A pedagogical introduction to Solitons in Classical field theory - II

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Outline

- Polyakov - tHooft Monopoles
- Dyons
- Instantons
- Quantum Information theory and Classical Solitons

Polyakov-tHooft Monopoles [1]

[1] David Tong. TASI lectures on solitons: Instantons, monopoles, vortices and kinks. 2005.

Monopoles

Following Dirac's interpretation on monopole, two scientists Polyakov and Gerad t' Hooft gave an interpretation for monopoles which was widely accepted in early 1970s. The method adopted by Polyakov and t'Hooft are discussed

t'Hooft approach

According to t'Hooft, *Gauge theories in which* $U(1)$ *taken as a subgroup with a compact covering group like,* $SU(2)$ *or* $SU(3)$*, genuine magnetic monopoles can be created (or) regular solutions of field equations.*
Let us consider $SU(3)$ or $SU(3)$ broken to $U(1)$. The system can be represented as,

$$\mathcal{L} = \frac{-1}{4} F^{\mu\nu a} F^{a}_{\mu\nu} - \frac{1}{2} \Pi^{\mu a} \Pi^{a}_{\mu} - U(\phi) \tag{1}$$

where, $F^{a}_{\mu\nu} = \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu}$
$\Pi^{a}_{\mu} = D_\mu \phi^a = \partial_\mu \phi^a + e\epsilon^{abc} A^{b}_{\mu} \phi^{c}_{\nu}$
$U(\phi) = \frac{\lambda}{4} \left[\phi^2 - \frac{\mu^2}{\lambda} \right]^2$
The minima occurs when $\frac{\partial U}{\partial \phi} = 0$. So we get $\phi^2 = \frac{\mu^2}{\lambda}$. When $\lambda \to 0$ $\mu \to$ constant inorder to make ϕ^2 constant. The lagrangian through the Higgs mechanism gives rise to 2 masses through 2 vector fields namely, $W^{\pm}$ Bosons. The mass of W bosons,

$$M_w = \frac{e\mu}{\sqrt{\lambda}} \tag{2}$$

t'Hooft ansatz

The $U(1)$ interactions would give rise to the Higgs mass. The solution we are interested are time independent and with finite energy. Initially we transform the co-ordinates in spherical polar co-ordinate system,

$$(x, y, z) \rightarrow (r, \theta, \phi)$$

So we want the solution to be spherically symmetrical.Therefore,

$$\phi_a(x, t) = r_a \phi(r) \tag{3}$$

By the ansatz of t'Hooft we have ,

$$A_0^a = \frac{J(r)}{er}$$

$$A_\mu^a = \epsilon_{\mu ab} r_b A(r)$$

The derivation of this ansatz will be explained in the Bogomolnyi trick section of Dyons. Thus converting the lagrangian in to spherical polar co-ordinates we have,

$$L = 4\pi \int_0^\infty r^2 dr \Big[- r^2 (\frac{dA}{dr})^2 - 4rA\frac{dA}{dr} - 6A^2 - 2er^2A^3 - \frac{1}{2}r^2(\frac{d\phi}{dr})^2 - rA\frac{d\phi}{dr} - \frac{3}{2}\phi^2 - 2e \tag{4}$$

t'Hooft approach - Continued

Since, $A(r), \phi(r)$ doesnot depend on time. The Lagrangian can be considered stationary. When $r \to \infty$ ϕ^a should vanish.

$$\phi_a(r) = \frac{\mu}{\sqrt{\lambda}} . \frac{1}{r} \tag{5}$$

$$A(r) = ar^{-n} \tag{6}$$

Substituting the value of $A(r)$ in the previous equation with the condition $r \to \infty$ we get the value of $n = 2$ and $a = \frac{-1}{e}$. So we get the following ,

$$A^a_\mu = \frac{\epsilon_{\mu ab} r_b}{er^2}$$

$$\phi_a = \frac{\mu}{\sqrt{\lambda}} \frac{r_a}{r}$$

$$A^a_0 = \frac{J(r)}{er^2} r^a$$

$$A^a_i = \epsilon_{aij} r_j \frac{1 - k(r)}{er^2}$$

$J(r) = 0$ will give us the time independent solution that nullifies the electric field component and gives us the mmagnetic monopole solution. To find physically observable fields, particularly the EM ones, $F_{\mu\nu}$. We must define gauge invariance which would be true,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e} \epsilon^{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \tag{7}$$

This equation satisfies the Maxwell equation except the source free terms, For,
$F_{0i} = B_i = \frac{-r_a}{er^2}$
Following this we get the Non-Minimal Dirac quantisation,

$$ge = -4\pi \tag{8}$$

with the total flux of $\frac{4\pi}{e}$. This was the approach of t'Hooft. Polyakov gave slightly a modified approach to the monopole solutions.

