# The Chern Simon's action and Quantum Hall effect


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# Motivation

## Why this work ?

- Luttinger liquid model of electrons and Hubbard models - well understood through Field theory.
- To understand Quantum Hall effect as an effective theory [1] of the background EM gauge field $A$.
- To study the *emergent degrees of freedom*[2] - In FQHE, it gives rise to 'fractional charges'.

[1] Interactions of electrons and magnetic field is considered as an *effective action*

[2] These emergent degrees of freedom is seen coupled with EM field with the gauge field $A$.

- Overview - Hall effect
- Chern - Simon's effective action and QHE correspondence
- Parity Anomaly

Notations Used:
QHE - Quantum Hall Effect
IQHE - Integer Quantum Hall Effect
FQHE - Fractional Quantum Hall Effect
CS - Chern Simon

Overview - Hall effect

# Classical Hall effect

- A constant current $I$ is made to flow in the x-direction. The Hall effect is the statement that this induces a voltage $V_H$ in the y-direction.

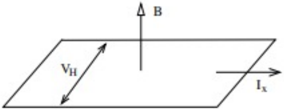


Figure: Classical Hall effect

- Classical Hall effect resistivity[3]: $\rho_{xx} = \frac{m}{ne^2\tau}; \rho_{xy} = \frac{B}{ne}$

[3] D. Tong, *Quantum Hall effect*, Infosys TIFR Lectures

# Quantum Hall effect

## Because world is governed by quantum mechanics !

- Physics of electrons in semiconductors provides information about the behaviour of Fermions in lower dimensions.
- Two types - IQHE and FQHE

# Integer Quantum Hall Effect

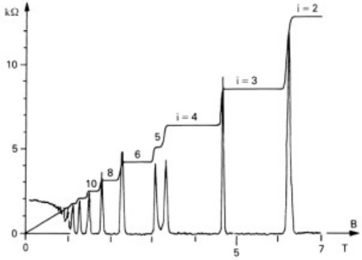


Figure: IQHE[4]

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}; \nu \in \mathcal{Z}$$

$$\sigma_{xy} = \frac{1}{\rho_{xy}} = k\nu; \nu \in \mathcal{Z}$$

where, $k = \frac{e^2}{2\pi\hbar}$

[4] K. v Klitzing, G. Dorda, M. Pepper, *New Method for High-Accuracy Determination of the Fine Structure Constant Based on Quantized Hall Resistance*, Phys. Rev. Lett. 45 494.

# Fractional Quantum Hall effect

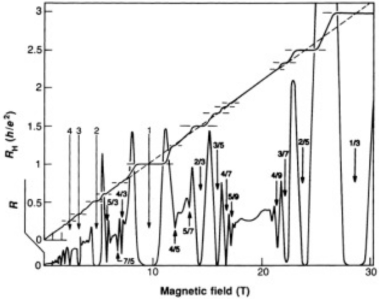


Figure: FQHE[5]

$$\sigma_{xy} = k\nu; \nu \in \mathcal{Q}$$

where, $k = \frac{e^2}{2\pi\hbar}$

[5] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Two-Dimensional Magnetotransport in the Extreme Quantum Limit*, Phys. Rev. Lett. 48 (1982)1559.

Chern - Simon's effective action and QHE correspondence

# Energy levels

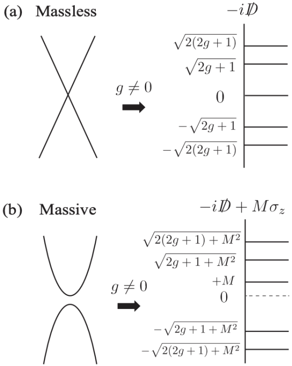


FIG. 1. Schematic of the energy spectrum and Landau level. (a) and (b) show the massless and the massive cases ($M > 0$), respectively. ($g$ represents a monopole charge.) The right figure of (b) shows asymmetry between the positive and negative energy levels due to the absence of $-M$. (In general, there exists the energy level $E = +\text{sgn}(g \cdot M)\ |M|$ while not $E = -\text{sgn}(g \cdot M)\ |M|$.) The original reflection symmetry of the energy levels with respect to the zero-energy is broken due to the mass term.

