Interpolating Manifestly Covariant Conformal Algebra $(1+1)$ between IFD and LFD

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Outline

Conformal Transformations

Let us consider a flat space in d dimensions and transformations thereof, which locally preserves the angle between any two lines. A conformal transformation is a smooth, invertible map $x \to x'$ such that

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x), \tag{1.1}$$

where the positive function $\Lambda(x)$ is called the scale factor.
Furthermore, for flat spaces, the scale factor $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of translations and rotations, respectively, Lorentz transformations.
Let us next consider the infinitesimal coordinate transformations which up to first order in a small parameter $\epsilon(x) << 1$ read

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2). \tag{1.2}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$ of $x'_\mu = x_\mu + \epsilon_\mu(x) + \mathcal{O}(\epsilon^2)$.

$$\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} \tag{1.3}$$

$$\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = M^\mu_\nu x^\nu \tag{1.4}$$

$$\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu \tag{1.5}$$

$$\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu \tag{1.6}$$

The Finite conformal transformations are:

$$\text{(translation)} \quad x'^\mu = x^\mu + a^\mu$$

$$\text{(rotation)} \quad x'^\mu = M^\mu_\nu x^\nu$$

$$\text{(dilatation)} \quad x'^\mu = \alpha x^\mu$$

$$\text{(SCT)} \quad x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$$

SCT

Inversions are given by

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \tag{1.7}$$

The SCTs can be understood as an inversion of x^μ, followed by a translation b^μ, and followed again by an inversion.

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \longrightarrow \boxed{x''^\mu = \frac{x^\mu}{x^2} - b^\mu} \longrightarrow \boxed{x'''^\mu = \frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}}$$

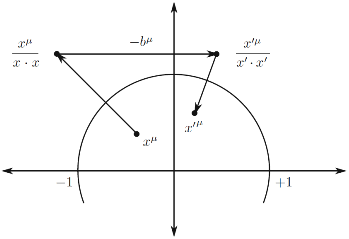

Figure 1: Illustration of a finite SCT

Conformal algebra

The generators of conformal transformations are: $P^\mu = i\partial^\mu$ (translation), $M^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$ (rotation), $D = ix_\mu\partial^\mu$ (dilation or scaling), and $\mathfrak{K}^\mu = i\left(2x^\mu x_\nu\partial^\nu - x^2\partial^\mu\right)$ (SCT).
Therefore, the full Conformal algebra is given by

$$\begin{aligned}
[P_\mu, P_\nu] &= 0;\ [\mathfrak{K}_\mu, \mathfrak{K}_\nu] = 0; \\
[D, P_\mu] &= -iP_\mu;\ [D, \mathfrak{K}_\mu] = i\mathfrak{K}_\mu; \\
[P_\rho, M_{\mu\nu}] &= i\left(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu\right); \\
[\mathfrak{K}_\rho, M_{\mu\nu}] &= i\left(g_{\rho\mu}\mathfrak{K}_\nu - g_{\rho\nu}\mathfrak{K}_\mu\right); \\
[M_{\alpha\beta}, M_{\rho\sigma}] &= -i\left(g_{\beta\sigma}M_{\alpha\rho} - g_{\beta\rho}M_{\alpha\sigma} + g_{\alpha\rho}M_{\beta\sigma} - g_{\alpha\sigma}M_{\beta\rho}\right); \\
[\mathfrak{K}_\mu, P_\nu] &= -2i\left(g_{\mu\nu}D + M_{\mu\nu}\right);\ [D, M_{\mu\nu}] = 0.
\end{aligned}$$

Full conformal algebra in Interpolation

A comprehensive table of the **105 commutation** relations among the co-variant components of the Conformal generators is presented below:

	$P_{\hat{+}}$	$P_{\hat{1}}$	$P_{\hat{2}}$	$K^{\hat{3}}$	$\mathcal{D}^{\hat{1}}$	$\mathcal{D}^{\hat{2}}$	$J^{\hat{3}}$	$\mathcal{K}^{\hat{1}}$	$\mathcal{K}^{\hat{2}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$\mathfrak{K}_{\hat{1}}$	$\mathfrak{K}_{\hat{2}}$	$\mathfrak{K}_{\hat{-}}$	D
$P_{\hat{+}}$	0	0	0	$i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	$i\mathbb{C}P_{\hat{1}}$	$i\mathbb{C}P_{\hat{2}}$	0	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$2i(\mathbb{S}D - K^{\hat{3}})$	$iP_{\hat{+}}$
$P_{\hat{1}}$	0	0	0	0	$iP_{\hat{+}}$	0	$-iP_{\hat{2}}$	$iP_{\hat{-}}$	0	0	$2i\mathcal{D}^{\hat{1}}$	$-2iD$	$-2iJ^{\hat{3}}$	$2i\mathcal{K}^{\hat{1}}$	$iP_{\hat{1}}$
$P_{\hat{2}}$	0	0	0	0	0	$iP_{\hat{+}}$	$iP_{\hat{1}}$	0	$iP_{\hat{-}}$	0	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$-2iD$	$2i\mathcal{K}^{\hat{2}}$	$iP_{\hat{2}}$
$K^{\hat{3}}$	$-i\left(\mathbb{C}P_{\hat{-}} - \mathbb{S}P_{\hat{+}}\right)$	0	0	0	$i\mathbb{S}\mathcal{D}^{\hat{1}} - i\mathbb{C}\mathcal{K}^{\hat{1}}$	$i\mathbb{S}\mathcal{D}^{\hat{2}} - i\mathbb{C}\mathcal{K}^{\hat{2}}$	0	$-i\mathbb{S}\mathcal{K}^{\hat{1}} - i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-i\mathbb{S}\mathcal{K}^{\hat{2}} - i\mathbb{C}\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	0	0	$-i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	0
$\mathcal{D}^{\hat{1}}$	$-i\mathbb{C}P_{\hat{1}}$	$-iP_{\hat{+}}$	0	$-i\mathbb{S}\mathcal{D}^{\hat{1}} + i\mathbb{C}\mathcal{K}^{\hat{1}}$	0	$-i\mathbb{C}J^{\hat{3}}$	$-i\mathcal{D}^{\hat{2}}$	$iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathbb{S}P_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{+}}$	0	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{D}^{\hat{2}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$-iP_{\hat{+}}$	$-i\mathbb{S}\mathcal{D}^{\hat{2}} + i\mathbb{C}\mathcal{K}^{\hat{2}}$	$i\mathbb{C}J^{\hat{3}}$	0	$i\mathcal{D}^{\hat{1}}$	$i\mathbb{S}J^{\hat{3}}$	$iK^{\hat{3}}$	$-i\mathbb{S}P_{\hat{2}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{+}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0
$J^{\hat{3}}$	0	$iP_{\hat{2}}$	$-iP_{\hat{1}}$	0	$i\mathcal{D}^{\hat{2}}$	$-i\mathcal{D}^{\hat{1}}$	0	$i\mathcal{K}^{\hat{2}}$	$-i\mathcal{K}^{\hat{1}}$	0	0	$i\mathfrak{K}_{\hat{2}}$	$-i\mathfrak{K}_{\hat{1}}$	0	0
$\mathcal{K}^{\hat{1}}$	$-i\mathbb{S}P_{\hat{1}}$	$-iP_{\hat{-}}$	0	$i\mathbb{S}\mathcal{K}^{\hat{1}} + i\mathbb{C}\mathcal{D}^{\hat{1}}$	$-iK^{\hat{3}}$	$-i\mathbb{S}J^{\hat{3}}$	$-i\mathcal{K}^{\hat{2}}$	0	$i\mathbb{C}J^{\hat{3}}$	$i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$-i\mathfrak{K}_{\hat{-}}$	0	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	0
$\mathcal{K}^{\hat{2}}$	$-i\mathbb{S}P_{\hat{2}}$	0	$-iP_{\hat{-}}$	$i\mathbb{S}\mathcal{K}^{\hat{2}} + i\mathbb{C}\mathcal{D}^{\hat{2}}$	$i\mathbb{S}J^{\hat{3}}$	$-niK^{\hat{3}}$	$i\mathcal{K}^{\hat{1}}$	$-i\mathbb{C}J^{\hat{3}}$	0	$i\mathbb{C}P_{\hat{2}}$	$-i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathfrak{K}_{\hat{-}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0
$P_{\hat{-}}$	0	0	0	$i\left(\mathbb{S}P_{\hat{-}} + \mathbb{C}P_{\hat{+}}\right)$	$i\mathbb{S}P_{\hat{1}}$	$i\mathbb{S}P_{\hat{2}}$	0	$-i\mathbb{C}P_{\hat{1}}$	$-i\mathbb{C}P_{\hat{2}}$	0	$2i(\mathbb{S}D + K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$-2i\mathbb{C}D$	$iP_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$-2i\mathbb{C}D$	$-2i\mathcal{D}^{\hat{1}}$	$-2i\mathcal{D}^{\hat{2}}$	$-i\left(\mathbb{S}\mathfrak{K}_{\hat{+}} - \mathbb{C}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$i\mathbb{C}\mathfrak{K}_{\hat{2}}$	0	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	$-2i(\mathbb{S}D + K^{\hat{3}})$	0	0	0	0	$-i\mathfrak{K}_{\hat{+}}$
$\mathfrak{K}_{\hat{1}}$	$2i\mathcal{D}^{\hat{1}}$	$2iD$	$-2iJ^{\hat{3}}$	0	$i\mathfrak{K}_{\hat{+}}$	0	$-i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0	$2i\mathcal{K}^{\hat{1}}$	0	0	0	0	$-i\mathfrak{K}_{\hat{1}}$
$\mathfrak{K}_{\hat{2}}$	$2i\mathcal{D}^{\hat{2}}$	$2iJ^{\hat{3}}$	$2iD$	0	0	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	0	$i\mathfrak{K}_{\hat{-}}$	$2i\mathcal{K}^{\hat{2}}$	0	0	0	0	$-i\mathfrak{K}_{\hat{2}}$
$\mathfrak{K}_{\hat{-}}$	$-2i(\mathbb{S}D - K^{\hat{3}})$	$-2i\mathcal{K}^{\hat{1}}$	$-2i\mathcal{K}^{\hat{2}}$	$i\left(\mathbb{C}\mathfrak{K}_{\hat{+}} + \mathbb{S}\mathfrak{K}_{\hat{-}}\right)$	$i\mathbb{S}\mathfrak{K}_{\hat{1}}$	$i\mathbb{S}\mathfrak{K}_{\hat{2}}$	0	$-i\mathbb{C}\mathfrak{K}_{\hat{1}}$	$-i\mathbb{C}\mathfrak{K}_{\hat{2}}$	$2i\mathbb{C}D$	0	0	0	0	$-i\mathfrak{K}_{\hat{-}}$
D	$-iP_{\hat{+}}$	$-iP_{\hat{1}}$	$-iP_{\hat{2}}$	0	0	0	0	0	0	$-iP_{\hat{-}}$	$i\mathfrak{K}_{\hat{+}}$	$i\mathfrak{K}_{\hat{1}}$	$i\mathfrak{K}_{\hat{2}}$	$i\mathfrak{K}_{\hat{-}}$	0

Kinematic and dynamic generators of the Conformal group[1,2]

The generators of conformal transformations:

$$\text{(translation)} \quad P^{\hat{\mu}} = i\partial^{\hat{\mu}} \tag{2.1}$$

$$\text{(dilation)} \quad D = ix_{\hat{\mu}}\partial^{\hat{\mu}} \tag{2.2}$$

$$\text{(rotation)} \quad M^{\hat{\mu}\hat{\nu}} = i\left(x^{\hat{\mu}}\partial^{\hat{\nu}} - x^{\hat{\nu}}\partial^{\hat{\mu}}\right) \tag{2.3}$$

$$\text{(SCT)} \quad \mathfrak{K}^{\hat{\mu}} = i\left(2x^{\hat{\mu}}x_{\hat{\nu}}\partial^{\hat{\nu}} - x^2\partial^{\hat{\mu}}\right) \tag{2.4}$$

Since $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \rightarrow \ i(x^0.x^0 + \vec{x}.\vec{x})$ as $\delta \rightarrow 0$, and $\left[\mathfrak{K}^{\hat{+}}, x^{\hat{+}}\right] = i\left(2x^{\hat{+}}x^{\hat{+}} - (x^{\hat{\alpha}}.x_{\hat{\alpha}})\mathbb{C}\right) \rightarrow \ i(2x^{+}.x^{+})$ as $\delta \rightarrow \pi/4$, the conformal generator (LF time component) $\mathfrak{K}_{-}$ is Kinematic in LFD, but Dynamic in IFD. And $[D, x^{\hat{+}}] = ix^{\hat{+}}$, so D is always Kinematic in both IFD and LFD.