Polyakov approach

The value of A_μ, ϕ_a which was obtained by the ansatz of t"Hooft. The following differential equations are obtained.

$$r^2 K'' = K(K^2 - 1) + KH^2 \tag{9}$$

$$r^2 H'' = 2HK^2 + \frac{\lambda}{e^2}(H^3 - c^2 r^2 H) \tag{10}$$

When $r \to \infty$ $\lambda \to 0$ to get a finite, time independent solution. Therefore we get,

$$r^2 H'' = 2HK^2 \tag{11}$$

On solving this by taking Prasad - Sommerfield method,

$$H = \left[Crcoth(cr) - 1 \right] \tag{12}$$

$$K = \left[\frac{Cr}{sinhcr} \right] \tag{13}$$

Prasad - Sommerfield obtained the result by the back substitution of these results. By following the steps we get the mass of monopoles. The result obtained by t'Hooft and Polyakov were consistent.

Dyons

A year after Polyakov and t' Hooft gave their interpretation on monopoles. Julia and Zee gave their discovery which generalised the existence of Monopoles. Let us discuss some prerequistes which would be required to understand Dyons better. Consider an Abelian Higgs model ,

$$F_{\mu\nu} = \alpha\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right] + \beta\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c \tag{14}$$

We are interested in Time independent, Finite energy configuration,

$$E = \int d^3x\left[\frac{1}{4}F^a_{ij}F^a_{ij} + \Pi^a_i\Pi^a_i + U(\phi)\right] \tag{15}$$

where, $\Pi^a_i = D_i\phi^a$. Prasad - Sommerfield limit, $\lambda \to 0$ and the minima occurs at $\phi^2 = \frac{\mu^2}{\lambda}$.

Bogomolnyi's method

Bogomolnyi's method of completing the square was very much helpful in getting finite energy solutions. Following is the bogomolnyi's trick in arriving the solutions.

$$\frac{1}{2}|\Pi_i^a \pm \frac{1}{2}\varepsilon_{ijk}F_{jk}^a|^2 \geq 0 \tag{16}$$

$$\implies \frac{1}{2}\Pi_i^a\Pi_i^a + \frac{1}{4}F_{ij}^aF_{ij}^a \pm \frac{1}{2}\Pi_i^a\varepsilon_{ijk}F_{jk}^a \geq 0$$

$$\implies E = \int d^3x\left[\frac{1}{4}F^2 + \frac{1}{2}\Pi^2\right] \geq \mp\frac{1}{2}\int \left(d^3x\varepsilon_{ijk}\Pi_i^aF_{jk}^a\right)$$

$$\implies E = d^3x\left\{\frac{1}{4}F^2 + \frac{1}{2}\Pi^2\right\} \geq \left|\frac{1}{2}\int d^3x\varepsilon_{ijk}\Pi_i^aF_{jk}^a\right|$$

$$\implies E \geq \left|\frac{1}{2}\int d^3x\varepsilon_{ijk}D_i\phi^aF_{jk}^a\right| \tag{17}$$

The equality is saturated. This equation breaks in to two forms.

$$\Pi_i^a + \frac{1}{2}\varepsilon_{ijk}F_{jk}^a = 0 \qquad Q < 0 \tag{18}$$

$$\Pi_i^a - \frac{1}{2}\varepsilon_{ijk}F_{jk}^a = 0 \qquad Q > 0 \tag{19}$$

Dyons

Dyons are substances that carry both electrical and charges. The following conditions are prescribed for dyons:

- P. S. limit : $U(\phi)$ is ignored when $\lambda \to 0$
- We take electric charges, so $A_0{}^a \neq 0$
- Time independent and finite energy