## QED Effective action

Lagrangian for massive $\mathrm{QED}_{2+1}$ is

$$\mathcal{L}_{2+1} = \frac{-1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi \tag{1}$$

where, $\not{D} = \gamma_\mu D^\mu$, $D_\mu = \partial_\mu + i(-e)A_\mu, \mu = 0, 2, 3$

Action : $S[\Psi, \not{A}] = \int d^3x \left\{ \frac{-1}{4} F^2 + \bar{\Psi}(i\not{\partial} - \not{A} - m)\Psi \right\}$ The generating functional is given by

$$Z[\eta, \eta^\dagger] = N \int d\Psi d\bar{\Psi} e^{iS[\Psi, \not{A}] + \int d^3x \bar{\eta}(x)\Psi(x) + \int d^3x \bar{\Psi}(x)\eta(x)} \tag{2}$$

Integrating $\eta, \bar{\eta}$ we get

$$Z[\not{A}] = N \int d\Psi d\bar{\Psi} e^{iS[\Psi, \not{A}]} \tag{3}$$

$$Z[\not{A}] = N \int d\Psi d\bar{\Psi} e^{\int d^3x \left\{ \frac{-1}{4} F^2 + \bar{\Psi}(i\not{\partial} - \not{A} - m)\Psi \right\}} \tag{4}$$

where, $N$ is a normalization factor. Now let us use some review on the properties of matrices. Using similarity transformation of matrices we write two matrices as

$$A = P^{-1} B P \tag{5}$$

$$e^A = P^{-1} e^B P$$

$$\det(e^A) = \det(P^{-1}) \det(e^B) \det(P)$$

$$\det(e^A) = \det(e^B) = e^{b_1 + b_2 + \dots}$$

$$\therefore \det(e^A) = e^{\mathrm{Tr}(B)}$$

Now we go ahead in calculating the normalization factor. By using the above relations we write (4) as

$$Z[\not{A}] = e^{iS} = Ne^{\int d^3x \frac{-1}{4} F^2} \det[-i(i\not{\partial} - \not{A} - m)] \tag{6}$$

This can be obtained when $A = 0 \implies Z[0] = 1$. Thus

$$1 = N\det[-i(i\not{\partial} - m)], \because F = dA \tag{7}$$

$$\therefore N = \det\frac{1}{[-i(i\not{\partial} - m)]}$$

Now we denote $(i\partial\!\!\!/ - m)^{-1} = iX \implies N = \det\frac{iX}{-i}$. We get the generating functional as

$$e^{iS} = e^{\int d^3x \frac{-1}{4}F^2} \det\left[\frac{iX}{-i}\right] \det\left[-i\left(\frac{1}{iX} - A\!\!\!/\right)\right] \tag{8}$$

Using the property of matrices, $\det(AB) = \det A.\det B$. Thus

$$e^{iS} = e^{\int d^3x \frac{-1}{4}F^2} \det(\mathcal{I} - iXA\!\!\!/) \tag{9}$$

Taking log on both sides we get

$$iS = \int d^3x \frac{-1}{4}F^2 + ln\left[\det(\mathcal{I} - iXA\!\!\!/)\right]$$

Again using the property of matrices, $ln[\det A] = Tr[lnA]$

$$iS = \int d^3x \frac{-1}{4}F^2 + Tr\left[ln(\mathcal{I} - iXA\!\!\!/)\right] \tag{10}$$

Using expansion of $ln(1-x) = -x - \frac{x^2}{2} - ..$ we get

$$iS = \int d^3x \frac{-1}{4}F^2 + Tr\left[-iXA\!\!\!/ + \frac{1}{2}XA\!\!\!/XA\!\!\!/ + ...\right] \tag{11}$$

