematic operators in all interpolation angle and K^3 is a kinematic operator exactly at the LF
- We observe that all kinematic operators exclusively independent of interpolation angle in the new basis and they are form invariant.

$$\mathcal{J}_3|p;j,m\rangle_\delta = TJ_3T^{-1}T|0;j,m\rangle = m|p;j,m\rangle_\delta,$$

$$\mathcal{J}_3 = TJ_3T^{-1}$$

$$T = T_{12}T_3 = e^{i\beta_1\mathcal{K}^{\hat 1}+i\beta_2\mathcal{K}^{\hat 2}}e^{-i\beta_3K^3}$$

$$\mathcal{J}_3 = J_3\cos\alpha + (\beta_1J_1+\beta_2J_2)\frac{\sin\alpha}{\sqrt{\beta_1^2+\beta_2^2}}$$

$$\mathcal{J}_3 = \frac{1}{\sqrt{\mathbf{P}_\perp^2+\frac{P_{\hat -}^2}{\mathbb{C}}}}\left(P^1J_1+P^2J_2+\frac{P_{\hat -}}{\sqrt{\mathbb{C}}}J_3\right)$$

$$\mathcal{J}_3 = \frac{\boldsymbol{P^N}.\boldsymbol{J}}{|\boldsymbol{P^N}|}$$

Form invariant with Jacob-Wick Helicity

$$\cos\alpha = \frac{P_{\hat -}}{\mathbb{P}}$$

$$\sin\alpha = \frac{\sqrt{\mathbf{P}_\perp^2\mathbb{C}}}{\mathbb{P}}$$

$$\frac{\beta_j}{\alpha} = \frac{P^j}{\sqrt{\mathbf{P}_\perp^2\mathbb{C}}}$$

$$\alpha = \sqrt{\mathbb{C}(\beta_1^2+\beta_2^2)}$$

$$\frac{\sqrt{\mathbb{C}}}{\mathbb{P}} = \frac{1}{\sqrt{\mathbf{P}_\perp^2+\frac{P_{\hat -}^2}{\mathbb{C}}}}$$

Magnitude of the total momentum in the new basis

$$\vec{P}^N = (P^1, P^2, \frac{P_{\hat -}}{\sqrt{\mathbb{C}}})$$

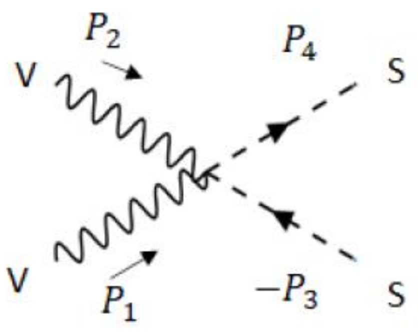
H. Leutwyler and J. Stern, Ann Phys. (N.Y.) **112** 94 (1978)

M. Jacob and G. Wick, Ann. Phys.**7**, 404 (1959)

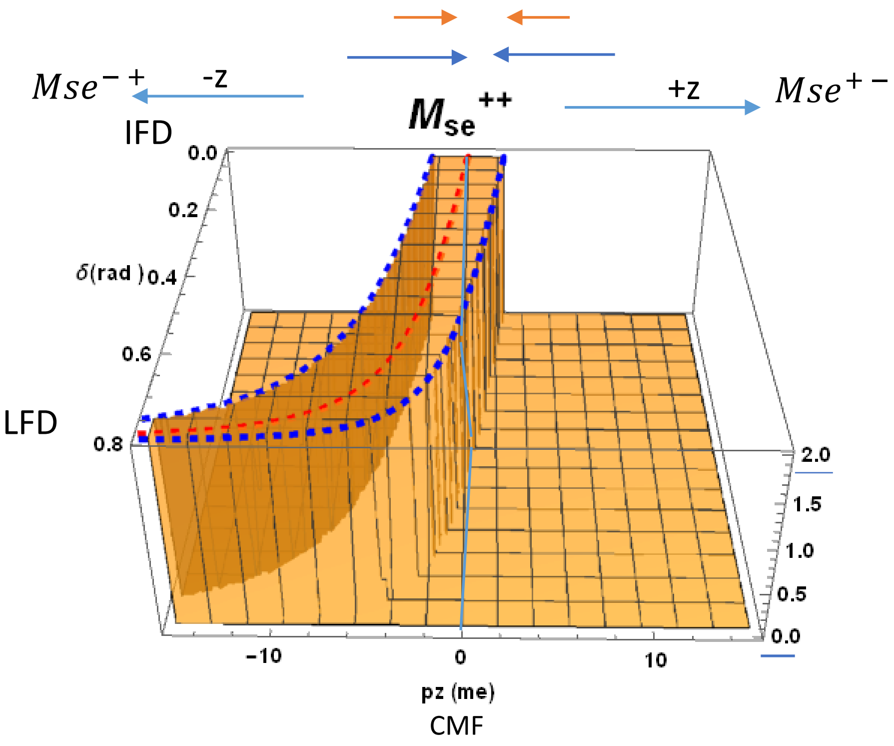
Z.Li, M.An, and C.-R.Ji, Phys. Rev. D. **92**, 105014 (2015)