We know that,

$$F_{0i}^a = -\partial_i A_0 + e\varepsilon^{abc} A_0^b A_i^c \implies F_{0i}^a = -D_i A_0^a$$

$$F_{\mu\nu}^a F^{\mu\nu a} = -F_{ij}^a F_{ij}^a + 2(D_i A_0^a)^2$$

$$\mathcal{H} = \frac{1}{2}(D_i A_0^a)^2 + \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2}(D_i \phi^a)^2 \geq 0$$

where $\mathcal{H}$ is finite energy. Let's take a presence of Lorentz symmetry (boosts) of the Lagrangian,

$$\begin{pmatrix} \tilde{A}_0^a \\ \tilde{\phi}^a \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} A_0^a \\ \phi^a \end{pmatrix}$$

Dyons- continued

By the ansatz from t'Hooft monopole,

$$A_i{}^a = \frac{\varepsilon_{aij} r_j (1 - k(r))}{e r^3}$$

$$A_0{}^a = \frac{r_a J(r)}{e r^2}$$

$$\phi^a = \frac{r_a . H(r)}{e r^2}$$

The BPS solution would give rise to monopole solution,

$$K = \frac{C_r}{\sinh C_r}$$

$$J = 0$$

$$H = C_r \coth C_r - 1$$

By boosts,

$$K' = \frac{C_r}{\sinh C_r}$$

$$J' = \left[C_r \coth C_r - 1 \right] \sinh \gamma$$

$$H' = \left[C_r \coth C_r - 1 \right] \cosh \gamma$$

Non-minimal dirac quantisation

This solution will have non-zero electric and magnetic charges,

$$Q_{\text{elec}} = \frac{4\pi}{e} \sinh \gamma$$

We have two dyons of charges (e_1, g_1) and (e_2, g_2). By Dirac quantisation,

$$\Big[e_1 g_1 - e_2 g_2 \Big] = 4n\pi$$

$$\implies eg = 4n\pi$$

This implies non- minimal Dirac quantisation.

Instantons

Instantons

1. The solution of classical field equations can be applied (or) observed in some problems in Quantum Mechanics like Tunneling.
2. Semi-Classical approach can be used in theories with small coupling constants which would give rise to Instantons or solutions in Euclidean space time which would play an important role.
3. The tunneling probabilty can be obtained from these instantons. The term *Instantons* was coined by *t'Hooft*.
4. The study of instantons plays an important role in the study early universe processes in cosmology, study of the ground state or the vacua state in the gauge theories in particle physics and also in condensed matter systems.

Wick Rotation

It is a method of finding the solution to a mathematical problem in a space time from a solution in Euclidean space by a transformation.

$$t \to i\tau$$

When $t \to -\infty$ the system tends to $-a$ and when $t \to \infty$ the system tends to a. The Wick rotated solution appear to capture the process of tunneling from one vacuum to the other. The tunneling probability can be obtained through the study of instantons.

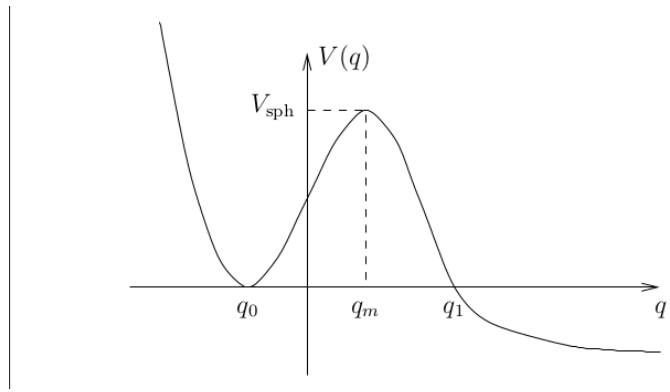


Figure: Tunneling of Instantons

The Wick rotated solution appears to capture the process of tunneling from one vacuum to another. Consider the system in $1+1$ dimensions the equation of the one dimensional system can be given by,

$$\frac{\bar{h}\kappa^2}{2m} = V(q) - E \tag{20}$$

where, $\kappa = \frac{\sqrt{2m(V(q)-E)}}{\bar{h}}$. By WKB approximation the transmission probability which occurs by the process of tunneling,

$$T \equiv exp\left[\int dq\kappa(q)\right]\left[1+\theta(\bar{h})\right] \tag{21}$$

When $h \to 0$ the solution is not defined, the perturbation approach cannot be used. So we can use path integral formulation to compute this non- perturbation stuff.