# Emergence of Chern Simon term

$$
\begin{aligned}
iS &= \int d^3x \frac{-1}{4} F^2 + Tr\left[-iX \not{A}\right] + \frac{1}{2} Tr\left[X \not{A} X \not{A}\right] + \mathcal{O}(\not{A}^3) \\
iS &= \int d^3x \frac{-1}{4} F^2 + Tr\left[\frac{-1}{i\not{\partial} - m} \not{A}\right] - \frac{1}{2} Tr\left[\frac{1}{i\not{\partial} - m} \not{A} \frac{1}{i\not{\partial} - m} \not{A}\right] + \mathcal{O}(\not{A}^3) \\
iS &= (-1)\left[\int d^3x \frac{1}{4} F^2 + Tr\left[\frac{1}{i\not{\partial} - m} \not{A}\right] + \frac{1}{2} Tr\left[\frac{1}{i\not{\partial} - m} \not{A} \frac{1}{i\not{\partial} - m} \not{A}\right] + \mathcal{O}(\not{A}^3)\right]
\end{aligned}
\tag{12}
$$

- First term give rise to the Maxwell term i.e. $\int d^3x \frac{-1}{4} F^2$. Chern Simon term is quadratic in $A$. So, we should look for the third term i.e. $\frac{1}{2} Tr\left[\frac{1}{i\not{\partial}-m} \not{A} \frac{1}{i\not{\partial}-m} \not{A}\right]$.
- The second term gives rise to the tadpole term (one-loop Feynman diagram with one external leg) i.e. $Tr\left[\frac{1}{i\not{\partial}-m} \not{A}\right]$.
- Therefore we are interested in $\frac{1}{2} Tr\left[\frac{1}{i\not{\partial}-m} \not{A} \frac{1}{i\not{\partial}-m} \not{A}\right]$.

- Trace is a combination of space-time and spinors. We remove the space-time trace as follows.

$$\frac{1}{2}Tr\left[\frac{1}{i\not\partial - m}\not A \frac{1}{i\not\partial - m}\not A\right] = \frac{1}{2}tr(\text{spinors})\int d^3x \langle x| \left[\frac{1}{i\not\partial - m}\not A \frac{1}{i\not\partial - m}\not A\right]|x\rangle \tag{13}$$

$$tr(\text{spinors})\int d^3x \langle x| \left[\frac{1}{i\not\partial - m}\not A \frac{1}{i\not\partial - m}\not A\right]|x\rangle = tr(\text{spinors}) \int d^3x d^3y d^3z d^3w \langle x| \frac{1}{i\not\partial - m}|y\rangle \langle y|\not A|z\rangle \langle z|\frac{1}{i\not\partial - m}|w\rangle\langle w|\not A|x\rangle \tag{14}$$

where,

$$\langle x|\frac{1}{i\not\partial - m}|y\rangle = \int \bar{d}^3k \frac{e^{ik(x-y)}}{i\not k - m}$$

$$\langle y|\not A|z\rangle = \not A(y)\int \bar{d}^3l e^{il(y-z)}$$

$$\langle z|\frac{1}{i\not\partial - m}|w\rangle = \int \bar{d}^3k' \frac{e^{ik'(z-w})}{i\not k' - m}$$

$$\langle w|\not A|x\rangle = \not A(w)\int \bar{d}^3l' e^{il'(w-x)}$$

Thus,

$$\frac{1}{2}Tr\left[\frac{1}{i\partial\!\!\!/ - m}A\!\!\!/\frac{1}{i\partial\!\!\!/ - m}A\!\!\!/\right] = \frac{1}{2}tr(\text{spinors})\int d^3xd^3yd^3zd^3w đ^3k\frac{e^{ik(x-y)}}{ik\!\!\!/ - m}$$
$$A\!\!\!/(y)\int đ^3le^{il(y-z)}\int đ^3k'\frac{e^{ik'(z-w)}}{ik\!\!\!/' - m}A\!\!\!/(w)\int đ^3l'e^{il'(w-x)} \quad (15)$$

$$= \quad \frac{1}{2}tr(\text{spinors})\int d^3yd^3w\int đ^3k\frac{e^{i(w-y)k}}{ik\!\!\!/ - m}A\!\!\!/(y)\int đ^3k'\frac{e^{i(y-w)k'}}{ik\!\!\!/' - m}A\!\!\!/(w)$$

$$= (-ie)^2\frac{1}{2}tr(\text{spinors})\int d^3yd^3wA^\mu(y)A^\nu(w)\int đ^3k\frac{e^{i(w-y)k}}{ik\!\!\!/ - m}\gamma_\mu\int đ^3k'\frac{e^{i(y-w)k'}}{ik\!\!\!/' - m}\gamma_\nu$$

