Closed vs. Open Spacetime and Homogeneous vs. Inhomogeneous Spacetime Transformations

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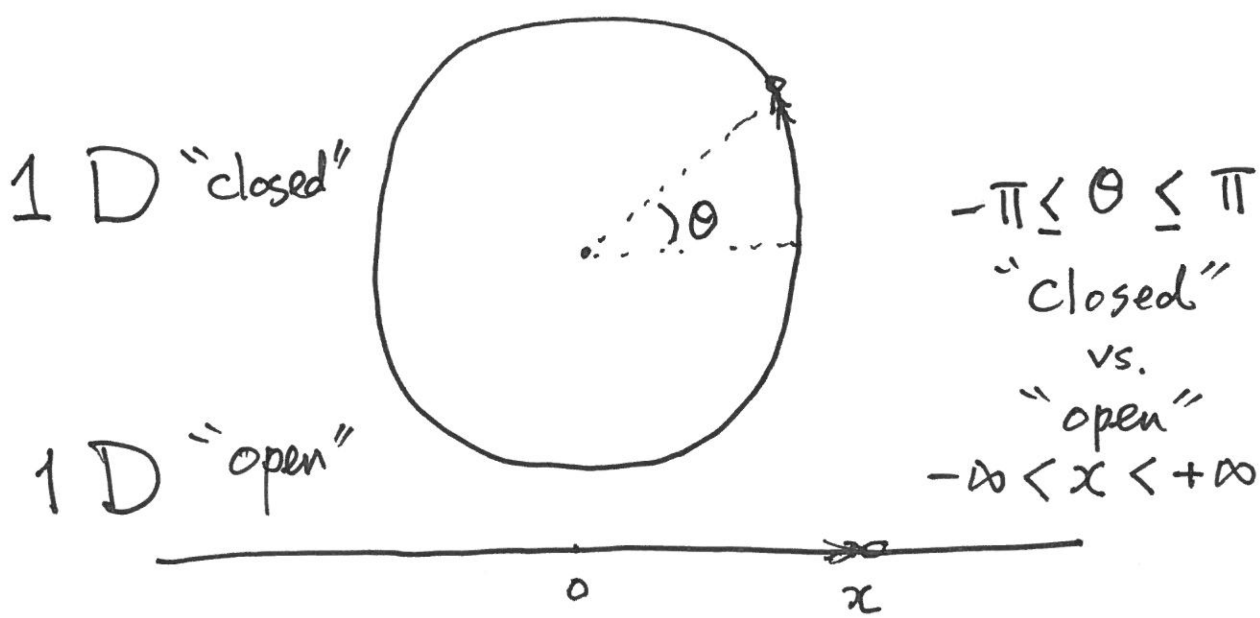
In memory of Prof. Yongseok Oh



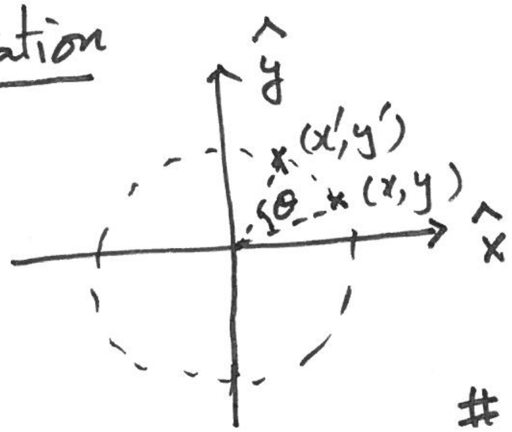
Key Aspects of Spacetime Transformation

From Lower to Higher Dimensions

(1). Number of Homogeneous Transformations

Matrix Representation of Homogeneous Transf.

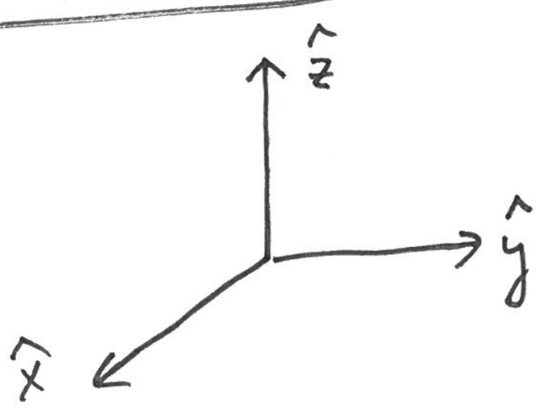
2D Rotation



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

# of Rotation = 1

3D Rotation



${}_3C_2 = 3$ Rotations

(e.g. 3 Euler Angle Rotations)

4D Homogeneous Transfs

$$ {}_4C_2 = 6 = 3 \text{ Rotations} + 3 \text{ Boosts} $$

$$ M^{\mu\nu} = \begin{bmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{bmatrix} $$

$$ K^1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K^2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K^3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} $$

$$ J^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad J^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad J^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

In general, $x'^{\mu} = \underbrace{\Lambda^{\mu}{}_{\nu}}_{\text{"Homogeneous"}} x^{\nu} + \underbrace{a^{\mu}}_{\text{"Inhomogeneous"}}$

10 Poincaré Generators = 6 (Homogeneous) + 4 (Inhomogeneous)

= 3 Rotations + 3 Boosts + 4 Translations

$(J^1, J^2, J^3) + (K^1, K^2, K^3) + (P^0, P^1, P^2, P^3)$

(2) How do we understand Inhomogeneous Transfs?