Interpolation angle	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2,\ \mathcal{K}^{\hat{2}} = J^1,\ J^3, P^1, P^2, P^3,\ D$	$\mathcal{D}^{\hat{1}} = -K^1,\ \mathcal{D}^{\hat{2}} = -K^2,\ K^3, P^0,\ \mathfrak{K}_0,\ \mathfrak{K}_1,\ \mathfrak{K}_2,\ \mathfrak{K}_3$
$0 \leq \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}},\ D$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}},\ \mathfrak{K}_{\hat{+}},\ \mathfrak{K}_{\hat{1}},\ \mathfrak{K}_{\hat{2}},\ \mathfrak{K}_{\hat{-}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1,\ \mathcal{K}^{\hat{2}} = -E^2,\ J^3, K^3, P^1, P^2, P_{-},\ D,\ \mathfrak{K}_{-}$	$\mathcal{D}^{\hat{1}} = -F^1,\ \mathcal{D}^{\hat{2}} = -F^2,\ P_{+},\ \mathfrak{K}_{+},\ \mathfrak{K}_1,\ \mathfrak{K}_2$

[1] Chueng-Ryong Ji and Chad Mitchell, Phys. Rev. **D 64**, 085013 (2001).

[2] Chueng-Ryong Ji, Ziyue Li, and Alfredo Takashi Suzuki, Phys. Rev. **D 91**, 065020 (2015).

Boost as Rotations in 4D

The 4D Angular momentum tensor is given by:

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}. \tag{3.1}$$

$$\left[M^{\alpha\beta}, M^{\rho\sigma}\right] = -i\left(g^{\beta\sigma} M^{\alpha\rho} - g^{\beta\rho} M^{\alpha\sigma} + g^{\alpha\rho} M^{\beta\sigma} - g^{\alpha\sigma} M^{\beta\rho}\right) \tag{3.2}$$

There are $\frac{n(n-1)}{2}$ number of planes in n dimension:

Dimension	# of planes
1D	0
2D	1
3D	3
4D	6
5D	10
6D	15

(3.3)

Interpolating Manifestly Covariant Conformal Algebra $(3+1)$

We define the following 6×6 tensor in the projective-space-time:

$$J_{\hat{a},\hat{b}} = \begin{pmatrix} 0 & -D & -\frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{\mathfrak{K}_1}{\sqrt{2}} & -\frac{\mathfrak{K}_2}{\sqrt{2}} & -\frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} \\ D & 0 & \frac{P_{\hat{+}}}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_{\hat{-}}}{\sqrt{2}} \\ \frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{P_{\hat{+}}}{\sqrt{2}} & 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & -\frac{P_1}{\sqrt{2}} & -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & -\frac{P_2}{\sqrt{2}} & -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ \frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} & -\frac{P_{\hat{-}}}{\sqrt{2}} & -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}_{(6\times 6)} \tag{3.4}$$

Then, the simplified conformal algebra in interpolation is:

$$\left[J_{\hat{a}\hat{b}}, J_{\hat{c}\hat{d}}\right] = -i\left(g_{\hat{b}\hat{d}}J_{\hat{a}\hat{c}} - g_{\hat{b}\hat{c}}J_{\hat{a}\hat{d}} + g_{\hat{a}\hat{c}}J_{\hat{b}\hat{d}} - g_{\hat{a}\hat{d}}J_{\hat{b}\hat{c}}\right) \tag{3.5}$$

where,

$$g_{\hat{a},\hat{b}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}_{6\times 6} \tag{3.6}$$

Isomorphism with Dirac matrices

Dirac[3] has shown the existence of isomorphism between $SO(4,2)$ conformal group and Dirac matrices. Later, Hepner[4] has explicitly shown the isomorphism between the group of Dirac's four-row γ-matrices and the continuous conformal group in Euclidean space.
We show

$$J'_{a,b} = \begin{pmatrix} 0 & \gamma_5 & \frac{(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_3}{\sqrt{2}} \\ -\gamma_5 & 0 & \frac{(1+\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_3}{\sqrt{2}} \\ \frac{-(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_0}{\sqrt{2}} & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_1}{\sqrt{2}} & \gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_2}{\sqrt{2}} & \gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_3}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_3}{\sqrt{2}} & \gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{pmatrix}. \tag{3.7}$$

This $J'_{a,b}$ obeys the $SO(4+1,1)$ algebra
$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right)$

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[3] Dirac, P. A. M. Annals Math. 37, 429–442 (1936)
[4] Hepner, W. A. Nuovo Cim. 26, 351–368 (1962).

Isomorphism with Dirac matrices

The representation of the conformal group in terms of 4×4 gamma matrices are the following;

$$P_\mu = \frac{i}{2}(1+\gamma_5)\gamma_\mu; \tag{3.8}$$

$$\mathfrak{K}_\mu = \frac{-i}{2}(1-\gamma_5)\gamma_\mu; \tag{3.9}$$

$$K^1 = \frac{i}{2}\gamma_1\gamma_0; \quad K^2 = \frac{i}{2}\gamma_2\gamma_0; \quad K^3 = \frac{i}{2}\gamma_3\gamma_0; \tag{3.10}$$

$$J^1 = \frac{i}{2}\gamma_2\gamma_3; \quad J^2 = \frac{i}{2}\gamma_3\gamma_1; \quad J^3 = \frac{i}{2}\gamma_1\gamma_2; \tag{3.11}$$

$$D = \frac{-i}{2}\gamma_5. \tag{3.12}$$

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

We have:

$$J_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & -D & -\frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} \\ D & 0 & \frac{P_{\hat{+}}}{\sqrt{2}} & \frac{P_{\hat{-}}}{\sqrt{2}} \\ \frac{\mathfrak{K}_{\hat{+}}}{\sqrt{2}} & -\frac{P_{\hat{+}}}{\sqrt{2}} & 0 & K^3 \\ \frac{\mathfrak{K}_{\hat{-}}}{\sqrt{2}} & -\frac{P_{\hat{-}}}{\sqrt{2}} & -K^3 & 0 \end{pmatrix}_{(4\times 4)} \tag{4.1}$$

Then, the $1+1$ conformal algebra in LFD is given by

$$\left[J_{\hat{a}\hat{b}}, J_{\hat{c}\hat{d}}\right] = -i\left(g_{\hat{b}\hat{d}}J_{\hat{a}\hat{c}} - g_{\hat{b}\hat{c}}J_{\hat{a}\hat{d}} + g_{\hat{a}\hat{c}}J_{\hat{b}\hat{d}} - g_{\hat{a}\hat{d}}J_{\hat{b}\hat{c}}\right) \tag{4.2}$$

where,

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & \mathbb{S} \\ 0 & 0 & \mathbb{S} & -\mathbb{C} \end{pmatrix}_{4\times 4} \tag{4.3}$$

From (4.2), one can write

$$J_{\hat{a}\hat{b}} = i(x_{\hat{a}}\partial_{\hat{b}} - x_{\hat{b}}\partial_{\hat{a}}) \tag{4.4}$$

where $\hat{a}, \hat{b} \in \{-2, -1, \hat{+}, \hat{-}\}$.