Rearranging and integrating we get

$$= \frac{-e^2}{2}\int đ^3k\int đ^3k'A^\mu(k-k')tr(\text{spinors})\left[\frac{1}{ik\!\!\!/ - m}\gamma_\mu\frac{1}{ik\!\!\!/' - m}\gamma_\nu\right]A^\nu(k'-k) \quad (16)$$

Let $k' - k = p \implies k - k' = -p$. So we get,

$$= \frac{-e^2}{2} \int \text{đ}^3 p A^\mu(-p) \left\{ \int \text{đ}^3 k tr(\text{spinors}) \left[ \frac{1}{i\not{k} - m} \gamma_\mu \frac{1}{i(\not{k} + \not{p}) - m} \gamma_\nu \right] \right\} A^\nu(p) \tag{17}$$

Since, $k' = k + p$. We know the chern simon term is quadratic in $A$ with a $\epsilon$ term which arises through three gamma matrices. Let us now look at
$\left\{ \int \text{đ}^3 k tr(\text{spinors}) \left[ \frac{1}{i\not{k} - m} \gamma_\mu \frac{1}{i(\not{k} + \not{p}) - m} \gamma_\nu \right] \right\}$

$$\int \text{đ}^3 k tr(\text{spinors}) \left[ \frac{1}{i\not{k} - m} \gamma_\mu \frac{1}{i(\not{k} + \not{p}) - m} \gamma_\nu \right] = \int \text{đ}^3 k tr(\text{spinors}) \left[ \frac{i\not{k} + m}{k^2 + m^2} \gamma_\mu \frac{i(\not{k} + \not{p}) + m}{(k + p)^2 + m^2} \gamma_\nu \right] \tag{18}$$

We now look for the terms which can give 3 gamma matrices so that,

$$tr(\gamma_\mu \gamma_\lambda \gamma_\nu) = 2\epsilon_{\mu\lambda\nu} \tag{19}$$

Thus we get the following terms,

$$\left[ \frac{i\not{k} + m}{k^2 + m^2} \gamma_\mu \frac{i(\not{k} + \not{p}) + m}{(k+p)^2 + m^2} \gamma_\nu \right] = \frac{i\not{k} \gamma_\mu m \gamma_\nu + m \gamma_\mu i(\not{k} + \not{p}) \gamma_\nu}{(k^2 + m^2)((k+p)^2 + m^2)}$$

$$\left[ \frac{i\not{k} + m}{k^2 + m^2} \gamma_\mu \frac{i(\not{k} + \not{p}) + m}{(k+p)^2 + m^2} \gamma_\nu \right] = im \left[ \frac{\gamma_\lambda k^\lambda \gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\lambda k^\lambda \gamma_\nu + \gamma_\mu \gamma_\lambda p^\lambda \gamma_\nu}{(k^2 + m^2)((k+p)^2 + m^2)} \right]$$

$$tr(\text{spinors}) \left[ \frac{i\not{k} + m}{k^2 + m^2} \gamma_\mu \frac{i(\not{k} + \not{p}) + m}{(k+p)^2 + m^2} \gamma_\nu \right] = \\ im \left[ tr(\text{spinors}) \left[ \frac{k^\lambda \gamma_\mu \gamma_\nu \gamma_\lambda + \gamma_\nu \gamma_\mu \gamma_\lambda k^\lambda + \gamma_\nu \gamma_\mu \gamma_\lambda p^\lambda}{(k^2 + m^2)((k+p)^2 + m^2)} \right] \right] \tag{20}$$

$$tr(\text{spinors})\left[\frac{i\not{k}+m}{k^2+m^2}\gamma_\mu\frac{i(\not{k}+\not{p})+m}{(k+p)^2+m^2}\gamma_\nu\right] =$$

$$im\left[tr(\text{spinors})\left[\frac{k^\lambda(2\epsilon_{\mu\nu\lambda})+(2\epsilon_{\nu\mu\lambda})k^\lambda+(2\epsilon_{\nu\mu\lambda})p^\lambda}{(k^2+m^2)((k+p)^2+m^2)}\right]\right] \quad (21)$$

$$tr(\text{spinors})\left[\frac{i\not{k}+m}{k^2+m^2}\gamma_\mu\frac{i(\not{k}+\not{p})+m}{(k+p)^2+m^2}\gamma_\nu\right] = -2im\left[\frac{\epsilon_{\mu\nu\lambda}}{(k^2+m^2)((k+p)^2+m^2)}\right]$$