$$< \frac{T}{2}, q_f / \frac{-T}{2}, q_i > \approx \int [dq] e^{\frac{iS(q(t))}{\bar{h}}} \tag{22}$$

Upon Wick's rotation $t \to i\tau$ the action transforms as following,

$$S[q(t)] = \int dt L(q, \dot{q}) \to \int d\tau L(-i\dot{q}, q)$$

Thus, $\frac{dq}{dt} = \frac{-idq}{d\tau}$. The transformed Lagrangian of the system ,

$$\tilde{L} = \frac{-m}{2}(\frac{dq}{d\tau})^2 - U(q) \tag{23}$$

Therefore, $U(\phi)$in L. Then in $\tilde{L}$ in $U(\phi)$ would correspond to inverted potential. The solution is the Euclidean Instantons. Instantons are localised in space time with Finite energy. Solitons are localised in space with Finite energy. The Euclidean action is given by,

$$S_E = \int d\tau \left[\frac{m}{2}(\frac{dq}{d\tau})^2 + U(\phi) \right] \tag{24}$$

Euclidean Instantons

At Saddle point,

$$m\frac{d^2q}{d\tau^2} = \frac{dU}{dq} \tag{25}$$

This is the equation of motion. For the mexican hat potential, $U(q) = \frac{\lambda}{2}(a^2 - q^2)^2$ and $\frac{dq}{d\tau} = \pm\sqrt{2V(q)}$. Therefore, the action at saddle point will be given by,

$$S_E = \int_{q=-a}^{q=a} dq\sqrt{2V(q)} \tag{26}$$

Substituting the value of the potential and integrating we get,

$$S_E = \frac{4a^3\sqrt{\lambda}}{3} \tag{27}$$

The Euclidean action occurs at the saddle point. The following observation is made:

- Kink Soliton also has the same mass.
- If $q \to \phi$ and $\tau \to x$ the kink soliton is replaced.
- Soliton in 'd spatial dimension' would tend to Euclidean Instanton in a 'd dimensional theory of finite action'.

Euclidean Instanton solution

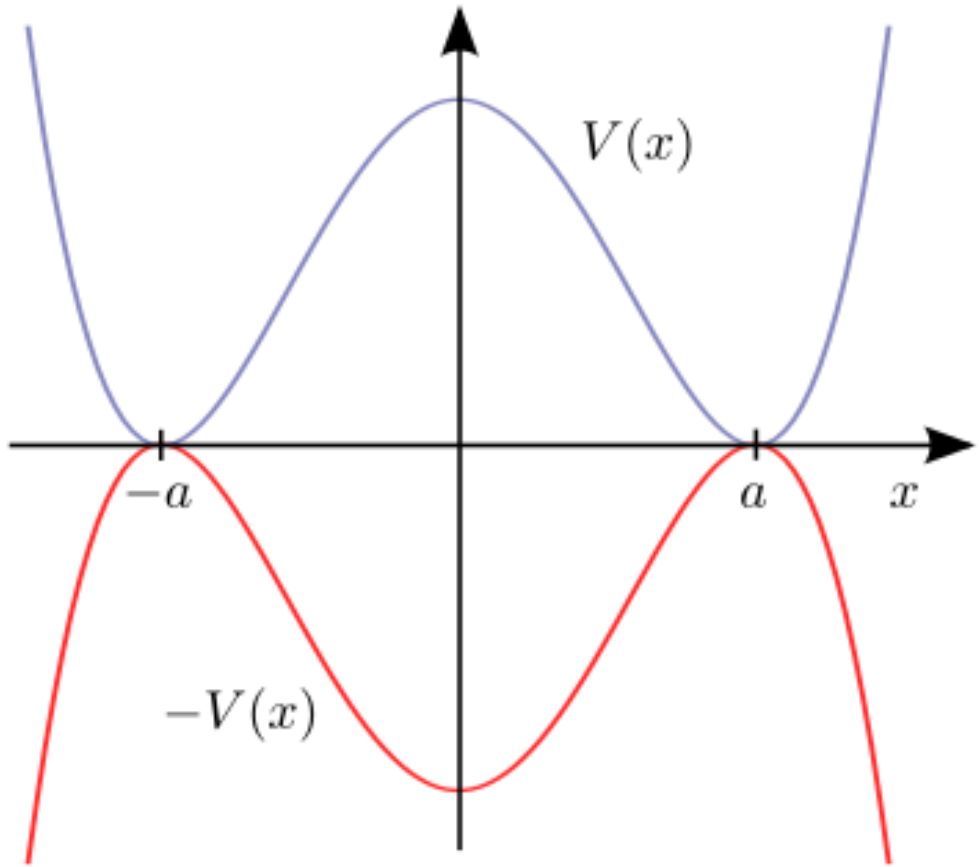


Figure: Euclidean Instanton solution

$$\mathcal{L} = \frac{-1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a} \tag{28}$$