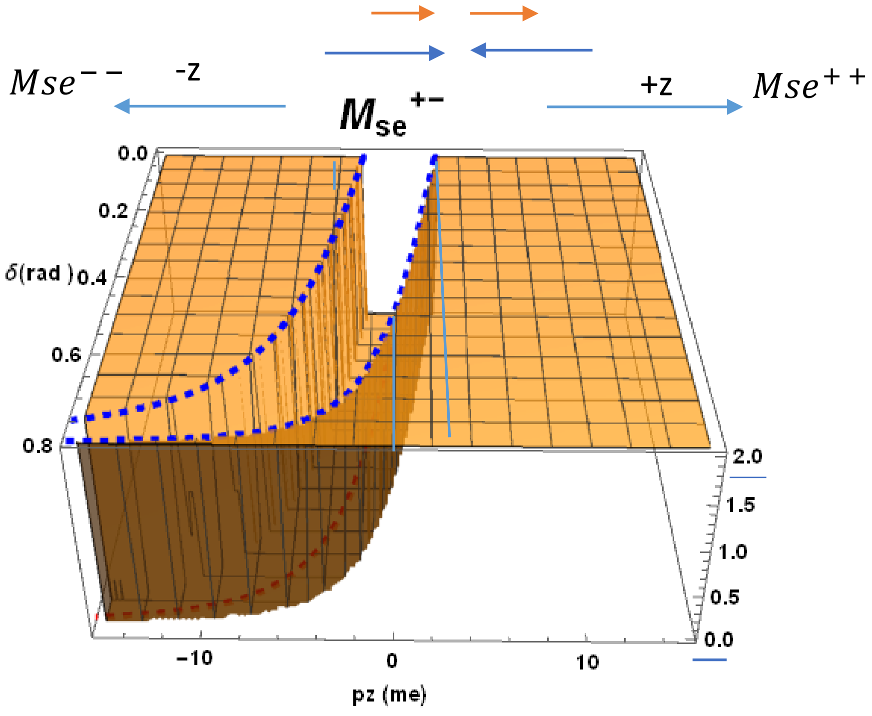
Scalar particle and its anti-particle production by two neutral massive spin-1 particles

- Seagull Channel



$$M_{se}^{\lambda_1\lambda_2} = -2g_{\hat{\mu}\hat{\nu}}\varepsilon^{\hat{\mu}}(p_1,\lambda_1)\,\varepsilon^{\hat{\nu}}(p_2,\lambda_2)$$





