The democracy of light-front components and the zero mode issue

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Dr. Ji's group meeting

Mar. 17, 2023

Scalar meson $\to \gamma^*\gamma^*$ Transition Form Factor in 1+1-d scalar model: Manifestly covariant calculation

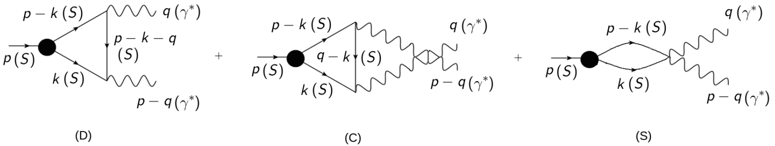


Figure: One-loop covariant Feynman Diagrams that contribute to the $S \to \gamma^*\gamma^*$ transition form factor

The total amplitude consists of these three Feynman diagrams, i.e., the direct (D), crossed (C), and the seagull (S) diagrams, where p is the momentum of the incident scalar meson, while q is the momentum of the emitted photon. As a result of momentum conservation, $q' = p - q$ is the momentum of the final state photon.

From gauge invariance argument, we can know that the total amplitude $\Gamma^{\mu\nu}$ is of the form

$$\Gamma^{\mu\nu} = F(q^2, q'^2)\left(g^{\mu\nu} q\cdot q' - q'^{\mu} q^{\nu}\right), \quad (1)$$

which satisfies both

$$q_{\mu}\left(g^{\mu\nu} q\cdot q' - q'^{\mu} q^{\nu}\right) = 0 \quad (2)$$

and

$$q'_{\nu}\left(g^{\mu\nu} q\cdot q' - q'^{\mu} q^{\nu}\right) = 0, \quad (3)$$

so that the form factor can be obtained by

$$F(q^2, q'^2) = \frac{\Gamma^{\mu\nu}}{g^{\mu\nu} q\cdot q' - q'^{\mu} q^{\nu}}. \quad (4)$$

The amplitude $\Gamma^{\mu\nu}$ is calculated as such, following the Feynman rules for the scalar field theory.

$$\begin{aligned}\Gamma^{\mu\nu} =& \Gamma_D^{\mu\nu} + \Gamma_C^{\mu\nu} + \Gamma_S^{\mu\nu} \\ =& ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{(2p-2k-q)^\mu (p-2k-q)^\nu}{((p-k-q)^2-m^2)((p-k)^2-m^2)(k^2-m^2)} \right. \\ &+ \frac{(q-2k)^\mu (p-2k+q)^\nu}{((p-k)^2-m^2)(k^2-m^2)((q-k)^2-m^2)} \\ &+ \left. \frac{-2g^{\mu\nu}}{((p-k)^2-m^2)(k^2-m^2)} \right\}, \end{aligned} \tag{5}$$

where the coupling constant of the simple scalar model g_s is fixed from the normalization condition. For simplicity, we take all the intermediate scalar particles' mass to be m and their charge to be e, but it can be easily generalized to unequal mass/charge cases. The initial scalar meson has mass M.

We finally obtain

$$F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \int_0^1 dx \int_0^{1-x} dy (1-2y) \left(\frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right), \quad (6)$$

where

$$\Delta_1 = x(x-1)q^2 + 2x(x+y-1) q\cdot q' + (x+y)(x+y-1)q'^2 + m^2, \quad (7)$$

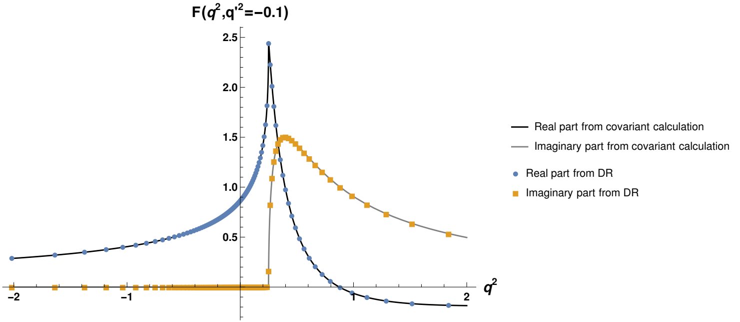
$$\Delta_2 = x(x-1)q'^2 + 2x(x+y-1) q\cdot q' + (x+y)(x+y-1)q^2 + m^2. \quad (8)$$

Doing the x and y integrations, we get the analytic formula for the transition form factor,

$$F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \times \frac{(2-\omega-\gamma'-\gamma)\frac{\sqrt{\omega}}{\sqrt{1-\omega}} \tan^{-1}\left(\frac{\sqrt{\omega}}{\sqrt{1-\omega}}\right) + (\gamma-\gamma'-\omega)\frac{\sqrt{1-\gamma'}}{\sqrt{\gamma'}} \tan^{-1}\left(\frac{\sqrt{\gamma'}}{\sqrt{1-\gamma'}}\right) + (\gamma'-\gamma-\omega)\frac{\sqrt{1-\gamma}}{\sqrt{\gamma}} \tan^{-1}\left(\frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}\right)}{m^4\left[4\omega\gamma'\gamma + \omega^2 + (\gamma'-\gamma)^2 - 2\omega(\gamma'+\gamma)\right]}, \quad (9)$$

where $\gamma = \frac{q^2}{4m^2}$, $\gamma' = \frac{q'^2}{4m^2}$, and $\omega = \frac{M^2}{4m^2}$.

Now, taking $m = 0.25\ GeV$, $M = 0.14\ GeV$, and normalizing the form factor so that $F(q^2 = 0, q'^2 = 0) = 1$ (thus fixing g_s), and taking the value of $q'^2 = -0.1\ GeV^2$, we show below the numerical results of the form factor as a function of q^2. The agreement of the lines with the dots show the agreement of our result with the Dispersion Relation (DR)



Scalar meson $\to \gamma^*\gamma^*$ Transition Form Factor in 1+1-d scalar model: LFTO calculation

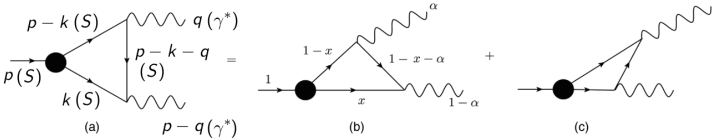
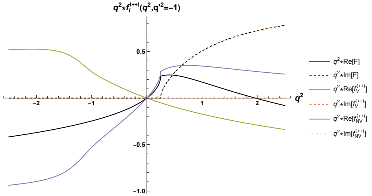


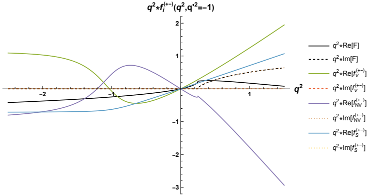
Figure: (Take the direct diagram as an example). The covariant diagram (a) is sum of the two LF x^+-ordered diagrams (b) and (c).