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

Let's say that $A^{\hat{a}}$ is hyper-4-vector; suppose $A^{\hat{a}}$ and B^{a} transform under 6d rotation:

$$A'^{\hat{a}} = R^{\hat{a}}{}_{\hat{b}} A^{\hat{b}}, \quad B'^{\hat{a}} = R^{\hat{a}}{}_{\hat{b}} B^{\hat{b}}. \tag{4.5}$$

Then the inner products $A' \cdot B'$ and $A \cdot B$ can be written as

$$A'_{\hat{b}} B'^{\hat{b}} = (g_{\hat{a}\hat{b}} R^{\hat{a}}{}_{\hat{c}} R^{\hat{b}}{}_{\hat{d}}) A^{\hat{c}} B^{\hat{d}}, \tag{4.6}$$

$$A_{\hat{b}} B^{\hat{b}} = g_{\hat{c}\hat{d}} A^{\hat{c}} B^{\hat{d}}. \tag{4.7}$$

In order for $A'.B' = A.B$ to hold for any A and B, the coefficients of $A^{\hat{c}} B^{\hat{d}}$ should be the same term by term:

$$g_{\hat{a}\hat{b}} R^{\hat{a}}{}_{\hat{c}} R^{\hat{b}}{}_{\hat{d}} = g_{\hat{c}\hat{d}}. \tag{4.8}$$

Let's start by looking at a hyper-4d rotation transformation, which is (has to be) infinitesimally close to the identity:

$$R^{\hat{a}}{}_{\hat{b}} = g^{\hat{a}}{}_{\hat{b}} + \omega^{\hat{a}}{}_{\hat{b}} \ , \tag{4.9}$$

where $\omega^{\hat{a}}{}_{\hat{b}}$ is a set of small (real) numbers.

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

Inserting this into the defining condition, we get

$$g^{\hat{c}\hat{d}} = R^{\hat{c}}{}_{\hat{b}} R^{\hat{b}\hat{d}} \ , \tag{4.10}$$

$$= (g^{\hat{c}}{}_{\hat{b}} + \omega^{\hat{c}}{}_{\hat{b}})(g^{\hat{b}\hat{d}} + \omega^{\hat{b}\hat{d}}) \ ,$$

$$= g^{\hat{c}\hat{d}} + \omega^{\hat{c}\hat{d}} + \omega^{\hat{d}\hat{c}} + \mathcal{O}(\omega^2). \tag{4.11}$$

Keeping terms to the first order in ω, we then obtain

$$\omega^{\hat{a}\hat{b}} = -\omega^{\hat{b}\hat{a}} \ . \tag{4.12}$$

Thus, it has 6 independent parameters:

$$\omega^{\hat{a}\hat{b}} = \begin{pmatrix} 0 & \omega^{\hat{-2},\hat{-1}} & \omega^{\hat{-2},\hat{+}} & \omega^{\hat{-2},\hat{-}} \\ -\omega^{\hat{-2},\hat{-1}} & 0 & \omega^{\hat{-1},\hat{+}} & \omega^{\hat{-1},\hat{-}} \\ -\omega^{\hat{-2},\hat{+}} & -\omega^{\hat{-1},\hat{+}} & 0 & \omega^{\hat{+},\hat{-}} \\ -\omega^{\hat{-2},\hat{-}} & -\omega^{\hat{1},\hat{-}} & -\omega^{\hat{+},\hat{-}} & 0 \end{pmatrix}_{(4\times4)} \ . \tag{4.13}$$

This can be conveniently parameterized using 6 anti-symmetric matrices as

$$\omega^{\hat{a}\hat{b}} = -i \sum_{\hat{c}<\hat{d}} \omega^{\hat{c}\hat{d}} (J_{\hat{c}\hat{d}})^{\hat{a}\hat{b}} \ , \tag{4.14}$$

Interpolating Manifestly Covariant Conformal Algebra (1 + 1)

where

$$(J_{\hat{c}\hat{d}})^{\hat{a}\hat{b}} = i(g^{\hat{a}}{}_{\hat{c}} g^{\hat{b}}{}_{\hat{d}} - g^{\hat{b}}{}_{\hat{c}} g^{\hat{a}}{}_{\hat{d}}). \tag{4.15}$$

then

$$\omega^{\hat{a}\hat{b}} = -i \sum_{\hat{c}<\hat{d}} \omega^{\hat{c}\hat{d}} (J_{\hat{c}\hat{d}})^{\hat{a}\hat{b}} = -i \sum_{\hat{c}>\hat{d}} \omega^{\hat{c}\hat{d}} (J_{\hat{c}\hat{d}})^{\hat{a}\hat{b}} = -\frac{i}{2} \omega^{\hat{c}\hat{d}} (J_{\hat{c}\hat{d}})^{\hat{a}\hat{b}} , \tag{4.16}$$

where in the last expression, the sum over all values of $\hat{c}$ and $\hat{d}$ is implied. The infinitesimal transformation (hyper-4d rotation) ((**??**)) can then be written a

$$R^{\hat{a}}{}_{\hat{b}} = g^{\hat{a}}{}_{\hat{b}} - \frac{i}{2} \omega^{\hat{c}\hat{d}} (J_{\hat{c}\hat{d}})^{\hat{a}}{}_{\hat{b}} , \tag{4.17}$$

The generator representation $(J_{\hat{c}\hat{d}})^{\hat{a}}{}_{\hat{b}}$ can be obtained by

$$(J_{\hat{c}\hat{d}})^{\hat{a}}{}_{\hat{b}} = (J_{\hat{c}\hat{d}})^{\hat{a}\hat{f}} g_{\hat{f}\hat{b}} \tag{4.18}$$

Interpolating Manifestly Covariant Conformal Algebra (1 + 1)