where, $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$ in the fundamental representation. If $SU(2)$ is taken as a group then $T_a = \frac{i\sigma_a}{2}$ where σ_a is a pauli's matrix and for $SU(3)$ T_a is Gellman matrix. Therefore,

$$Tr(F_{\mu\nu} F^{\mu\nu}) = F^a_{\mu\nu} F^{\mu\nu b} Tr(T_a T_b) \tag{29}$$

$$Tr(F_{\mu\nu} F^{\mu\nu}) = \frac{-1}{2} F^a_{\mu\nu} F^{\mu\nu a} \tag{30}$$

Therefore,

$$\mathcal{L} = \frac{1}{2g^2} Tr_F(F_{\mu\nu} F^{\mu\nu}) \tag{31}$$

Let the action of that system,

$$S = \frac{-1}{2g^2} \int d^4x Tr(F_{\mu\nu} F^{\mu\nu}) \tag{32}$$

By the process of completing the squares by Bogomolnyi, we define the dual of $F^{\mu\nu} = F^{\tilde{\mu}\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$$(F^{\mu\nu} \pm F^{\tilde{\mu}\nu})^2 = F_{\mu\nu} F^{\mu\nu} + F^{\tilde{}}_{\mu\nu} F^{\tilde{\mu}\nu} \pm 2 F_{\mu\nu} F^{\tilde{\mu}\nu} \tag{33}$$

Therefore,

$$S_E = \frac{-1}{4g^2} \int d^4x (F^{\mu\nu} \pm F^{\tilde{\mu\nu}})^2 \geqslant 0 \tag{34}$$

By working out the integration we get,

$$S_E \geqslant \frac{8\pi^2}{g^2} n \tag{35}$$

If $n = 1$ it gives the instanton solution and $n = -1$ it gives the anti-instanton solution.
When $n > 0$ $(F^{\mu\nu} + F^{\tilde{\mu\nu}}) = 0$, Anti-self dual.
When $n < 0$ $(F^{\mu\nu} - F^{\tilde{\mu\nu}}) = 0$, Self dual.

Tunneling for a periodic potential

In Condensed matter physics, Bloch theorem states that the solutions to the Schrodinger equation in a periodic potential take the form of plane wave modulated by periodic function.

$$V(x+a) = V(x) \tag{36}$$

which is the consquence of periodic potential. By Bloch theorem ,

$$\psi_\theta(x+a) = e^{i\theta}\psi_\theta(x) \tag{37}$$

where, $\frac{-\pi}{a} < \theta < \frac{\pi}{a}$ which is the first Brillouin zone. $\theta = ka$ k is the crystal momentum. For Pure gauge, An instanton solution interpolates between n and $n+k$. The true ground state is an analog of θ *vacua*.