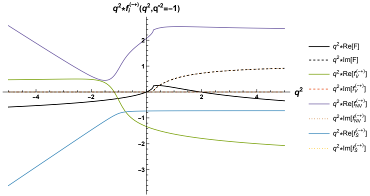
If one assumes each individual LFTO diagram contribution is of the gauge invariant form, i.e. $\Gamma^{\mu\nu} = F(q^2, q'^2)\,(g^{\mu\nu} q\cdot q' - q'^{\mu} q^{\nu})$, one can obtain the LFTO contributions by calculating just the plus-plus current: $F_{(b)} = \frac{\Gamma^{++}_{(b)}}{g^{++} q\cdot q' - q'^{+} q^{+}}$, $F_{(c)} = \frac{\Gamma^{++}_{(c)}}{g^{++} q\cdot q' - q'^{+} q^{+}}$.

However, this way defined LFTO form factors (or GPDs, since they are essentially form factors with unintegrated x), changes with the component. (You never see this if you don't look at other components than the $++$)

Taking $q'^2 = -1.0 GeV^2$.







I'm missing the $--$ component because of difficulty of calculation. More on that later.

Spurious form factors

The most general way, is to write the LFTO diagrams as 4 form factors

$$\Gamma_i^{\mu\nu} = f_i^A(q^2, q'^2)A^{\mu\nu} + f_i^B(q^2, q'^2)B^{\mu\nu} + f_i^C(q^2, q'^2)C^{\mu\nu} + f_i^D(q^2, q'^2)D^{\mu\nu},$$

where $i = D(b),\ D(c),\ C(b),\ C(c),\ or\ S$. Only $A^{\mu\nu}$ is gauge invariant, while $B^{\mu\nu}$, $C^{\mu\nu}$, and $D^{\mu\nu}$ are not. Of course the individual form factors must satisfy

$$\sum_i f_i^B(q^2, q'^2) = \sum_i f_i^C(q^2, q'^2) = \sum_i f_i^D(q^2, q'^2) = 0. \tag{10}$$

(Now the component-dependence is completely in those forms).
The four forms are found to be

$$\begin{aligned}
A^{\mu\nu} &= g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}, \\
B^{\mu\nu} &= q^{\mu} q'^{\nu}, \\
C^{\mu\nu} &= q^{\mu} \left(q^{\nu} - \frac{q \cdot q'}{q'^2} q'^{\nu} \right), \\
D^{\mu\nu} &= \left(q'^{\mu} - \frac{q \cdot q'}{q^2} q^{\mu} \right) q'^{\nu}.
\end{aligned}$$

where we select them so that each two out of the four are orthogonal.

So that we can obtain the individual form factors by

$$f_i^A(q^2, q'^2) = \frac{A_{\mu\nu}\Gamma_i^{\mu\nu}}{A_{\mu\nu}A^{\mu\nu}} = \frac{A_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2 q'^2} \tag{11}$$

$$f_i^B(q^2, q'^2) = \frac{B_{\mu\nu}\Gamma_i^{\mu\nu}}{B_{\mu\nu}B^{\mu\nu}} = \frac{B_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2 q'^2} \tag{12}$$

$$f_i^C(q^2, q'^2) = \frac{C_{\mu\nu}\Gamma_i^{\mu\nu}}{C_{\mu\nu}C^{\mu\nu}} = \frac{C_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2\left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} \tag{13}$$

$$f_i^D(q^2, q'^2) = \frac{D_{\mu\nu}\Gamma_i^{\mu\nu}}{D_{\mu\nu}D^{\mu\nu}} = \frac{D_{\mu\nu}\Gamma_i^{\mu\nu}}{q'^2\left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)}. \tag{14}$$

For $i = S$, we then have

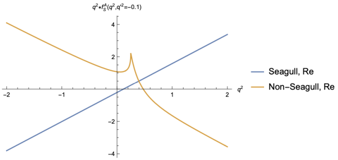
$$f_S^A = f_S^B = \frac{q \cdot q'}{q^2 q'^2} \Gamma_S^{+-} \tag{15}$$

$$f_S^C = \frac{1}{q^2} \Gamma_S^{+-} \tag{16}$$

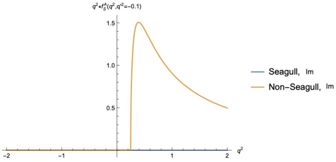
$$f_S^D = \frac{1}{q'^2} \Gamma_S^{+-} \tag{17}$$

where

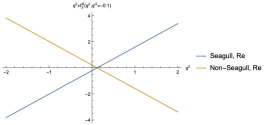
$$\Gamma_S^{+-} = \frac{e^2 g_s}{4\pi} \int_0^1 dx \frac{2}{(1-x)x\left(\frac{m^2}{1-x} + \frac{m^2}{x} - M^2\right)} = \Gamma_S^{-+}. \tag{18}$$



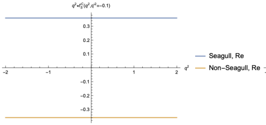
Form factor A --- real part



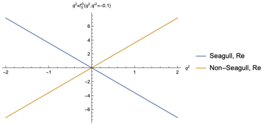
Form factor A --- imaginary part



Spurious form factor B



Spurious form factor C



Spurious form factor D

Going to do this for the separation into Valence, Non-Valence, and Seagull, but need the “minus-minus” component.

The light-front zero-mode issue

The two-point function in the picture below is responsible for the axial anomaly in 1+1-d QED.

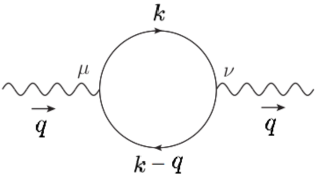


Figure: Feynman diagram for the photon self-energy at one-loop order.

The two vertices being one axial one vector can be obtained from the one with both vertices being vector

$$T_5^{\mu\nu} = \varepsilon^{\nu\lambda} T_\lambda^\mu, \tag{19}$$

So to get $T_5^{\mu\nu}$, it is enough to compute $T^{\mu\nu}$, the vacuum polarization tensor.

Covariant calculation of the two-point function in 1+1-d QED with dimensional regularization

$$\begin{aligned}
T^{\mu\nu}(q) =& ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{Tr\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{q}+m)\right]}{[k^2-m^2]\left[(k-q)^2-m^2\right]} \\
=& ie^2 \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{(k_\alpha k_\beta - k_\alpha q_\beta)\, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2\, Tr\left[\gamma^\mu\gamma^\nu\right]}{\left[k^2+2(x-1)k\cdot q-(x-1)q^2-m^2\right]^2} \\
=& ie^2 \int_0^1 dx \left(\frac{i(-\pi)}{(2\pi)^2(-(x-1)q^2-m^2-(x-1)^2q^2)} \right) \\
& \times \left((x-1)^2 q_\alpha q_\beta + g_{\alpha\beta} \frac{-(x-1)q^2-m^2-(x-1)^2q^2}{2-n} \right) Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] \\
& + ie^2 \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{m^2\, Tr\left[\gamma^\mu\gamma^\nu\right] - k_\alpha q_\beta\, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{\left[k^2+2(x-1)k\cdot q-(x-1)q^2-m^2\right]^2}
\end{aligned}$$