The representation matrices of conformal generators are defined by taking the first index to be superscript and the second subscript:

$$\frac{-\mathfrak{K}_{\hat{\mu}}}{\sqrt{2}} \equiv (J_{\hat{-2}\hat{\mu}})^{\hat{a}}{}_{\hat{b}} \; ; \quad \frac{P_{\hat{\mu}}}{\sqrt{2}} \equiv (J_{\hat{-1}\hat{\mu}})^{\hat{a}}{}_{\hat{b}} \; ; \quad -D \equiv (J_{\hat{-2}\hat{-1}})^{\hat{a}}{}_{\hat{b}} \; ; \quad K^3 \equiv (J_{\hat{+},\hat{-}})^{\hat{a}}{}_{\hat{b}}. \tag{4.19}$$

where $a, b \in \{-2, -1, \hat{+}, \hat{-}\}$ and $\mu \in \{\hat{+}, \hat{-}\}$.
Explicitly,

$$\mathfrak{K}_{\hat{+}} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ i\mathbb{C} & 0 & 0 & 0 \\ i\mathbb{S} & 0 & 0 & 0 \end{pmatrix} ; \quad \mathfrak{K}_{\hat{-}} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ i\mathbb{S} & 0 & 0 & 0 \\ -i\mathbb{C} & 0 & 0 & 0 \end{pmatrix} ; \quad D = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_{\hat{+}} = \sqrt{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i\mathbb{C} & 0 & 0 \\ 0 & -i\mathbb{S} & 0 & 0 \end{pmatrix} ; \quad P_{\hat{-}} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i\mathbb{S} & 0 & 0 \\ 0 & i\mathbb{C} & 0 & 0 \end{pmatrix} ; \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i\mathbb{S} & i\mathbb{C} \\ 0 & 0 & i\mathbb{C} & i\mathbb{S} \end{pmatrix} \tag{4.20}$$

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

We define the conformal transformations in interpolation as those transformations preserving the light cone. This is equivalent to preserving angles and also equivalent to preserving ratios of lengths. Let's consider a hyper-4d vector;

$$\tilde{x}_{\hat{-1}} = \frac{-\lambda}{\sqrt{2}}; \tag{4.21}$$

$$\tilde{x}_{\hat{-2}} = \frac{-\lambda}{\sqrt{2}}(x^{\hat{\mu}}.x_{\hat{\mu}}); \tag{4.22}$$

$$\tilde{x}_{\hat{\mu}} = \lambda x_{\hat{\mu}}; \tag{4.23}$$

Now, let's consider the hyper-4d dot product,

$$g_{\hat{a}\hat{b}}\tilde{x}^{\hat{a}}\tilde{x}^{\hat{b}} = \tilde{x}^{\hat{-2}}\tilde{x}_{\hat{-2}} + \tilde{x}^{\hat{-1}}\tilde{x}_{\hat{-1}} + \tilde{x}^{\hat{\mu}}\tilde{x}_{\hat{\mu}} \tag{4.24}$$

$$\tilde{x}_{\hat{a}}.\tilde{x}^{\hat{a}} = -2\tilde{x}_{\hat{-2}}\tilde{x}_{\hat{-1}} + \tilde{x}^{\hat{\mu}}\tilde{x}_{\hat{\mu}} \qquad (\because\ \tilde{x}_{\hat{-1}} = -\tilde{x}^{\hat{-2}}\ \&\ \tilde{x}_{\hat{-2}} = -\tilde{x}^{\hat{-1}}) \tag{4.25}$$

$$\tilde{x}_{\hat{a}}.\tilde{x}^{\hat{a}} = -2\frac{-\lambda}{\sqrt{2}}(x^{\hat{\mu}}.x_{\hat{\mu}})\frac{-\lambda}{\sqrt{2}} + \lambda^2 x^{\hat{\mu}}x_{\hat{\mu}} = -\lambda^2 x^{\hat{\mu}}x_{\hat{\mu}} + \lambda^2 x^{\hat{\mu}}x_{\hat{\mu}} \tag{4.26}$$

$$\tilde{x}_{\hat{a}}.\tilde{x}^{\hat{a}} = 0 \tag{4.27}$$

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

then,

$$\boxed{x_{\hat{\mu}} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_{\hat{\mu}}}{\tilde{x}_{\hat{-1}}}} \tag{4.28}$$

$$\boxed{\frac{x_{\hat{\mu}}}{x^2}} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_{\hat{\mu}}}{\tilde{x}_{\hat{-1}}}\frac{2\tilde{x}_{\hat{-1}}\tilde{x}_{\hat{-1}}}{\tilde{x}_{\hat{\mu}}\tilde{x}^{\hat{\mu}}} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_{\hat{\mu}}}{\tilde{x}_{\hat{-1}}}\frac{2\tilde{x}_{\hat{-1}}\tilde{x}_{\hat{-1}}}{2\tilde{x}_{\hat{-2}}\tilde{x}_{\hat{-1}}} = \boxed{-\frac{1}{\sqrt{2}}\frac{\tilde{x}_{\hat{\mu}}}{\tilde{x}_{\hat{-2}}}} \tag{4.29}$$

Also, $\boxed{x^2 = \frac{\tilde{x}_{\hat{-2}}}{\tilde{x}_{\hat{-1}}}}$.

Let's find the space-time transformation under each conformal generators in $1+1$.

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

For $\mathfrak{K}_{\hat{+}}$:

$$\begin{pmatrix} \tilde{x}'_{\hat{-}2} \\ \tilde{x}'_{\hat{-}1} \\ \tilde{x}'_{\hat{+}} \\ \tilde{x}'_{\hat{-}} \end{pmatrix} = \exp\left(-ib^{\hat{+}}\mathfrak{K}_{\hat{+}}\right) \begin{pmatrix} \tilde{x}_{\hat{-}2} \\ \tilde{x}_{\hat{-}1} \\ \tilde{x}_{\hat{+}} \\ \tilde{x}_{\hat{-}} \end{pmatrix} \tag{4.30}$$

$$\begin{pmatrix} \tilde{x}'_{\hat{-}2} \\ \tilde{x}'_{\hat{-}1} \\ \tilde{x}'_{\hat{+}} \\ \tilde{x}'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbb{C}(b^{\hat{+}})^2 & 1 & \sqrt{2}b^{\hat{+}} & 0 \\ \sqrt{2}\mathbb{C}b^{\hat{+}} & 0 & 1 & 0 \\ \sqrt{2}\mathbb{S}b^{\hat{+}} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{\hat{-}2} \\ \tilde{x}_{\hat{-}1} \\ \tilde{x}_{\hat{+}} \\ \tilde{x}_{\hat{-}} \end{pmatrix} \tag{4.31}$$

$$\begin{pmatrix} \tilde{x}'_{\hat{-}2} \\ \tilde{x}'_{\hat{-}1} \\ \tilde{x}'_{\hat{+}} \\ \tilde{x}'_{\hat{-}} \end{pmatrix} = \begin{pmatrix} \tilde{x}_{\hat{-}2} \\ \mathbb{C}(b^{\hat{+}})^2\tilde{x}_{\hat{-}2} + \tilde{x}_{\hat{-}1} + \sqrt{2}b^{\hat{+}}\tilde{x}_{\hat{+}} \\ \sqrt{2}\mathbb{C}b^{\hat{+}}\tilde{x}_{\hat{-}2} + \tilde{x}_{\hat{+}} \\ \sqrt{2}\mathbb{S}b^{\hat{+}}\tilde{x}_{\hat{-}2} + \tilde{x}_{\hat{-}} \end{pmatrix} \tag{4.32}$$