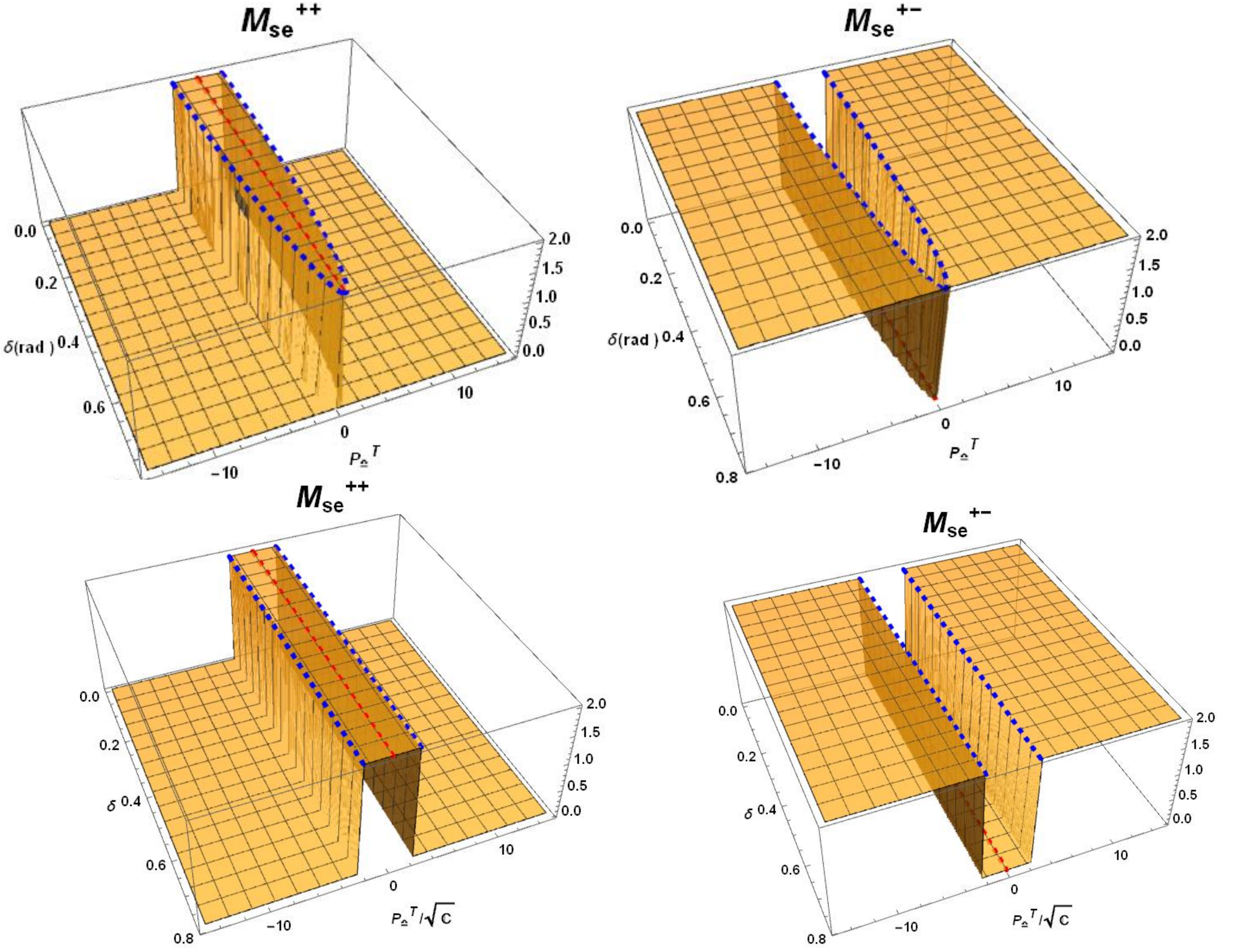
$$(P_{\hat{-}})^T = \sqrt{\bar{E}^2 + (P^z)^2}\sin\delta + P^z\cos\delta$$