The key is to take the space-time dimension $n \to 2$ *after* the momentum integration

$$
\begin{aligned}
T^{\mu\nu}(q) = &- \frac{e^2}{4\pi}\int_0^1 dx \frac{(x-1)^2 q_\alpha q_\beta \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{x(x-1)q^2 + m^2} \\
&- \frac{e^2}{4\pi}\int_0^1 dx \frac{m^2 \, Tr\left[\gamma^\mu\gamma^\nu\right] + (x-1) q_\alpha q_\beta \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{x(x-1)q^2 + m^2} \\
&+ \frac{e^2}{4\pi}\int_0^1 dx \frac{1}{2-n} g_{\alpha\beta} \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]. \qquad (20)
\end{aligned}
$$

According to the n-dimensional formula

$$
g_{\alpha\beta} \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] = 2(2-n) g^{\mu\nu}, \qquad (21)
$$

one gets

$$
T^{\mu\nu}(q) = - \frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu} q^2) + g^{\mu\nu} m^2}{x(x-1)q^2 + m^2} + \frac{e^2}{2\pi} g^{\mu\nu}. \qquad (22)
$$

which satisfies gauge invariance

$$T^{\mu\nu}(q) = T(q^2)\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) \tag{23}$$

where

$$T(q^2) = -\frac{e^2}{\pi}\left(1 - \frac{m^2/q^2}{\sqrt{1/4 - m^2/q^2}} \ln\left(\frac{1 - \frac{1}{2\sqrt{1/4 - m^2/q^2}}}{1 + \frac{1}{2\sqrt{1/4 - m^2/q^2}}}\right)\right) \tag{24}$$

assuming $q^2 > 4m^2$.

There are other regularization methods to get this.

Axial Anomaly through Analytic Regularization

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PHYSICAL REVIEW D VOLUME 3, NUMBER 4 15 FEBRUARY 1971

Anomalies of the Axial-Vector Current in Two Dimensions*

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(Received 27 October 1970)

In a world with one space and one time dimension, it is shown that the axial-vector current in a vector-gluon model exhibits anomalies in perturbation theory analogous to those found by Adler in fermion electrodynamics. The analysis is extended to include a four-fermion (Thirring) interaction; this model has been solved exactly by Sommerfield, permitting an explicit verification of the perturbation-theory calculations. In analogy to results in four dimensions, a model is presented in which the anomalous properties of the axial-vector current, both in its divergence and in its commutation relations, follow immediately from the canonical structure of the theory.

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Dispersion relation approach to the anomaly in 2 dimensions*

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Two-dimensional Chiral Anomaly in Differential Regularization

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Abstract

The two-dimensional chiral anomaly is calculated using differential regularization. It is shown that the anomaly emerges naturally in the vector and axial Ward identities on the same footing as the four-dimensional case. The vector gauge symmetry can be achieved by an appropriate choice of the mass scales without introducing the seagull term. We have analyzed the reason why such a universal result can be obtained in differential regularization.

PACS: 11.10.Kk ; 11.15.-q
Keywords: Two-dimensional chiral anomaly; Differential regularization; arbitrary local term; Ward identity.

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Light-front view of the axial anomaly

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(Received 7 September 1995)

Motivated by an apparent puzzle of the light-front vacua incompatible with the axial anomaly, we have considered the two-dimensional massless Schwinger model for an arbitrary interpolating angle of Hornbostel's interpolating quantization surface. By examining spectral deformation of the Dirac sea under an external electric field semiclassically, we have found that the axial anomaly is quantization angle independent. This indicates an intricate nontrivial vacuum structure present even in the light-front limit. [S0556-2821(96)04110-0]

PACS number(s): 11.40.Ha, 11.10.Ef, 11.10.Kk, 11.30.Rd

Recently the idea of using light-front quantization [1], which has been applied successfully in the context of current algebra [2] and the parton model [3] in the past, was revived as a promising method for solving QCD [4,5]. While the hope is partly based on the observation that the perturbative vacuum becomes extremely simple, it has been a challenge to understand nontrivial vacuum states such as chiral symmetry breaking and θ-vacuum structures. As these aspects are essential to low-energy hadron physics, it is important to understand how these aspects come about in light-front quantized QCD. In simpler models, this issue has been studied only very recently [6], and it has been found that the $k^+=0$ zero modes are responsible for nontrivial vacuum phenomena.

In this paper, we address another particular aspect of the nontrivial vacuum structure: the axial anomaly [7]. It is well known that the regularization procedure in quantum field

There have been previous studies of the axial anomaly on the light front from various approaches. Bergknoff [9] has studied the Schwinger model on the light front. He has shown that the particle mass of the Schwinger boson results from the axial anomaly so that the nonconservation of the axial vector current is equivalent to the massive Klein-Gordon equation in the bosonized theory. A more recent but similar result was obtained by Heinzl *et al*. [10]. Both works [9,10] have taken the light-front limit as the first step, subsequently performed the quantization *on* the light-front hypersurface (which, in 1+1 dimensions, is purely lightlike) and finally calculated physical observables. As the light-front limit is taken already, however, this approach necessarily involves light-front constraint equations. A proper and careful treatment of these constraint equations is essential to obtain the correct result for physical observables. Various at-

With the light-front calculation still under investigation, and all the physics about axial anomaly very interesting by itself, today I just want to focus on the calculation techniques, and show the way to get the minus-minus component correct in this two-point function calculation. Let

$$T^{\mu\nu}(q) = I^{\mu\nu}_{(1)}(q) + I^{\mu\nu}_{(2)}(q) + I^{\mu\nu}_{(3)}(q)$$

where

$$I^{\mu\nu}_{(1)}(q) = \frac{ie^2}{4\pi^2}\int dk^+ \int dk^- \frac{k_\alpha k_\beta \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]};$$

$$I^{\mu\nu}_{(2)}(q) = -\frac{ie^2}{4\pi^2}\int dk^+ \int dk^- \frac{k_\alpha q_\beta \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]};$$

and

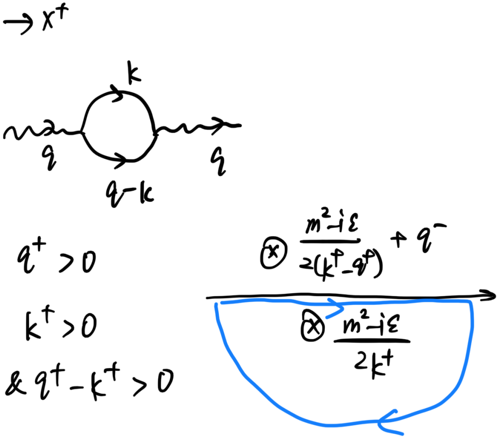
$$I^{\mu\nu}_{(3)}(q) = \frac{ie^2}{4\pi^2}\int dk^+ \int dk^- \frac{m^2 \, Tr\left[\gamma^\mu\gamma^\nu\right]}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]}.$$

We see that $I^{\mu\nu}_{(1)}(q)$ is logarithmically divergent while $I^{\mu\nu}_{(2)}(q)$ and $I^{\mu\nu}_{(3)}(q)$ are not divergent. The simplest term, $I^{\mu\nu}_{(3)}(q)$, can be calculated easily:

$$
\begin{aligned}
I^{\mu\nu}_{(3)}(q) &= \frac{ie^2}{4\pi^2} 2g^{\mu\nu} m^2 \int dk^+ \int dk^- \frac{1}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]} \\
&= \frac{ie^2}{4\pi^2} 2g^{\mu\nu} m^2 (-2\pi i) q^+ \int_0^1 dx \frac{1}{2k^+ 2(k-q)^+ \left(\frac{m^2}{2k^+} - \frac{m^2}{2(k-q)^+} - \frac{q^2}{2q^+}\right)} \\
&= g^{\mu\nu} \frac{e^2 m^2}{2\pi} \int_0^1 dx \frac{1}{x(x-1)\left(\frac{m^2}{x} - \frac{m^2}{x-1} - q^2\right)} \\
&= g^{\mu\nu} \frac{e^2 m^2}{2\pi} \int_0^1 dx \frac{-1}{x(x-1)q^2 + m^2}.
\end{aligned}
$$

For "$--$" component, this term is zero.