Interpolating Manifestly Covariant Conformal Algebra $(1+1)$

On space-time, this transformation gives

$$x'_{\hat{+}} = \frac{-1}{\sqrt{2}} \frac{\tilde{x}'_{\hat{+}}}{\tilde{x}'_{\hat{-}1}} \tag{4.33}$$

$$= \frac{-1}{\sqrt{2}} \frac{\sqrt{2}\mathbb{C}b^{\hat{+}}\tilde{x}_{\hat{-}2} + \tilde{x}_{\hat{+}}}{\mathbb{C}(b^{\hat{+}})^2\tilde{x}_{\hat{-}2} + \tilde{x}_{\hat{-}1} + \sqrt{2}b^{\hat{+}}\tilde{x}_{\hat{+}}} \tag{4.34}$$

$$= \frac{-1}{\sqrt{2}} \frac{\sqrt{2}\mathbb{C}b^{\hat{+}}\frac{\tilde{x}_{\hat{-}2}}{\tilde{x}_{\hat{-}1}} + \frac{\tilde{x}_{\hat{+}}}{\tilde{x}_{\hat{-}1}}}{\mathbb{C}(b^{\hat{+}})^2\frac{\tilde{x}_{\hat{-}2}}{\tilde{x}_{\hat{-}1}} + 1 + \sqrt{2}b^{\hat{+}}\frac{\tilde{x}_{\hat{+}}}{\tilde{x}_{\hat{-}1}}} \tag{4.35}$$

$$= -\frac{1}{\sqrt{2}} \frac{\sqrt{2}\mathbb{C}b^{\hat{+}}x^2 + \left(-\sqrt{2}x_{\hat{+}}\right)}{\mathbb{C}(b^{\hat{+}})^2x^2 + 1 + \sqrt{2}b^{\hat{+}}\left(-\sqrt{2}x_{\hat{+}}\right)} \tag{4.36}$$

$$x'_{\hat{+}} = \frac{x_{\hat{+}} - \mathbb{C}b^{\hat{+}}x^2}{1 - 2b^{\hat{+}}x_{\hat{+}} + \mathbb{C}(b^{\hat{+}})^2x^2} \tag{4.37}$$

Transformation of $x_{\hat{\pm}}$ under conformal transformation

In the interpolation form:

Generators	$x'_{\hat{+}}$	$x'_{\hat{-}}$
$\mathfrak{K}_{\hat{+}}$	$x'_{\hat{+}} = \dfrac{x_{\hat{+}} - \mathbb{C} b^{\hat{+}} x^2}{1 - 2b^{\hat{+}} x_{\hat{+}} + \mathbb{C}(b^{\hat{+}})^2 x^2}$	$x'_{\hat{-}} = \dfrac{x_{\hat{-}} - \mathbb{S} b^{\hat{+}} x^2}{1 - 2b^{\hat{+}} x_{\hat{+}} + \mathbb{C}(b^{\hat{+}})^2 x^2}$
$\mathfrak{K}_{\hat{-}}$	$x'_{\hat{+}} = \dfrac{x_{\hat{+}} - \mathbb{S} b^{\hat{-}} x^2}{1 - 2b^{\hat{-}} x_{\hat{-}} - \mathbb{C}(b^{\hat{-}})^2 x^2}$	$x'_{\hat{-}} = \dfrac{x_{\hat{-}} + \mathbb{C} b^{\hat{-}} x^2}{1 - 2b^{\hat{-}} x_{\hat{-}} - \mathbb{C}(b^{\hat{-}})^2 x^2}$
$P_{\hat{+}}$	$x'_{\hat{+}} = x_{\hat{+}} + \mathbb{C} a^{\hat{+}}$	$x'_{\hat{-}} = x_{\hat{-}} + \mathbb{S} a^{\hat{+}}$
$P_{\hat{-}}$	$x'_{\hat{+}} = x_{\hat{+}} + \mathbb{S} a^{\hat{-}}$	$x'_{\hat{-}} = x_{\hat{-}} - \mathbb{C} a^{\hat{-}}$
D	$x'_{\hat{+}} = e^{-\alpha} x_{\hat{+}}$	$x'_{\hat{-}} = e^{-\alpha} x_{\hat{-}}$
K^3	$x'_{\hat{+}} = (\cosh\eta_3 - \mathbb{S}\sinh\eta_3) x_{\hat{+}} + (\mathbb{C}\sinh\eta_3) x_{\hat{-}}$	$x'_{\hat{-}} = (\mathbb{C}\sinh\eta_3) x_{\hat{+}} + (\cosh\eta_3 + \mathbb{S}\sinh\eta_3) x_{\hat{-}}$

where, $x_{\hat{+}} = x_0 \cos\delta + x_3 \sin\delta$, and $x_{\hat{-}} = x_0 \sin\delta - x_3 \cos\delta$

Transformation of x_0 and x_3 under conformal transformation

In the instant form limit: $\delta \longrightarrow 0$, $\mathbb{S} \longrightarrow 0$, and $\mathbb{C} \longrightarrow 1$.

Generators	x'_0	x'_3
$\mathfrak{K}_0$	$x'_0 = \dfrac{x_0 - b_0 x^2}{1 - 2b_0 x_0 + (b_0)^2 x^2}$	$x'_3 = \dfrac{x_3}{1 - 2b_0 x_0 + (b_0)^2 x^2}$
$-\mathfrak{K}_3$	$x'_0 = \dfrac{x_0}{1 - 2b_3 x_{\underline{\hat{\ }}} - (b_3)^2 x^2}$	$x'_3 = \dfrac{x_3 + b_3 x^2}{1 - 2b_3 x_3 - (b_3)^2 x^2}$
P_0	$x'_0 = x_0 + a_0$	$x'_3 = x_3$
$-P_3$	$x'_0 = x_0$	$x'_3 = x_3 - a_3$
D	$x'_0 = e^{-\alpha} x_0$	$x'_3 = e^{-\alpha} x_3$
K^3	$x'_0 = (\cosh \eta_3) x_0 - (\sinh \eta_3) x_3$	$x'_3 = -(\sinh \eta_3) x_0 + (\cosh \eta_3) x_3$

Transformation of $x_\pm$ under conformal transformation

In the light front limit: $\delta \longrightarrow \frac{\pi}{4}$, $\mathbb{C} \longrightarrow 0$, and $\mathbb{S} \longrightarrow 1$.