- Zero-mode

$$P^z \to -\infty\,,\ (P_{\hat{-}})^T \to 0$$

At $\delta = \frac{\pi}{4}$

$\frac{(P_{\hat{-}})^T}{\sqrt{\mathbb{C}}}$ → Finite Value



Energy-Momentum Dispersion Relation

Instant-form dynamic

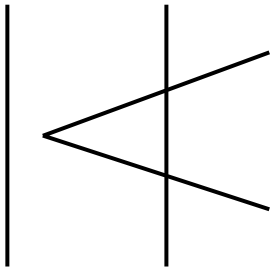
$$P^0 = \sqrt{\vec{P}^2 + m^2}$$

Light-front form dynamic

$$P^- = \frac{P_\perp^2 + m^2}{2P^+}$$

Time-ordered diagrams : Vacuum Fluctuation

Instant time →

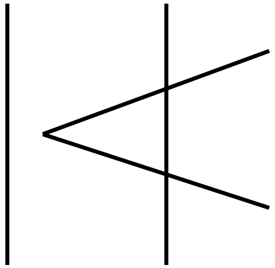


$+|P_z|$

$-|P_z|$

Allowed

Light-front time →

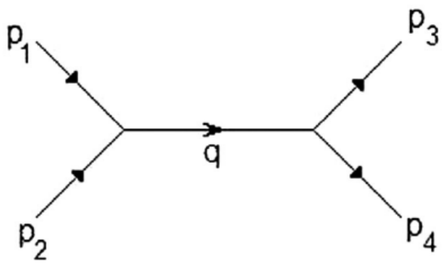


$P_1^+ > 0$

$P_2^+ > 0$

Not Allowed
unless $P_1^+ = P_2^+ = 0$

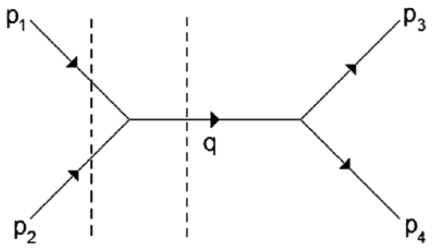
Annihilation and creating of scalar particles



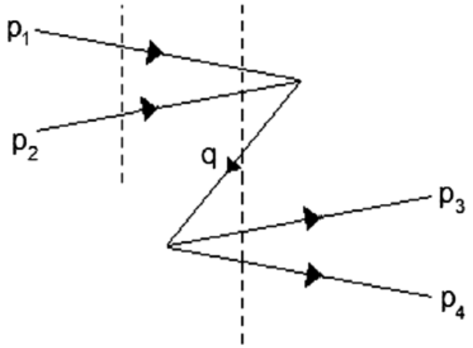
$$\Sigma = \frac{1}{s - m^2},$$

$$s = (p_1 + p_2)^2$$

(a)



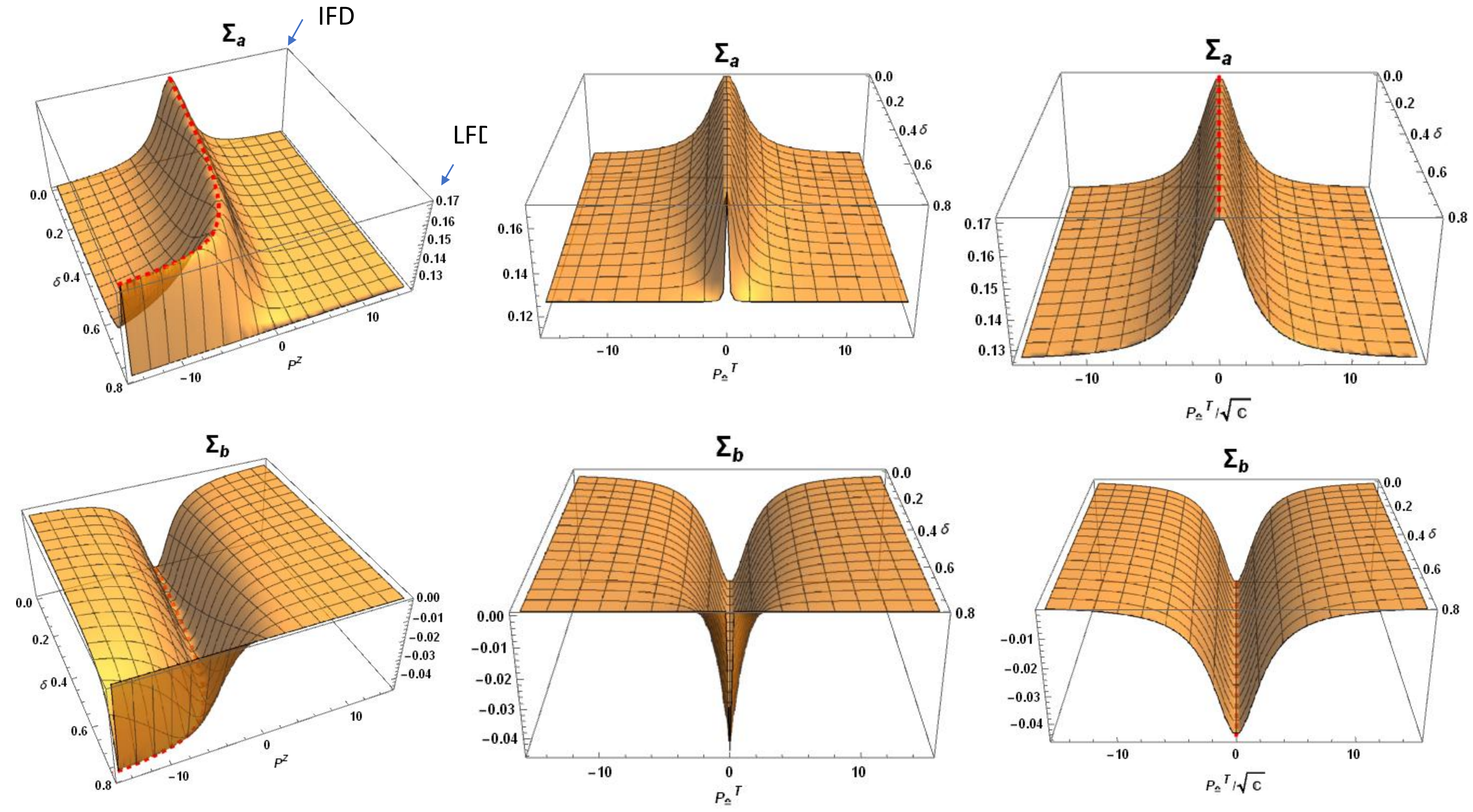
(b)