Bloch theorem

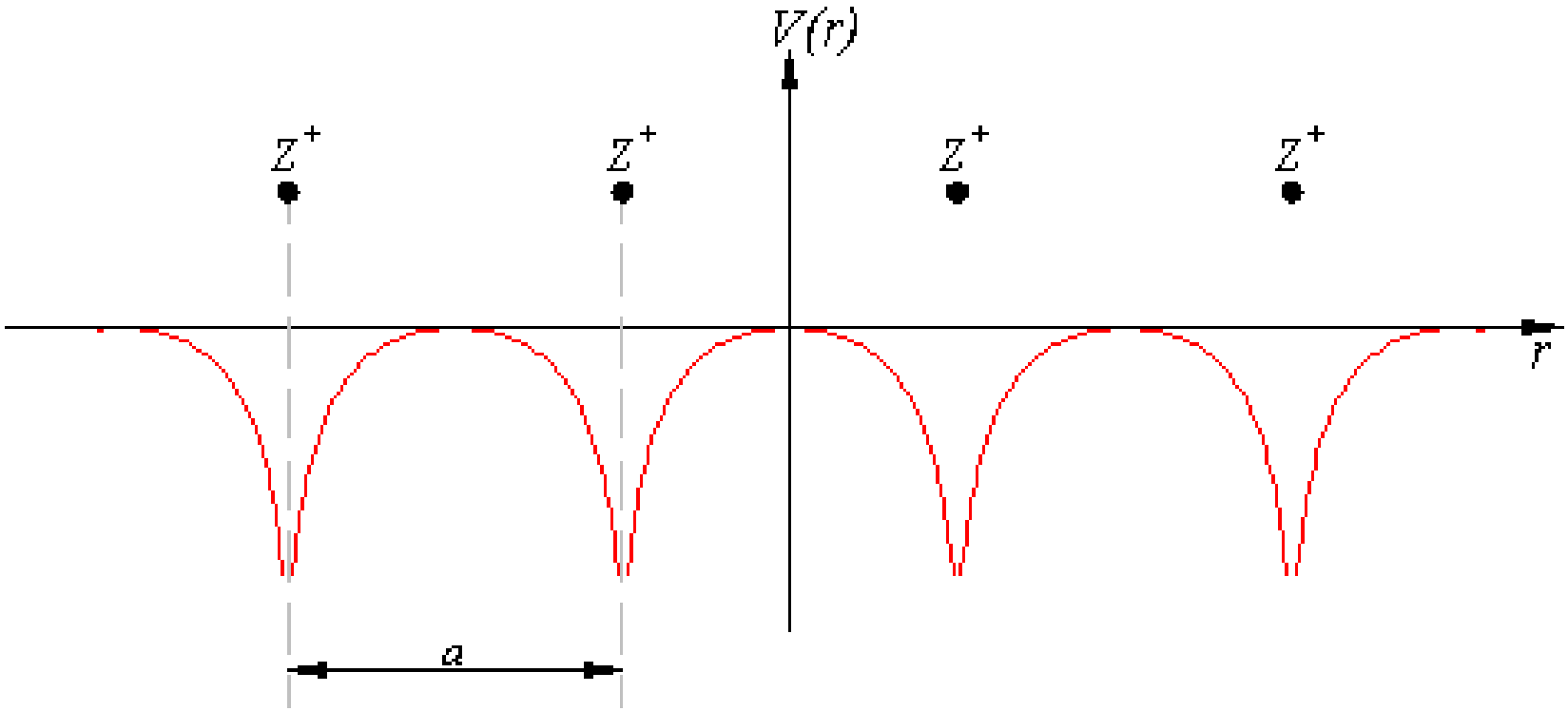


Figure: Periodic Potential

Quantum Information theory and Classical Solitons

Based on the work with V.K.Ojha and Subashish Banerjee [arxiv:2205.02531]

The charge distribution $(Q_W(x))$ is obtained by

$$Q_W(x) = \frac{1}{h}\int dp \int dy \left[\psi(x+y)\psi^*(x-y)e^{\frac{iPy}{\hbar}}\right] \tag{38}$$

$$Q_W(x) = \int dp W(x,p) \tag{39}$$

By neglecting the higher order powers of p_0^3 we obtain

$$Q_W(x) = \frac{4a^3 p_0}{h}\tanh^2\left(\sqrt{\frac{\lambda}{2}}ax\right) \approx |\psi(x)|^2 \tag{40}$$

The computation of current density from Wigner distribution is given by

$$J_W(x) = \int dp.p.W(x,p)$$

Integrating and substituting the boundary values gives

$$J_W(x) = \frac{2a^3}{h}\left[\frac{1}{2}\tanh^2\left(\sqrt{\frac{\lambda}{2}}ax\right)(p_2^2 - p_1^2) - \frac{2a^2}{36\hbar}\tanh^2\left(\sqrt{\frac{\lambda}{2}}ax\right)(p_2^4 - p_1^4)\right] \tag{41}$$

If we take the value of p_1, p_2 to be $-p_0, p_0$ respectively, then eq.(41) reduces to

$$J_W(x) = 0$$

1. Charge distribution

$$Q_W(x) = \frac{4b}{h}\left[p_0\left([\tan^{-1}(e^{\sqrt{\alpha}\beta x})]^2 - \frac{\alpha\beta^2 b^2 e^{2\sqrt{\alpha}\beta x}}{3(e^{2\sqrt{\alpha}\beta x}+1)^2}\right) - \frac{b^2 p_0^3}{9\hbar^2}\left(\tan^{-1}(e^{\sqrt{\alpha}\beta x})\right)^2\right] \approx |\psi(x)|^2 \quad (42)$$