Generators	x'_+	x'_-
$\mathfrak{K}_+$	$x'_+ = \dfrac{x_+}{1-2b^+x_+}$	$x'_- = x_-$
$\mathfrak{K}_-$	$x'_+ = x_+$	$x'_- = \dfrac{x_-}{1-2b^-x_-}$
P_+	$x'_+ = x_+$	$x'_- = x_- + a^+$
P_-	$x'_+ = x_+ + a^-$	$x'_- = x_-$
D	$x'_+ = e^{-\alpha}x_+$	$x'_- = e^{-\alpha}x_-$
K^3	$x'_+ = (\cosh\eta_3 - \sinh\eta_3)x_+$	$x'_- = (\cosh\eta_3 + \sinh\eta_3)x_-$

where, $x_+ = \frac{x_0+x_3}{\sqrt{2}}$, and $x_- = \frac{x_0-x_3}{\sqrt{2}}$

Interpolating D and K^3

Let's introduce the interpolating $D_{\hat{+}}$ and $D_{\hat{-}}$ as:

$$\begin{bmatrix} D_{\hat{+}} \\ D_{\hat{-}} \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\delta \\ \sin\delta & -\cos\delta \end{bmatrix} \begin{bmatrix} D \\ K^3 \end{bmatrix} \tag{5.1}$$

therefore,

$$D_{\hat{+}} = \cos\delta D + \sin\delta K^3, \tag{5.2}$$

$$D_{\hat{-}} = \sin\delta D - \cos\delta K^3. \tag{5.3}$$

We can also write

$$\begin{bmatrix} D \\ K^3 \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\delta \\ \sin\delta & -\cos\delta \end{bmatrix} \begin{bmatrix} D_{\hat{+}} \\ D_{\hat{-}} \end{bmatrix} \tag{5.4}$$

therefore,

$$D = \cos\delta D_{\hat{+}} + \sin\delta D_{\hat{-}}, \tag{5.5}$$

$$K^3 = \sin\delta D_{\hat{+}} - \cos\delta D_{\hat{-}}. \tag{5.6}$$

Interpolating D and K^3

The commutation relations among all interpolating conformal generators in two dimensional are given below:

Table 1: $1+1$ conformal algebra in the interpolation form.

	$P_{\hat{+}}$	$\mathfrak{K}_{\hat{-}}$	$D_{\hat{-}}$	$P_{\hat{-}}$	$\mathfrak{K}_{\hat{+}}$	$D_{\hat{+}}$
$P_{\hat{+}}$	0	$2i((\mathbb{S}\cos\delta-\sin\delta)D_{\hat{+}}+(\mathbb{S}\sin\delta+\cos\delta)D_{\hat{-}})$	$i\left((\sin\delta+\mathbb{S}\cos\delta)P_{\hat{+}}-\mathbb{C}\cos\delta P_{\hat{-}}\right)$	0	$2i\mathbb{C}(\cos\delta D_{\hat{+}}+\sin\delta D_{\hat{-}})$	$i\left((\cos\delta-\mathbb{S}\sin\delta)P_{\hat{+}}+\mathbb{C}\sin\delta P_{\hat{-}}\right)$
$\mathfrak{K}_{\hat{-}}$	$-2i((\mathbb{S}\cos\delta-\sin\delta)D_{\hat{+}}+(\mathbb{S}\sin\delta+\cos\delta)D_{\hat{-}})$	0	$-i\left((\sin\delta+\mathbb{S}\cos\delta)\mathfrak{K}_{\hat{-}}+\mathbb{C}\cos\delta\mathfrak{K}_{\hat{+}}\right)$	$2i\mathbb{C}(\cos\delta D_{\hat{+}}+\sin\delta D_{\hat{-}})$	0	$-i\left((\cos\delta-\mathbb{S}\sin\delta)\mathfrak{K}_{\hat{-}}-\mathbb{C}\sin\delta\mathfrak{K}_{\hat{+}}\right)$
$D_{\hat{-}}$	$-i\left((\sin\delta+\mathbb{S}\cos\delta)P_{\hat{+}}-\mathbb{C}\cos\delta P_{\hat{-}}\right)$	$i\left((\sin\delta+\mathbb{S}\cos\delta)\mathfrak{K}_{\hat{-}}+\mathbb{C}\cos\delta\mathfrak{K}_{\hat{+}}\right)$	0	$-i\left((\sin\delta-\mathbb{S}\cos\delta)P_{\hat{-}}-\mathbb{C}\cos\delta P_{\hat{+}}\right)$	$i\left((\sin\delta-\mathbb{S}\cos\delta)\mathfrak{K}_{\hat{+}}+\mathbb{C}\cos\delta\mathfrak{K}_{\hat{-}}\right)$	0
$P_{\hat{-}}$	0	$-2i\mathbb{C}(\cos\delta D_{\hat{+}}+\sin\delta D_{\hat{-}})$	$i\left((\sin\delta-\mathbb{S}\cos\delta)P_{\hat{-}}-\mathbb{C}\cos\delta P_{\hat{+}}\right)$	0	$2i((\mathbb{S}\cos\delta+\sin\delta)D_{\hat{+}}+(\mathbb{S}\sin\delta-\cos\delta)D_{\hat{-}})$	$i\left((\cos\delta+\mathbb{S}\sin\delta)P_{\hat{-}}+\mathbb{C}\sin\delta P_{\hat{+}}\right)$
$\mathfrak{K}_{\hat{+}}$	$-2i\mathbb{C}(\cos\delta D_{\hat{+}}+\sin\delta D_{\hat{-}})$	0	$-i\left((\sin\delta-\mathbb{S}\cos\delta)\mathfrak{K}_{\hat{+}}+\mathbb{C}\cos\delta\mathfrak{K}_{\hat{-}}\right)$	$-2i((\mathbb{S}\cos\delta+\sin\delta)D_{\hat{+}}+(\mathbb{S}\sin\delta-\cos\delta)D_{\hat{-}})$	0	$-i\left((\cos\delta+\mathbb{S}\sin\delta)\mathfrak{K}_{\hat{+}}-\mathbb{C}\sin\delta\mathfrak{K}_{\hat{-}}\right)$
$D_{\hat{+}}$	$-i\left((\cos\delta-\mathbb{S}\sin\delta)P_{\hat{+}}+\mathbb{C}\sin\delta P_{\hat{-}}\right)$	$i\left((\cos\delta-\mathbb{S}\sin\delta)\mathfrak{K}_{\hat{-}}-\mathbb{C}\sin\delta\mathfrak{K}_{\hat{+}}\right)$	0	$-i\left((\cos\delta+\mathbb{S}\sin\delta)P_{\hat{-}}+\mathbb{C}\sin\delta P_{\hat{+}}\right)$	$i\left((\cos\delta+\mathbb{S}\sin\delta)\mathfrak{K}_{\hat{+}}-\mathbb{C}\sin\delta\mathfrak{K}_{\hat{-}}\right)$	0