$$\Sigma_\delta^a = \frac{1}{2q^{\hat{+}}}\left(\frac{\mathbb{C}}{p_1^{\hat{+}} + p_2^{\hat{+}} - q^{\hat{+}}}\right)$$

$$\Sigma_\delta^b = -\frac{1}{2q^{\hat{+}}}\left(\frac{\mathbb{C}}{p_1^{\hat{+}} + p_2^{\hat{+}} + q^{\hat{+}}}\right).$$

Time-ordered diagram (b) does not exist in LFD



IFD
LF
Σa
Σb
δ
P^z
P⊥^T
P⊥^T/√C

Conclusion

- Novel scaled interpolating basis can recover usual dispersion relations in IFD , LFD also in between
- It can produce form-invariance characteristics of kinematic operators which clearly distinguishes kinematic operators from dynamic operators.
- The functions of kinematic operators also can preserve the form invariance in the scaled interpolating basis.- Generalized helicity operator
- Scale interpolating variables can be used to magnify the vicinity of zero modes which enables us to understand more physics in the zero mode.

Thank you

Poincare Algebra in the new basis (45 commutation relations)

	$\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}$	$P^{\hat{1}}$	$P^{\hat{2}}$	$\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	J^3	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{1}}$	K^3
$\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}$	0	0	0	0	$\frac{iP^{\hat{1}}}{\sqrt{\mathbb{C}}}$	$\frac{iP^{\hat{2}}}{\sqrt{\mathbb{C}}}$	0	0	0	$\frac{iP_{\hat{-}}}{\sqrt{\mathbb{C}}}$
$P^{\hat{1}}$	0	0	0	0	$\frac{-iP^{\hat{+}}}{\mathbb{C}} + \frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$	0	$-iP^{\hat{2}}$	$-iP_{\hat{-}}$	0	0
$P^{\hat{2}}$	0	0	0	0	0	$\frac{-iP^{\hat{+}}}{\mathbb{C}} + \frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$	$iP^{\hat{1}}$	0	$-iP_{\hat{-}}$	0
$\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}$	0	0	0	0	$\frac{-i\mathbb{S}P^{\hat{1}}}{\sqrt{\mathbb{C}}}$	$\frac{-i\mathbb{S}P^{\hat{2}}}{\sqrt{\mathbb{C}}}$	0	$i\sqrt{\mathbb{C}}P^{\hat{1}}$	$i\sqrt{\mathbb{C}}P^{\hat{2}}$	$\frac{iP^{\hat{+}}}{\sqrt{\mathbb{C}}}$
$\mathcal{D}^{\hat{1}}$	$-\frac{iP^{\hat{1}}}{\sqrt{\mathbb{C}}}$	$\frac{iP^{\hat{+}}}{\mathbb{C}} - \frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}})$	0	$\frac{i\mathbb{S}P^{\hat{1}}}{\sqrt{\mathbb{C}}}$	0	$-i\mathbb{C}J^3$	$-i\mathcal{D}^{\hat{2}}$	$-iK^3$	$-i\mathbb{S}J^3$	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$
$\mathcal{D}^{\hat{2}}$	$-\frac{iP^{\hat{2}}}{\sqrt{\mathbb{C}}}$	0	$\frac{iP^{\hat{+}}}{\mathbb{C}} - \frac{i\mathbb{S}P_{\hat{-}}}{\mathbb{C}}$	$\frac{i\mathbb{S}P^{\hat{2}}}{\sqrt{\mathbb{C}}}$	$i\mathbb{C}J^3$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^3$	$-iK^3$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$
J^3	0	$iP^{\hat{2}}$	$-iP^{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0
$\mathcal{K}^{\hat{1}}$	0	$iP_{\hat{-}}$	0	$-i\sqrt{\mathbb{C}}P^{\hat{1}}$	iK^3	$-i\mathbb{S}J^3$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^3$	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$
$\mathcal{K}^{\hat{2}}$	0	0	$iP_{\hat{-}}$	$-i\sqrt{\mathbb{C}}P^{\hat{2}}$	$i\mathbb{S}J^3$	iK^3	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^3$	0	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$
K^3	$\frac{-iP_{\hat{-}}}{\sqrt{\mathbb{C}}}$	0	0	$\frac{-iP^{\hat{+}}}{\sqrt{\mathbb{C}}}$	$\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	0