There is only one LFTO diagram:



Now we turn to $I^{\mu\nu}_{(2)}(q)$. We will focus on the "$--$" component, as other components can be easily computed without any trouble.

$$I^{\mu\nu}_{(2)}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \frac{k^\mu q^\nu - g^{\mu\nu} k\cdot q + q^\mu k^\nu}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]}.$$

$$I^{--}_{(2)}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \frac{2k^- q^-}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]}.$$

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Restoring the equivalence between the light-front and manifestly covariant formalisms

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(Received 7 September 2005; published 21 October 2005)

We discuss a treacherous point in light-front dynamics (LFD) which should be taken into account to restore complete equivalence with the manifestly covariant formalism. We present examples that require an inclusion of the arc contribution in the light-front energy contour integration in order to achieve the equivalence between the LFD result and the manifestly covariant result.

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which I will refer to as "the flying pole paper"

Here, because of equal mass in the two propagators, taking care of the arc contribution is enough to obtain the correct answer.

$$
\begin{aligned}
I^{--}_{(2)}(q) &= -\frac{ie^2}{\pi^2} q^- \int dk^+ \int dk^- \frac{k^-}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]} \\
&= -\frac{ie^2}{\pi^2} q^- (-2\pi i) q^+ \int_0^1 dx \frac{\frac{m^2}{2k^+}}{2k^+ 2(k-q)^+ \left(\frac{m^2}{2k^+} - \frac{m^2}{2(k-q)^+} - \frac{q^2}{2q^+}\right)} \\
&+ \frac{ie^2}{\pi^2} q^- \int dk^+ \lim_{R\to\infty} \int_0^{-\pi} iRe^{i\theta} d\theta \frac{Re^{i\theta}}{2k^+ 2(k-q)^+ \left(Re^{i\theta}\right)^2} \\
&= \frac{e^2}{\pi} q^- q^- \frac{1}{q^2} \int_0^1 dx \left\{ \frac{m^2}{x\,[x(x-1)q^2 + m^2]} + \frac{1}{2x(x-1)} \right\}. \qquad (25)
\end{aligned}
$$

In which

$$\frac{m^2}{x\left[x(x-1)q^2+m^2\right]} = \frac{(1-x)q^2}{x(x-1)q^2+m^2} + \frac{1}{x} \tag{26}$$

and

$$\begin{aligned}
&\int_0^1 dx \frac{1}{2x(x-1)} \\
&= -\frac{1}{2}\left(\int_0^1 dx \frac{1}{x} + \int_0^1 dx \frac{1}{1-x}\right) \\
&= -\frac{1}{2}\left(\int_0^1 dx \frac{1}{x} + \int_0^1 dx \frac{1}{x}\right) \\
&= -\int_0^1 dx \frac{1}{x}.
\end{aligned} \tag{27}$$

Thus, the answer is

$$\begin{aligned}
I_{(2)}^{--}(q) &= \frac{e^2}{\pi} q^- q^- \frac{1}{q^2} \int_0^1 dx \left\{ \frac{(1-x)q^2}{x(x-1)q^2+m^2} + \frac{1}{x} - \frac{1}{x} \right\} \\
&= \frac{e^2}{\pi} q^- q^- \int_0^1 dx \frac{(1-x)}{x(x-1)q^2+m^2}.
\end{aligned} \tag{28}$$

Now, let us tackle the difficult, divergent term, $I^{\mu\nu}_{(1)}(q)$.

$$I^{\mu\nu}_{(1)}(q) = \frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \frac{2k^\mu k^\nu - g^{\mu\nu}k^2}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]}.$$

$$\begin{aligned} I^{--}_{(1)}(q) &= \frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \frac{2k^-k^-}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]} \\ &= \frac{ie^2}{\pi^2}\int dk^+ \int dk^- \frac{(k^-)^2}{D_1 D_2}, \end{aligned}$$

where

$$D_1 = 2k^+k^- - m^2 + i\epsilon,$$

$$D_2 = 2(k-q)^+(k-q)^- - m^2 + i\epsilon.$$

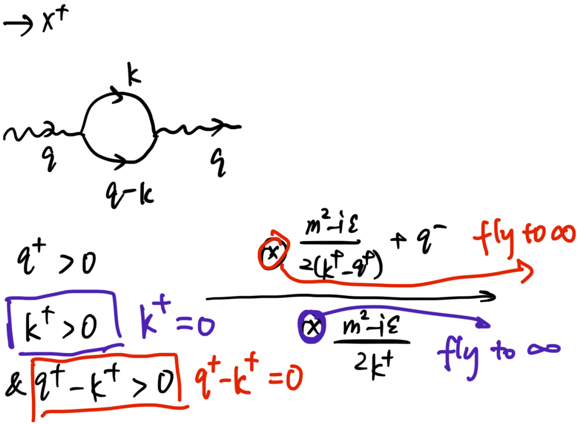
We will utilize the "asymptotic method" discussed in the flying pole paper.
When $k^- \to \infty$ and $k^+ \to 0$,

$$V_{asy1} = \frac{ie^2}{\pi^2} \int dk^+ \int dk^- \frac{(k^-)^2}{D_1 2(-q^+)k^-} = -\frac{ie^2}{2\pi^2 q^+} \int dk^+ \int dk^- \frac{k^-}{D_1}.$$

When $k^- \to \infty$ and $k^+ \to q^+$,

$$V_{asy2} = \frac{ie^2}{\pi^2} \int dk^+ \int dk^- \frac{(k^-)^2}{D_2 2q^+ k^-} = \frac{ie^2}{2\pi^2 q^+} \int dk^+ \int dk^- \frac{k^-}{D_2}.$$

This is the so-called "catching the flying pole"



We subtract the two asymptotic contributions from $I^{--}_{(1)}(q)$ and then add them back.

$$\begin{aligned} I^{--}_{(1)}(q) &= \left[I^{--}_{(1)}(q) - V_{asy1} - V_{asy2} \right] + V_{asy1} + V_{asy2} \\ &= \frac{ie^2}{\pi^2}\frac{1}{2q^+}\int dk^+ \int dk^- k^- \frac{2k^- q^+ + D_2 - D_1}{D_1 D_2} + V_{asy1} + V_{asy2} \\ &= \frac{ie^2}{\pi^2}\frac{1}{2q^+}\int dk^+ \int dk^- k^- \frac{2q^-(q^+ - k^+)}{D_1 D_2} + V_{asy1} + V_{asy2} \\ &= \frac{ie^2}{\pi^2}\int dk^+ \int dk^- \frac{k^- q^-(1 - k^+/q^+)}{D_1 D_2} + V_{asy1} + V_{asy2}. \end{aligned} \tag{29}$$