Thus,

$$\frac{1}{2}Tr\left[\frac{1}{i\not{\partial}-m}\not{A}\frac{1}{i\not{\partial}-m}\not{A}\right] = -ime^2\int đ^3pA^\mu(-p)\epsilon_{\mu\nu\lambda}p^\lambda\left\{\int đ^3k\frac{1}{(k^2+m^2)((k+p)^2+m^2)}\right\}A^\nu(p) \quad (22)$$

# Feynman Trick

Now by using Feynman trick, $\int \frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2}$,
where $A = k^2 + m^2, B = (k+p)^2 + m^2$ The denominator becomes,
$\int d\!\!{}^-{}^3k \int_0^1 dx \frac{1}{[(k+px)^2+p^2x(1-x)+m^2]^2}$. To evaluate this we need to use beta and gamma functions. We can use this identity,

$$\int d\!\!{}^-{}^3k \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{3/2}} \frac{\Gamma(1/2)}{\Gamma(2)} \frac{1}{\Delta^{1/2}} \tag{23}$$

where, $k = k + px, \Delta = p^2x(1-x) + m^2$. This integral is evaluated by converting the given integral in spherical polar co ordinates and doing the wick rotations. Therefore, we get

$$\frac{1}{2}Tr\left[\frac{1}{i\partial\!\!\!/ - m} A\!\!\!/ \frac{1}{i\partial\!\!\!/ - m} A\!\!\!/\right] = \frac{me^2\sqrt{\pi}}{(4\pi)^{3/2}} \int d\!\!{}^-{}^3p A^\mu(-p)\epsilon_{\mu\nu\lambda}p^\lambda \left\{ \int_0^1 dx \frac{1}{\sqrt{p^2x(1-x)+m^2}} \right\} A^\nu(p) \tag{24}$$

$$\frac{1}{2}Tr\left[\frac{1}{i\partial\!\!\!/ - m} A\!\!\!/ \frac{1}{i\partial\!\!\!/ - m} A\!\!\!/\right] = \frac{me^2 f(m)}{8\pi} \int d\!\!{}^-{}^3p A^\mu(-p)\epsilon_{\mu\nu\lambda}p^\lambda A^\nu(p) \tag{25}$$

where, $f(m)$ is the result of the integration of $\int d\!\!{}^-{}^3k \int_0^1 dx \frac{1}{[(k+px)^2+p^2x(1-x)+m^2]^2}$.

$$\text{We know that}\Delta^{1/2} = \sqrt{p^2 x(1-x) + m^2}$$

$$\frac{1}{\Delta^{1/2}} = \frac{1}{\pm m}\int_0^1 dx \frac{1}{\sqrt{\frac{p^2}{m^2}x(1-x)+1}}$$

By using the procedure of completing the squares, the denominator changes as follows

$$\frac{1}{\Delta^{1/2}} = \frac{1}{\pm m}\int_0^1 dx \frac{1}{\sqrt{\left(\frac{1}{4}\frac{p^2}{m^2}+1\right) - \left(\frac{p^2}{m^2}(x-\frac{1}{2})^2\right)}}$$

There are no divergences, so we take the limit $\frac{p^2}{m^2} \to 0 \ni m \neq 0$

$$\frac{1}{\Delta^{1/2}} = \frac{1}{\pm m}$$

$$\int đ^3 k \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{3/2}}\frac{\Gamma(1/2)}{\Gamma(2)}\frac{1}{\pm m} \tag{26}$$