Current distribution

$$J_W(x) = \frac{2b}{h}\left[\left(\frac{p_2^2 - p_1^2}{2}\{\tan^{-1}(e^{2\sqrt{\alpha}\beta x}\}^2\right) - b^2\left(\frac{p_2^4 - p_1^4}{12\hbar^2}\{\tan^{-1}(e^{\sqrt{\alpha}\beta x})\}\right) + \frac{\alpha\beta^2 e^{2\sqrt{\alpha}\beta x}(p_2^2}{6[e^{2\sqrt{\alpha}\beta x} + \quad (43)$$

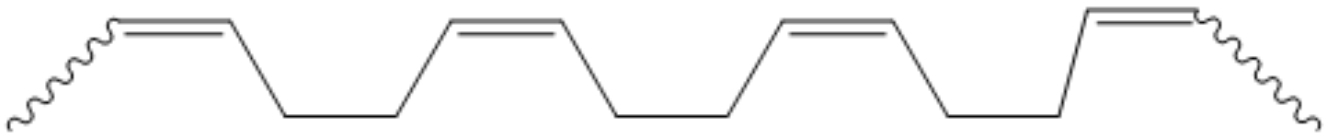
Assuming the bounds of 'p' to be symmetric, (i.e.) $p_1 = -p_0 \& p_2 = p_0$ we get the current density

$$J_W(x) = 0$$

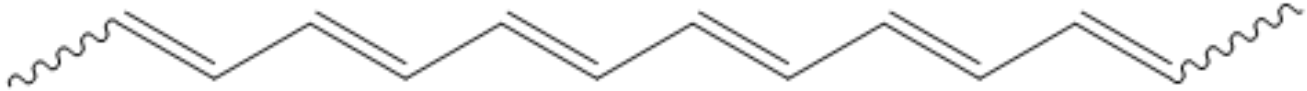
1. The uncertainty relation between energy and time has a special place as it has motivated researchers to understand how fast a quantum system can evolve. One such discovery was the breakthrough provided by Mandestamm-Tamm and Margolus–Levitin bounds , where they determined that the product of the energy variance and time required for a state to be orthogonal to its initial state was greater than Planck's constant.
2. This result implied that quantum mechanics has a fundamental speed limit characterized by Planck's constant, and, thus, this inequality is called the energy-time.
3. This result implied that quantum mechanics has a fundamental speed limit characterized by Planck's constant, and, thus, this inequality is called the energy-time uncertainty relation or *quantum speed limit time (QSL)*

$$\tau_{QSL} = max\{\frac{\pi}{2}\frac{\hbar}{\Delta H}, \frac{\pi}{2}\frac{\hbar}{< H >}\} \tag{44}$$

Polyaceteylene



cis-polyacetylene



trans-polyacetylene

Figure: Cis and Trans - Polyacetylene

Polyaceteylene as conducting polymer

Due to Solitons

1. Polyacetylene is the simplest conducting polymer with repeating $(CH)_x$ units. It is known for its high conductivity upon doping. This has opened up new horizons in the field of organic conducting polymers. Geometrically, it occurs in two forms viz, *cis-Polyacetylene* and *trans-Polyacetylene*.
2. The structure is characterised by the presence of alternating single and double bonds (dimerized pattern). This dimerized pattern constitutes one of the two degenerate structure (A), while the other ground state is formed by interchanging the single and the double bonds.
3. Due to the presence of two degenerate ground states we can expect a soliton excitations between the two degenerate ground states in the form of topological solitons or in the form of a moving domain wall.
4. This topological soliton or the moving domain wall separates both the states A and B. The model for trans-polyacetylene was first developed by Su, Schrieffer, and Heeger in 1979 inorder to describe soliton excitations. The alternating single and double bonds are σ & $\sigma + \pi$ bonds respectively.

Concluding Remarks

1. We calculated the Fisher information, Quantum speed limit time of Solitons in Polyaceteylene
2. Various entropies like Shannon Entropy was calculated for a kink soliton in Symmetric well.
3. The study of classical solitons can be also extended to supersymmetry and they are called as the *Supersymmetric solitons*.

Thank you very much for your attention