We notice now in terms of the k^- variable, the power has reduced from $\int dk^- \frac{(k^-)^2}{D_1 D_2}$ to $\int dk^- \frac{k^-}{D_1 D_2}$, due to the cancelation with the V_{asy}'s. Now this k^- integration, we've done before for $I^{--}_{(2)}(q)$.

$$
\begin{aligned}
& I^{--}_{(1)}(q) \\
=& \frac{ie^2}{\pi^2}\int dk^+ q^-(1-k^+/q^+)\left[(-2\pi i)\frac{\frac{m^2}{2k^+}}{2k^+2(k-q)^+\left(\frac{m^2}{2k^+}-\frac{m^2}{2(k-q)^+}-\frac{q^2}{2q^+}\right)}\right. \\
& \left. -\lim_{R\to\infty}\int_0^{-\pi} iRe^{i\theta}d\theta\frac{Re^{i\theta}}{2k^+2(k-q)^+\left(Re^{i\theta}\right)^2}\right] + V_{asy1} + V_{asy2} \\
=& -\frac{e^2}{2\pi}\frac{q^-}{q^+}\int_0^1 dx\,(1-x)\left\{\frac{m^2}{x\left[x(x-1)q^2+m^2\right]}+\frac{1}{2x(x-1)}\right\} + V_{asy1} + V_{asy2} \\
=& -\frac{e^2}{2\pi}\frac{2q^-q^-}{q^2}\int_0^1 dx\,(1-x)\left\{\frac{(1-x)q^2}{x(x-1)q^2+m^2}+\frac{1}{x}-\frac{1}{x}\right\} + V_{asy1} + V_{asy2} \\
=& -\frac{e^2}{\pi}q^-q^-\int_0^1 dx\frac{(1-x)^2}{x(x-1)q^2+m^2} + V_{asy1} + V_{asy2}
\end{aligned}
$$

Now what's left to do is to evaluate the two V_{asy}'s. There are a lot of methods to evaluate them in the flying pole paper, but for simplicity I will for now evaluate them as follows.

$$\begin{aligned}\frac{\partial}{\partial m^2} V_{asy1} &= -\frac{ie^2}{2\pi^2 q^+} \int dk^+ \int_{-R}^{R} dk^- \frac{k^-}{D_1^2} \\ &= -\frac{ie^2}{2\pi^2 q^+} \int dk^+ \left[\frac{-\frac{m^2}{2k^+k^- - m^2} + \ln\left(m^2 - 2k^+k^-\right)}{4(k^+)^2} \right]_{k^-=-R}^{R} \\ &= -\frac{ie^2}{2\pi^2 q^+} \int dk^+ \frac{i\pi}{4(k^+)^2}, \end{aligned} \tag{30}$$

where $k^+ \to 0$.

And

$$\frac{\partial}{\partial m^2} V_{asy2}$$
$$= \frac{ie^2}{2\pi^2 q^+} \int dk^+ \int_{-R}^{R} dk^- \frac{k^-}{D_2^2}$$
$$= \frac{ie^2}{2\pi^2 q^+} \int dk^+ \left[\frac{-\frac{2(k^+-q^+)q^-+m^2}{2(k^+-q^+)(k^--q^-)-m^2} + \ln\left(m^2 - 2(k^+ - q^+)(k^- - q^-)\right)}{4(k^+ - q^+)^2} \right]_{k^-=-R}^{R}$$
$$= \frac{ie^2}{2\pi^2 q^+} \int dk^+ \frac{i\pi}{4(k^+ - q^+)^2}, \tag{31}$$

where $k^+ - q^+ \to 0$.
So actually,

$$V_{asy1} + V_{asy2} = 0. \tag{32}$$

Thus, we obtain

$$I^{--}_{(1)}(q) = -\frac{e^2}{\pi} q^- q^- \int_0^1 dx \frac{(1-x)^2}{x(x-1)q^2 + m^2}.$$

Recall that

$$I^{--}_{(2)}(q) = \frac{e^2}{\pi} q^- q^- \int_0^1 dx \frac{(1-x)}{x(x-1)q^2 + m^2}.$$

So,

$$T^{--}(q) = I^{--}_{(1)}(q) + I^{--}_{(2)}(q) = -\frac{e^2}{2\pi}(2q^- q^-) \int_0^1 dx \frac{x(x-1)}{x(x-1)q^2 + m^2}.$$

In exact agreement with the covariant calculation.

If one ignores what's discussed in the flying pole paper, and calculates the $--$ component naively by the pole integration method,

$$
\begin{aligned}
T^{--}(q) &= \frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \frac{-2k^-q^- + 2k^-k^-}{[2k^+k^- - m^2]\,[2(k-q)^+(k-q)^- - m^2]} \\
&= \frac{ie^2}{2\pi^2}\int dk^+ (-2\pi i)\frac{-\frac{m^2}{k^+}q^- + 2\left(\frac{m^2}{2k^+}\right)^2}{2k^+2(k-q)^+\left(\frac{m^2}{2k^+} - \frac{m^2}{2(k-q)^+} - \frac{q^2}{2q^+}\right)} \\
&= -\frac{e^2}{2\pi}(2q^-q^-)\int_0^1 dx \frac{\frac{m^2}{xq^2}\left(\frac{m^2}{xq^2} - 1\right)}{x(x-1)q^2 + m^2}.
\end{aligned}
$$

In apparent disagreement with the covariant calculation.

The same kind of trouble comes in for the "$--$" component in the transition form factor calculation, where naive pole integration gives

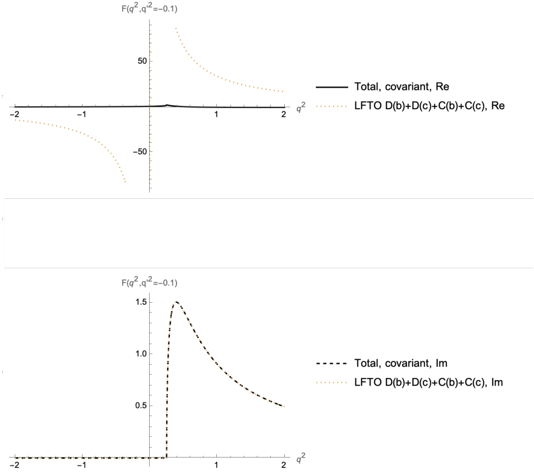
$$\Gamma^{--}_{D(b)} = \frac{e^2 g_s}{4\pi p^+ p^+} \int_0^{1-\alpha} dx \left(\frac{M^2}{2} + \frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \left(\frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \cdot \left[(1-x-\alpha)(1-x)x \left(\frac{m^2}{x} + \frac{m^2}{1-x-\alpha} - \frac{q'^2}{1-\alpha} \right) \left(\frac{m^2}{x} + \frac{m^2}{1-x} - M^2 \right) \right]^{-1}$$

and

$$\Gamma^{--}_{D(c)} = \frac{e^2 g_s}{4\pi p^+ p^+} \int_{1-\alpha}^{1} dx \left(\frac{q'^2}{2(1-\alpha)} - \frac{M^2}{2} + \frac{m^2}{1-x} \right) \left(\frac{q'^2}{2(1-\alpha)} - M^2 + \frac{m^2}{1-x} \right) \cdot \left[(1-x-\alpha)\,(1-x)\,x \left(\frac{m^2}{1-x-\alpha} - \frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha} \right) \left(\frac{m^2}{1-x} + \frac{m^2}{x} - M^2 \right) \right]^{-1}$$

Here, the $\int dx$ integrations could not be done simply using Mathematica like before, due to the end point singularities at $x=0$ and $x=1$, for $\Gamma^{--}_{D(b)}$ and $\Gamma^{--}_{D(c)}$, respectively.