- Apply to a rest particle of mass M with spin S $P = \{M, 0,0,0\}$ $S = \{0,0,0,1\}$

IFD

$e^{-i\beta_1 J^2}$

$P' = \{M, 0,0,0\}$

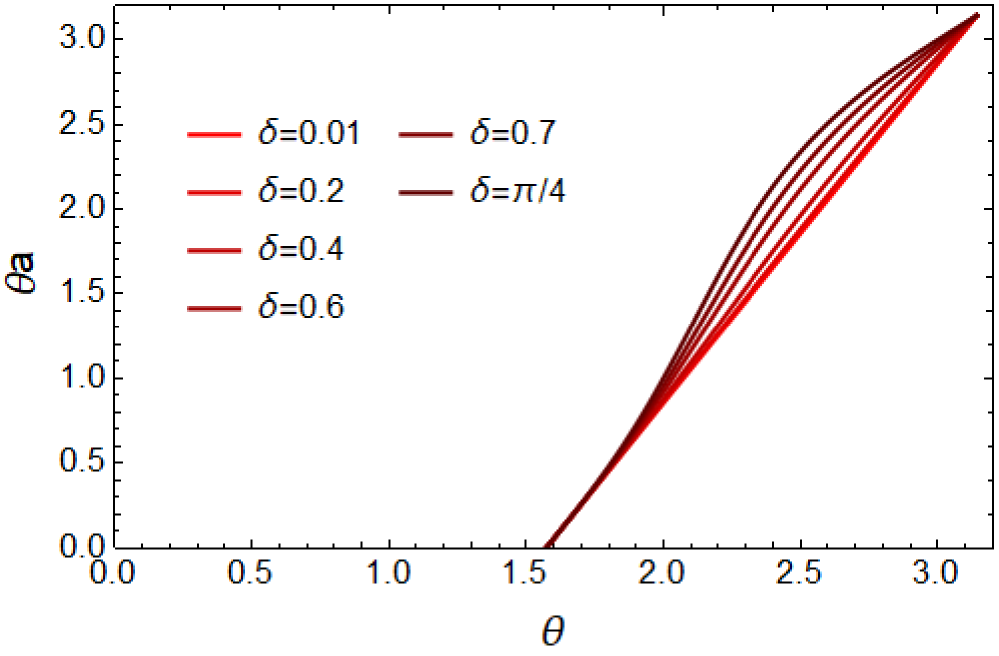
$S' = \{0, \mathrm{Sin}[\beta 1], 0, \mathrm{Cos}[\beta 1]\}$

LFD

$P' = \left\{\frac{1}{4}M(4+\beta 1^2), \frac{M\beta 1}{\sqrt{2}}, 0, -\frac{M\beta 1^2}{4}\right\}$

$S' = \left\{\frac{\beta 1^2}{4}, \frac{\beta 1}{\sqrt{2}}, 0, 1-\frac{\beta 1^2}{4}\right\}$

$\mathrm{Tan}[\theta] = -\frac{2\sqrt{2}}{\beta 1}$



$Cos[\theta a] = \frac{4-\beta 1^2}{\sqrt{16+\beta 1^4}}$

$Cos[\theta a] = \frac{1-2\mathrm{Cot}[\theta]^2}{\sqrt{1+4\mathrm{Cot}[\theta]^4}}$