4D Homogeneous Transfs $\left({}_4C_2 = 6 = \underset{\text{Rotations}}{3} + \underset{\text{Boosts}}{3} \right)$

$\downarrow$

3D Homogeneous Transfs + 3D Inhomogeneous Transfs

$\left({}_3C_2 = 3 \right)$ + 3 Non-relativistic Boosts.

Key Observation

3 Relativistic Boosts $\longrightarrow$ 3 Non-relativistic Boosts

Lorentz Transfs $\longrightarrow$ Galilean Transfs.

"Homogeneous" "Inhomogeneous"

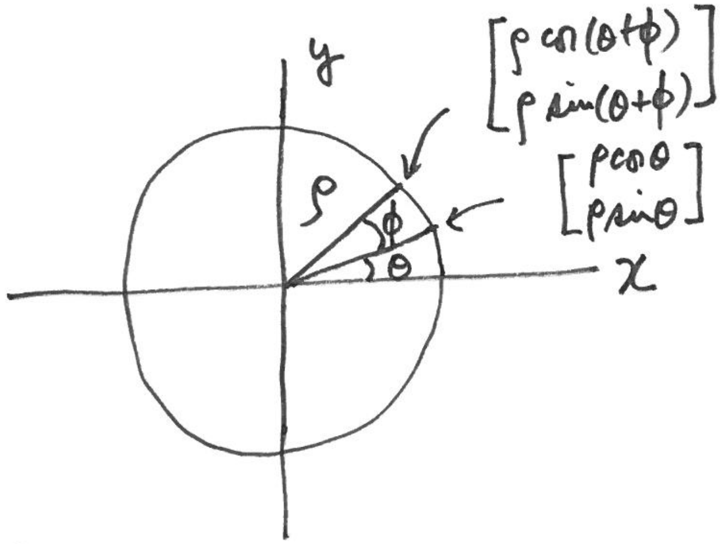
e.g. $\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \longrightarrow \begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$

$(c \to \infty)$ i.e. $t' = t$

$x' = x + vt$

Inhomogeneous Transfs in Lower Dimensions can be derived from Homogeneous Transfs in Higher Dimensions with an appropriate scaling.

e.g. Rotation in 2D $\rightarrow$ Translation in 1D.



$$\begin{bmatrix} \rho\cos(\theta+\phi) \\ \rho\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}$$

"Scaling"
$$\begin{bmatrix} \cos(\theta+\phi) \\ \rho\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi \\ \rho\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta \\ \rho\sin\theta \end{bmatrix}$$

$\rho \rightarrow \infty$, $\theta \rightarrow 0$, $\phi \rightarrow 0$ to get finite $y = \rho\theta$, $y_\phi = \rho\phi$

$$\begin{bmatrix} 1 \\ y + y_\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ y_\phi & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix}$$

(3). Applications

6 (Homogeneous) + 4 (Inhomogeneous) in 4D
may be understood as 10 (Homogeneous) in 5D
with an appropriate scaling.
i.e.

$$\underbrace{{}_5C_2}_{\text{Homogeneous}} = 10 = \underbrace{{}_4C_2}_{\text{Homogeneous}} + \underbrace{{}_4C_1}_{\text{Inhomogeneous}}$$

5D De Sitter Space ($\Lambda > 0$)

Anti-de Sitter Space ($\Lambda < 0$)

$$x'^{\alpha} = \Gamma^{\alpha}_{\beta}\, x^{\beta} \quad (\alpha, \beta = 0,1,2,3,4)$$

$\downarrow$

4D ($\Lambda = 0$) Poincaré Minkowski Space

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu} + a^{\mu} \quad (\mu, \nu = 0,1,2,3)$$

6D SO(4,2)

${}_6C_2 = 15$ Homogeneous Transformations.

$\boxed{\text{Dirac, Annals Math } \underline{37}, 429 \ (1936)}$

$$\underbrace{\gamma^0, \gamma^1, \gamma^2, \gamma^3}_{4D} \quad \underbrace{\gamma_5}_{5th\,D} \quad \underbrace{\gamma^0\gamma_5, \gamma^1\gamma_5, \gamma^2\gamma_5, \gamma^3\gamma_5}_{6th\,D}$$

Poincaré Minkowski + Dilation + Special Conformal Transfs.

$$\underbrace{{}_4C_2}_{3\text{ Rotations} + 3\text{ Boosts}} + \underbrace{{}_4C_1}_{4\text{ Translations}} + \underbrace{\text{Dilation}}_{1\text{ Scaling}} + \underbrace{\text{SCTs.}}_{4\text{ Translations}}$$

$$\Lambda^\mu{}_\nu x^\nu, \quad x^\mu + a^\mu, \quad \lambda x^\mu, \quad \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

$$\boxed{{}_6C_2 = {}_5C_2 + {}_5C_1 = \underbrace{{}_4C_2}_{\text{Lorentz Transfs.}} + \underbrace{{}_4C_1}_{\text{Translations}} + \underbrace{{}_1C_1}_{\text{Dilation}} + \underbrace{{}_4C_1}_{\text{SCTs}}}$$

(4) Conclusion / Discussion

1. Lower Dimension Inhomogeniety can be unified into Higher Dimension Homogeniety introducing a proper scaling.

e.g.

Non-relativistic Mechanics $\xleftrightarrow{c}$ Relativistic Mechanics

Translation $\xleftrightarrow{\rho}$ Rotation

Flat Minkowski $\xleftrightarrow{\Lambda}$ De Sitter/Anti-de Sitter

2. Causality/Relativity generating the time-dilation and length-contraction naturally introduce the scaling of spacetime, i.e. Dilation as the 5th Dimension, as well as SCTs with the 6th Dimension of the inversed 4 Dimensions.
3. Most efficient Dilation and Boost combination can be achieved in LFD.