Simply cutting out the singularities results in disagreement with the manifestly covariant calculation.



Without getting into details, we tried many other things, including

Way out of Γ^{--} difficulty

Jan. 30, 2023

As $\Gamma_D^{\mu\nu}$ and $\Gamma_C^{\mu\nu}$ are interchangiable by the change of variables $k \leftrightarrow p-k$, let me illustrate here only $\Gamma_D^{\mu\nu}$ for simplicity.

The integrand of $\Gamma_D^{\mu\nu}$ can be identified as

$$\frac{(2p-2k-q)^\mu (p-2k-q)^\nu}{D_{p-k-q}\, D_{p-k}\, D_k} \equiv \frac{N^{\mu\nu}}{D_{p-k-q}\, D_{p-k}\, D_k},$$

where

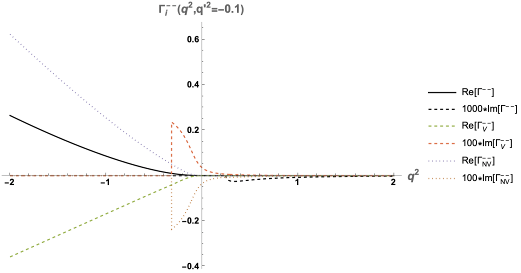
$$D_{p-k-q} = (p-k-q)^2 - m^2 + i\epsilon$$
$$D_{p-k} = (p-k)^2 - m^2 + i\epsilon$$
$$D_k = k^2 - m^2 + i\epsilon .$$

The key idea is to use the following equality

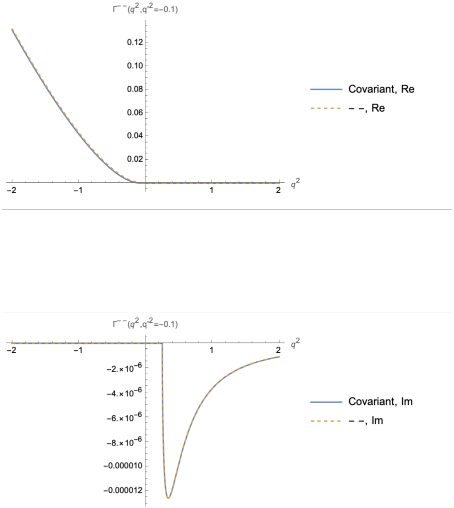
$q_\mu N^{\mu\nu} = D_{p-k} - D_{p-k-q}$ to reduce the number of denominators, while $\frac{k^-}{D_1 D_2}$ type computation was already done in PRD72, 076005(2005) and N^{+-} computation was also done previously.

$$\boxed{\frac{N^{--}}{D_{p-k-q}\, D_{p-k}\, D_k} = \frac{1}{D_{p-k-q}\, D_k} - \frac{1}{D_{p-k}\, D_k} - \frac{q^-}{q^+}\frac{N^{+-}}{D_{p-k}\, D_{p-k}\, D_k}}$$

Now because two denominators doesn't have different time-orderings, separating them into two different time-ordered contributions by hand will result in weird things.



And the total does not exactly agree, either.



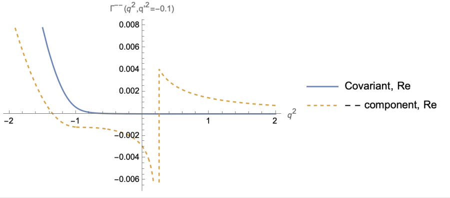
In[34]:=
$$\text{Table}\left[\left\{\left(\frac{\text{qpsqval}\left(\text{qpsqval}-(1-\alpha)M^2\right)}{4(1-\alpha)^2}\ /.\ \alpha\to\frac{M^2-\text{qpsqval}+\text{vqsq}+\sqrt{\left(-M^2+\text{qpsqval}-\text{vqsq}\right)^2-4M^2\,\text{vqsq}}}{2M^2}\right)*\text{FFcovRe}[\text{vqsq}]/\text{FFcov0}/2,\ \text{FLFrestmmRe}[\text{vqsq},\text{qpsqval}]/\text{FFcov0}\right\},\ \{\text{vqsq},-2,2,0.11\}\right]$$

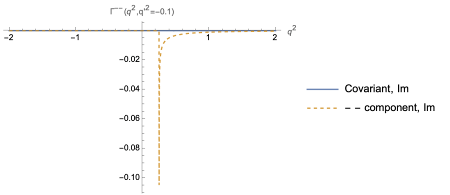
Out[34]=
```
{{0.131327, 0.132588}, {0.12023, 0.121487}, {0.109401, 0.110654}, {0.0988565, 0.100104}, {0.0886147, 0.0898568}, {0.0786958, 0.0799314},
 {0.069123, 0.070351}, {0.0599231, 0.0611423}, {0.0511273, 0.0523358}, {0.0427723, 0.0439679}, {0.0349018, 0.0360816}, {0.0275692, 0.0287288},
 {0.0208403, 0.0219734}, {0.0147987, 0.0158957}, {0.00955368, 0.0105985}, {0.00525288, 0.00621612}, {0.00210402, 0.00292474}, {0.000384404, 0.000929232},
 {9.29468×10^-6, 0.000226786}, {-0.0000120844, 0.0000851672}, {-0.0000154924, 0.0000405009}, {-0.0000151337, 0.0000218962}, {-6.64996×10^-6, 0.0000199425},
 {-3.06409×10^-6, 0.0000171008}, {-1.36938×10^-6, 0.0000145246}, {-5.06108×10^-7, 0.0000123897}, {-4.4858×10^-8, 0.0000106568},
 {2.07802×10^-7, 9.25034×10^-6}, {3.46376×10^-7, 8.10096×10^-6}, {4.20073×10^-7, 7.15287×10^-6}, {4.55836×10^-7, 6.36313×10^-6}, {4.6903×10^-7, 5.69895×10^-6},
 {4.68699×10^-7, 5.13533×10^-6}, {4.60305×10^-7, 4.65303×10^-6}, {4.47214×10^-7, 4.23711×10^-6}, {4.31527×10^-7, 3.87591×10^-6}, {4.14564×10^-7, 3.56017×10^-6}}
```