Interpolating D and K^3

In the limit $\delta \longrightarrow 0$, we recover the commutation relations among all instant form conformal generators in two dimensional as given below:

Table 2: $1+1$ conformal algebra in IFD

	P_0	$-\mathfrak{K}_3$	$-K^3$	$-P_3$	$\mathfrak{K}_0$	D
P_0	0	$-2iK^3$	iP_3	0	$2iD$	iP_0
$-\mathfrak{K}_3$	$2iK^3$	0	$-i\mathfrak{K}_0$	$2iD$	0	$i\mathfrak{K}_3$
$-K^3$	$-iP_3$	$i\mathfrak{K}_0$	0	iP_0	$-i\mathfrak{K}_3$	0
$-P_3$	0	$-2iD$	$-iP_0$	0	$2iK^3$	$-iP_3$
$\mathfrak{K}_0$	$-2iD$	0	$i\mathfrak{K}_3$	$-2iK^3$	0	$-i\mathfrak{K}_0$
D	$-iP_0$	$-i\mathfrak{K}_3$	0	iP_3	$i\mathfrak{K}_0$	0

Interpolating D and K^3

In the limit $\delta \longrightarrow \frac{\pi}{4}$, we recover the commutation relations among all light-front conformal generators in two dimensional as given below:

Table 3: $1+1$ conformal algebra in LFD

	P_+	$\mathfrak{K}_-$	D_-	P_-	$\mathfrak{K}_+$	D_+
P_+	0	$2\sqrt{2}iD_-$	$\sqrt{2}iP_+$	0	0	0
$\mathfrak{K}_-$	$-2\sqrt{2}iD_-$	0	$-\sqrt{2}i\mathfrak{K}_-$	0	0	0
D_-	$-\sqrt{2}iP_+$	$\sqrt{2}i\mathfrak{K}_-$	0	0	0	0
P_-	0	0	0	0	$2\sqrt{2}iD_+$	$\sqrt{2}iP_-$
$\mathfrak{K}_+$	0	0	0	$-2\sqrt{2}iD_+$	0	$-\sqrt{2}i\mathfrak{K}_+$
D_+	0	0	0	$-\sqrt{2}iP_-$	$\sqrt{2}i\mathfrak{K}_+$	0

where, $D_\pm = \frac{D \pm K^3}{\sqrt{2}}$, $\mathfrak{K}_\pm = \frac{\mathfrak{K}_0 \pm \mathfrak{K}_3}{\sqrt{2}}$, and $P_\pm = \frac{P_0 \pm P_3}{\sqrt{2}}$.

Interpolating D and K^3

We find[5] that $SO(2,2)$ splits into a direct sum of two identical algebras:

$$SO(2,2) \simeq SO(1,2) \oplus SO(1,2) \tag{5.7}$$

Lets make two new 3×3 anti symmetric tensors, namely J^+_{ab} and J^-_{ab}. Where, $a, b \in \{0, 1, 2\}$.

$$J^+_{ab} = \frac{1}{2}\begin{pmatrix} 0 & P_+ & D - K^3 \\ -P_+ & 0 & \mathfrak{K}_- \\ -D + K^3 & -\mathfrak{K}_- & 0 \end{pmatrix}_{3\times 3} ; \qquad J^-_{ab} = \frac{1}{2}\begin{pmatrix} 0 & P_- & D + K^3 \\ -P_- & 0 & \mathfrak{K}_+ \\ -D - K^3 & -\mathfrak{K}_+ & 0 \end{pmatrix}_{3\times 3} \tag{5.8}$$

and with the new metric g_{ab},

$$g_{ab} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{3\times 3} \tag{5.9}$$

they fulfill the $SO(1,2)$ commutation relations;

$$\left[J^{\pm}_{ab}, J^{\pm}_{cd}\right] = -i\left(g_{bd}J^{\pm}_{ac} - g_{bc}J^{\pm}_{ad} + g_{ac}J^{\pm}_{bd} - g_{ad}J^{\pm}_{bc}\right); \qquad \left[J^{+}_{ab}, J^{-}_{cd}\right] = 0$$

These are perfectly consistent with the above table in the LFD limit.

[5] Daniel Meise's Relations between 2D and 4D Conformal Quantum Field Theory

Interpolating D and K^3

Also, we have:

$$J_{a,b} = \begin{pmatrix} 0 & -D & -\frac{\mathfrak{K}_+}{\sqrt{2}} & -\frac{\mathfrak{K}_-}{\sqrt{2}} \\ D & 0 & \frac{P_+}{\sqrt{2}} & \frac{P_-}{\sqrt{2}} \\ \frac{\mathfrak{K}_+}{\sqrt{2}} & -\frac{P_+}{\sqrt{2}} & 0 & K^3 \\ \frac{\mathfrak{K}_-}{\sqrt{2}} & -\frac{P_-}{\sqrt{2}} & -K^3 & 0 \end{pmatrix}_{(4\times 4)} \tag{5.10}$$

Then, the simplified conformal algebra in LFD is:

$$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right) \tag{5.11}$$

where,

$$g_{a,b} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{4\times 4} \tag{5.12}$$

Current Progress

- Finding and verifying the basis for 4×4 projective-space-time representations and 4×4 gamma matrices (projective-space-time spinors) representations.
- Finding this new 3×3 projective-space-time representations.
- Understanding the split in conformal algebra in LFD: $SO(2,2) \simeq SO(1,2) \oplus SO(1,2)$
- Connecting our calculations to some suitable physical process to extract the physics of the conformal group.