The values of $\Gamma(1/2) = \sqrt{\pi}, \Gamma(2) = 1$

$$\int d\!\!\!{}^{-}{}^3k \frac{1}{(k^2 - \Delta)^2} = \frac{i}{8\pi} \frac{1}{|m|} \tag{27}$$

Thus we get the value of $f(m)$ as $|m|$.Transforming from momentum to position space again we get

$$\frac{1}{2} Tr \left[ \frac{1}{i\not{\partial} - m} \not{A} \frac{1}{i\not{\partial} - m} \not{A} \right] = \frac{me^2}{8\pi|m|} \int d^3x \epsilon_{\mu\nu\lambda} A^\mu \partial^\lambda A^\nu \tag{28}$$

The chern simon term has a zero mass dimension so there should be a mass dimensional term in the denominator which can be obtained through some regularization. Now, we write the whole action as follows:

$$iS = (-1) \int d^3x \left\{ \frac{1}{4} F^2 + \text{tadpole term} + \frac{me^2}{8\pi|m|} \epsilon_{\mu\nu\lambda} A^\mu \partial^\lambda A^\nu \right\} + \mathcal{O}(\not{A}^3) \tag{29}$$

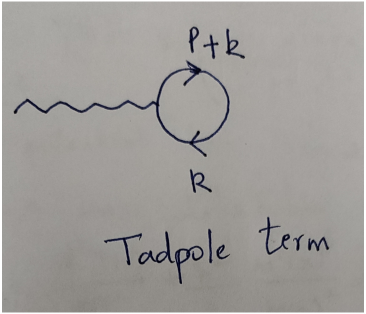


Figure: Tadpole term

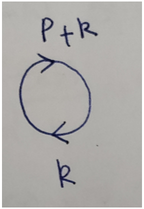


Figure: Propagators

Now we only focus the chern simon term,

$$iS = (-1)\int d^3x \frac{me^2}{8\pi|m|}\epsilon_{\mu\nu\lambda}A^\mu\partial^\lambda A^\nu \tag{30}$$

So, we get the action as

$$S = (i)\int d^3x \frac{me^2}{8\pi|m|}\epsilon_{\mu\nu\lambda}A^\mu\partial^\lambda A^\nu \tag{31}$$

Concerned Lagrangian which has the chern simon term is given by

$$\mathcal{L}(\text{chern simon}) = \frac{ime^2}{8\pi|m|}\epsilon_{\mu\nu\lambda}A^\mu\partial^\lambda A^\nu \tag{32}$$

## Gamma Matrices

For space-time dimension $d$, the matrices would be $n \times n$ where $n = 2^{\lfloor \frac{d}{2} \rfloor}$. I think the following matrices satisfies all the properties of gamma matrices.

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

From above we know, $tr(\gamma_\mu \gamma_\lambda \gamma_\nu) = 2\epsilon_{\mu\lambda\nu}$, where, $\mu = 0, \nu = 3, \lambda = 2$

$$tr(\gamma_\mu \gamma_\lambda \gamma_\nu) = tr\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$tr(\gamma_\mu \gamma_\lambda \gamma_\nu) = tr\left\{ \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

$$tr(\gamma_\mu \gamma_\lambda \gamma_\nu) = -2i \tag{33}$$

Therefore, we substitute $-i$ instead of $\epsilon_{\mu\nu\lambda}$. We get the Chern simon Lagrangian as

$$\mathcal{L}(\text{chern simon}) = \frac{me^2}{8\pi|m|} A^{\mu}\partial^{\lambda}A^{\nu} \tag{34}$$

Substituting $\mu = 0, \nu = 3, \lambda = 2$

$$\mathcal{L}(\text{chern simon}) = \frac{me^2}{8\pi|m|} A^{0}\partial^{2}A^{3} \tag{35}$$