In[35]:=
$$\text{Table}\left[\left\{\left(\frac{\text{qpsqval}\left(\text{qpsqval}-(1-\alpha)M^2\right)}{4(1-\alpha)^2}\ /.\ \alpha\to\frac{M^2-\text{qpsqval}+\text{vqsq}+\sqrt{\left(-M^2+\text{qpsqval}-\text{vqsq}\right)^2-4M^2\,\text{vqsq}}}{2M^2}\right)*\text{FFcovIm}[\text{vqsq}]/\text{FFcov0}/2,\ \text{FLFrestmmIm}[\text{vqsq},\text{qpsqval}]/\text{FFcov0}\right\},\ \{\text{vqsq},-2,2,0.11\}\right]$$

Out[35]=
```
{{0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},
 {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},
 {-0.0000121831, -0.0000121831}, {-0.000011436, -0.000011436}, {-9.09137×10^-6, -9.09137×10^-6}, {-7.16059×10^-6, -7.16059×10^-6},
 {-5.71702×10^-6, -5.71702×10^-6}, {-4.64417×10^-6, -4.64417×10^-6}, {-3.83603×10^-6, -3.83603×10^-6}, {-3.21636×10^-6, -3.21636×10^-6},
 {-2.73264×10^-6, -2.73264×10^-6}, {-2.34871×10^-6, -2.34871×10^-6}, {-2.03938×10^-6, -2.03938×10^-6}, {-1.78675×10^-6, -1.78675×10^-6},
 {-1.57791×10^-6, -1.57791×10^-6}, {-1.4034×10^-6, -1.4034×10^-6}, {-1.25614×10^-6, -1.25614×10^-6}, {-1.13077×10^-6, -1.13077×10^-6}}
```

Then we tried to take into account the asymptotic contributions without much success.





Finally we tried the following idea.

$$T_D^{\mu\nu} = i e^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^\mu (p-2k-q)^\nu}{D_1 D_2 D_3}$$

$$\frac{1}{D_1 D_2 D_3} \overset{\text{F.P.}}{=\!=} \int_0^1 dx \int_0^{1-x} dy \, \frac{2}{D_{cov}^3}$$

where $D_{cov} = [k-(x+y)p+yq]^2 - \underbrace{[(x+y)p+yq]^2 - m^2 + xp^2 + y(p-q)^2}_{\Delta}$

Then one could shift momentum to $l = k-(x+y)p+yq$.

For T_D^{--} calculation, only problematic part is $\int d^2k \frac{(k^-)^2}{D_1 D_2 D_3}$.

Let's separate it out.

$$\Gamma_D^{\mu\nu} = ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^\mu (p-q)^\nu - 2(2p-q)^\mu k^\nu}{D_1 D_2 D_3}$$

$$+ 8ie^2 g_s \int_0^1 dx \int_0^{1-x} dy \boxed{\int \frac{d^2k}{(2\pi)^2} \frac{k^\mu k^\nu}{D_{cov}^3}}$$

$$= \int \frac{d^2\ell}{(2\pi)^2} \frac{[\ell+(x+y)p-yq]^\mu [\ell+(x+y)p-yq]^\nu}{(\ell^2-\Delta)^3}$$

in which

$$\int \frac{d^2\ell}{(2\pi)^2} \frac{\ell^\mu \ell^\nu}{(\ell^2-\Delta)^3} = \int \frac{d^2\ell}{(2\pi)^2} \frac{\frac{1}{2} g^{\mu\nu} \ell^2}{(\ell^2-\Delta)^3}$$

for ++ and −−, this is 0.

$$\Gamma_D^{--} = ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^-(p-q)^- - 2(2p-q)^- k^-}{D_1 D_2 D_3}$$ no problem

$$+ 8ie^2 g_s \int_0^1 dx \int_0^{1-x} dy \int \frac{d^2k}{(2\pi)^2} \frac{[(x+y)p^- - y q^-]^2}{D_{cov}^3}$$ don't know

$$\Gamma_D^{++} = ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^+(p-q)^+ - 2(2p-q)^+ k^+}{D_1 D_2 D_3}$$ no problem

$$+ 8ie^2 g_s \int_0^1 dx \int_0^{1-x} dy \int \frac{d^2k}{(2\pi)^2} \frac{[(x+y)p^+ - y q^+]^2}{D_{cov}^3}$$ no problem

Now, the problem becomes :

Knowing that

$$8ie^2g_s\int_0^1 dx\int_0^{1-x}dy\int\frac{d^2k}{(2\pi)^2}\frac{[(x+y)p^+-yq^+]^2}{D_{cov}^3} = 8ie^2g_s(p^+)^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^2k}{(2\pi)^2}\frac{[(x+y)-y\alpha]^2}{D_{cov}^3}$$

$$= \frac{e^2g_s}{4\pi}(p^+)^2\int_0^{1-\alpha}dx\,\frac{4x^2}{(1-x-\alpha)(1-x)x\left(\frac{m^2}{x}+\frac{m^2}{1-x-\alpha}-\frac{q'^2}{1-\alpha}\right)\left(\frac{m^2}{x}+\frac{m^2}{1-x}-M^2\right)} \quad (b)$$

$$+\frac{e^2g_s}{4\pi}(p^+)^2\int_{1-\alpha}^1 dx\,\frac{4x^2}{(1-x-\alpha)(1-x)x\left(\frac{m^2}{1-x-\alpha}-\frac{m^2}{1-x}+M^2-\frac{q'^2}{1-\alpha}\right)\left(\frac{m^2}{1-x}+\frac{m^2}{x}-M^2\right)} \quad (c)$$

What is

$$8ie^2g_s\int_0^1 dx\int_0^{1-x}dy\int\frac{d^2k}{(2\pi)^2}\frac{[(x+y)p^--yq^-]^2}{D_{cov}^3} \;? \quad \text{———} \quad (*)$$

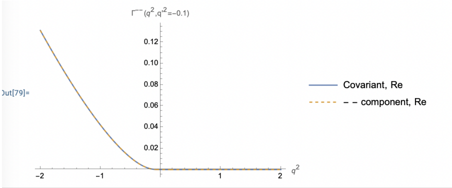
Obviously, if I define $\beta = \frac{q^-}{p^-} = 1 - \frac{q'^2}{M^2(1-\alpha)}$, then

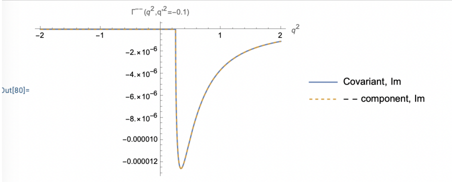
$$(*) = 8ie^2 g_s \int_0^1 dx \int_0^{1-x} dy \int \frac{d^2k}{(2\pi)^2} \frac{[(x+y)-y\beta]^2}{D_{cov}^3}(p^-)^2$$

$$= \frac{e^2 g_s}{4\pi}(p^-)^2 \int_0^{1-\beta} dx \frac{4x^2}{(1-x-\beta)(1-x)x\left(\frac{m^2}{x}+\frac{m^2}{1-x-\beta}-\frac{q'^2}{1-\beta}\right)\left(\frac{m^2}{x}+\frac{m^2}{1-x}-M^2\right)} \quad (b)$$

$$+ \frac{e^2 g_s}{4\pi}(p^-)^2 \int_{1-\beta}^1 dx \frac{4x^2}{(1-x-\beta)(1-x)x\left(\frac{m^2}{1-x-\beta}-\frac{m^2}{1-x}+M^2-\frac{q'^2}{1-\beta}\right)\left(\frac{m^2}{1-x}+\frac{m^2}{x}-M^2\right)} \quad (c)$$

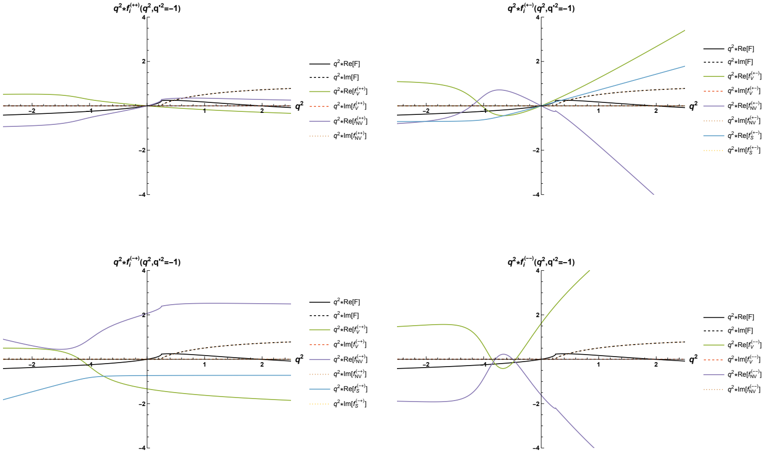
Then finally I got agreement



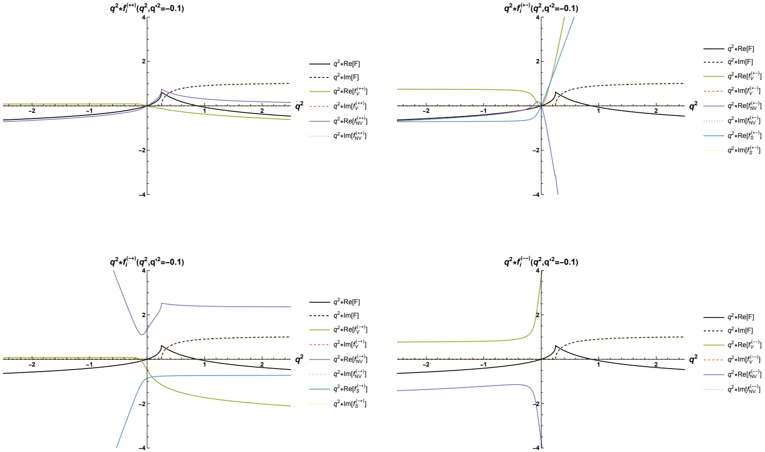


```
In[81]:= Table[{(qpsqval (qpsqval - (1 - α) M^2) / (4 (1 - α)^2) /. α → (M^2 - qpsqval + vqsq + Sqrt[(-M^2 + qpsqval - vqsq)^2 - 4 M^2 vqsq]) / (2 M^2)) * FFcovRe[vqsq] / FFcov0 / 2,
    (FLFDbmmRe[vqsq, qpsqval] + FLFDcmmRe[vqsq, qpsqval] + FLFDbmmReβ[vqsq, qpsqval] + FLFDcmmReβ[vqsq, qpsqval]) / FFcov0}, {vqsq, -2, 2, 0.11}]

Out[81]= {{0.131327, 0.131327}, {0.12023, 0.12023}, {0.109401, 0.109401}, {0.0988565, 0.0988565}, {0.0886147, 0.0886147}, {0.0786958, 0.0786958}, {0.069123, 0.069123},
    {0.0599231, 0.0599231}, {0.0511273, 0.0511273}, {0.0427723, 0.0427723}, {0.0349018, 0.0349018}, {0.0275692, 0.0275692}, {0.0208403, 0.0208403},
    {0.0147987, 0.0147987}, {0.00955368, 0.00955368}, {0.00525288, 0.00525288}, {0.00210402, 0.00210402}, {0.000384404, 0.000384404}, {9.29468×10^-6, 9.29468×10^-6},
    {-0.0000120844, -0.0000120844}, {-0.0000154924, -0.0000154924}, {-0.0000151337, -0.0000151337}, {-6.64996×10^-6, -6.64996×10^-6},
    {-3.06409×10^-6, -3.06409×10^-6}, {-1.36938×10^-6, -1.36938×10^-6}, {-5.06108×10^-7, -5.06108×10^-7}, {-4.4858×10^-8, -4.4858×10^-8},
    {2.07802×10^-7, 2.07802×10^-7}, {3.46376×10^-7, 3.46376×10^-7}, {4.20073×10^-7, 4.20073×10^-7}, {4.55836×10^-7, 4.55836×10^-7}, {4.6903×10^-7, 4.6903×10^-7},
    {4.68699×10^-7, 4.68699×10^-7}, {4.60305×10^-7, 4.60305×10^-7}, {4.47214×10^-7, 4.47214×10^-7}, {4.31527×10^-7, 4.31527×10^-7}, {4.14564×10^-7, 4.14564×10^-7}}

In[82]:= Table[{(qpsqval (qpsqval - (1 - α) M^2) / (4 (1 - α)^2) /. α → (M^2 - qpsqval + vqsq + Sqrt[(-M^2 + qpsqval - vqsq)^2 - 4 M^2 vqsq]) / (2 M^2)) * FFcovIm[vqsq] / FFcov0 / 2,
    (FLFDbmmIm[vqsq, qpsqval] + FLFDcmmIm[vqsq, qpsqval] + FLFDbmmImβ[vqsq, qpsqval] + FLFDcmmImβ[vqsq, qpsqval]) / FFcov0}, {vqsq, -2, 2, 0.11}]

Out[82]= {{0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.},
    {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {0., 0.}, {-0.0000121831, -0.0000121831}, {-0.000011436, -0.000011436}, {-9.09137×10^-6, -9.09137×10^-6},
    {-7.16059×10^-6, -7.16059×10^-6}, {-5.71702×10^-6, -5.71702×10^-6}, {-4.64417×10^-6, -4.64417×10^-6}, {-3.83603×10^-6, -3.83603×10^-6},
    {-3.21636×10^-6, -3.21636×10^-6}, {-2.73264×10^-6, -2.73264×10^-6}, {-2.34871×10^-6, -2.34871×10^-6}, {-2.03938×10^-6, -2.03938×10^-6},
    {-1.78675×10^-6, -1.78675×10^-6}, {-1.57791×10^-6, -1.57791×10^-6}, {-1.4034×10^-6, -1.4034×10^-6}, {-1.25614×10^-6, -1.25614×10^-6}, {-1.13077×10^-6, -1.13077×10^-6}}
```

Taking $q'^2 = -1.0 GeV^2$.



Taking $q'^2 = -0.1GeV^2$.



Taking $q'^2 = -0.01 GeV^2$.