Thus, the chern simon action is given by

$$S_{CS} = \int d^3x \frac{me^2}{8\pi|m|} A^{0}\partial^{2}A^{3} \tag{36}$$

# Vacuum Polarisation

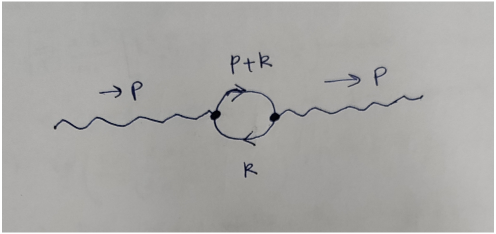


Figure: Vacuum Polarisation

$$\Pi^{\mu\nu}(p)(odd) = \int d^3p A^{\mu}(-p) \left\{ \int d^3k tr(\text{spinors}) \left[ \frac{1}{i\not{k}-m} \gamma_{\mu} \frac{1}{i(\not{k}+\not{p})-m} \gamma_{\nu} \right] \right\} A^{\nu}(p)$$

# Quantum Hall effect - CS term

$$Z[A] = N \int d\Psi e^{iS[\Psi, A] + iA^\mu J_\mu} \tag{37}$$

For the CS term this can be given by

$$Z[\not{A}] = N \int d\Psi e^{ikS_{CS}[\Psi, A] + iA^\mu J_\mu} \tag{38}$$

$$< J^\mu >= \frac{1}{i} \frac{\delta lnZ}{\delta A^\mu} = \frac{-k\delta S_{CS}}{i\delta A^\mu} = \frac{k}{4\pi} \epsilon_{\mu\lambda\nu} A^\mu A^\lambda A^\nu \tag{39}$$

# IQHE

$$< J^i >= \frac{k}{4\pi}(\epsilon_{ij0})\partial^j A^0 = \frac{k}{2\pi}\epsilon_{ij0}E^j$$

$$< J_0 >= \frac{k}{4\pi}(\epsilon_{0ij})\partial^i A^j = \frac{k}{2\pi}\epsilon_{0ij}B$$

$$\sigma = \frac{k}{2\pi}$$

where, $k$ is an integer. $k$ describes filled Landau levels. The quantised CS coupling (as a result of gauge invariance) therefore means the CS action necessarily describes the integer quantum Hall effect with $\nu = k$ filled Landau levels. Number of electrons in the Landau level is

$$n_e = k \int d^2x \frac{B}{2\pi} = kg \tag{40}$$

$g$- number of electrons in each Landau level.

# Parity Anomaly

Action:

$$S_{CS} = (i) \int d^3x \frac{me^2}{8\pi|m|} \epsilon_{\mu\nu\lambda} A^\mu \partial^\lambda A^\nu \tag{41}$$

- Parity operator - $P_\mu^\nu = \text{diag}(1, -1, 1)$. It acts on $x_\mu \rightarrow P_\mu^\nu x_\nu$.
- The gauge field transforms in the same way the coordinates since it is a covariant vector.

$$\epsilon_{\mu\nu\lambda} A^\mu \partial^\lambda A^\nu \rightarrow -\epsilon_{\mu\nu\lambda} A^\mu \partial^\lambda A^\nu$$

- Since we expect to use the CS theory to describe a system in a magnetic field which inherently breaks parity. We have now checked that the CS action possesses all of the required features needed for it to describe the integer quantum Hall effect.

# Parity anomaly - Detailed

- Parity inversion acts as $x^1 \to x^1$ and $x^2 \to -x^2$, and acts on the fermion as $\psi \to \sigma^2\psi$.
- The mass term $m\bar{\psi}\psi$ breaks this $\mathcal{Z}^2$ parity symmetry, and therefore an effective theory derived from a massive fermion may also break parity symmetry.
- The Chern–Simons term is one such parity-odd term, and we will indeed find that it arises as a quantum correction to the effective action.
- This result, wherein the CS term breaks the classical parity symmetry of the gauge field, is dubbed the 'parity anomaly'.

- An anomaly is a symmetry which is conserved in the classical action $S$, but is broken in the quantum path integral.
- In our example the tree-level effective action is parity-symmetric but if one tries to quantise the massless theory there are IR divergences which must be regulated with a mass.
- This mass term immediately breaks parity symmetry (and so do the one-loop vacuum polarisation bubble diagrams which are generated in perturbation theory), but this is an unavoidable step which must be taken to have a regularised quantum theory.
- Therefore it can be shown that the parity symmetry is not a true symmetry of the quantum theory, and the theory is anomalous.

Thank you for your attention !