Non-diagonal GPDs and the structure of hadrons

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in collaboration with Hyeondong Son, Sangyeong Son, and Marc Vanderhaeghen

A plan for today

1. Introduction and (some) general motivation
2. Kinematics of non-diagonal DVCS
3. A simple example: $N \to \Delta$ non-diagonal DVCS
4. $N \to \pi N$ transition GPDs
5. Dual parametrization of GPDs and GPD quintessence function;
6. Abel transform tomography and the Gribov-Froissart projection;
7. Some lessons from $\pi \to \pi\pi$ transition GPDs;
8. Omnes solution for dispersion relation;
9. Conclusions and Outlook.

What is non-diagonal DVCS/DVMP?

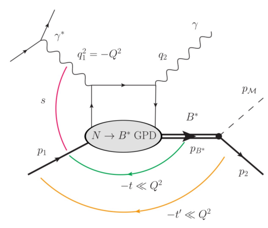
$$\gamma^*(q_1) + N(p_1) \to \left\{ \begin{matrix} \gamma^*(q_2) \\ \mathcal{M}'(p'_{\mathcal{M}}) \end{matrix} \right\} + \left[\mathcal{M}(p_{\mathcal{M}}) N(p') \right]; \ \mathcal{M} = \pi, \eta, \rho, \omega \ldots$$

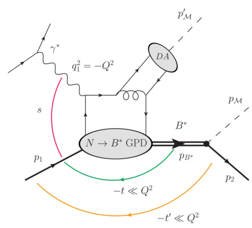
- Factorized description in terms of $N \to B^*$ GPDs in the generalized Bjorken kinematics:

$$-q_1^2; \ (p_1 + q_1)^2 - \text{large}; \qquad x_B = \frac{-q_1^2}{2p_1 \cdot q_1} - \text{fixed};$$

$$-t = -(p_{B^*} - p_1)^2; \ \ -t' = -(p_2 - p_1)^2; \ \ W_{\mathcal{M}N}^2 = (p_1 + p_{\mathcal{M}})^2 \quad \text{of hadronic scale.}$$

- Meson-nucleon system resonates at $W_{\mathcal{M}N} = M_{B^*}$.





- Status of factorization: same as for the DVCS&DVMP: X. Ji et al.'98, J. Collins et al.'97,99.

Some motivation

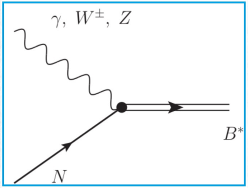
- Main goal is to understand B^* in terms of q, $\bar{q}$ and gluons.
- Available probes and their QCD structure:

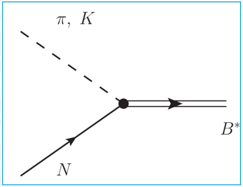
$$\text{E.m./weak probe}: \quad \text{QCD structure}:$$

$$\gamma \quad \Leftrightarrow \quad \langle B^* | \bar{q} \hat{Q}_{\rm e.m.} \gamma_\mu q | N \rangle$$

$$W^\pm, \; Z^0 \quad \Leftrightarrow \quad \langle B^* | \bar{q} \hat{Q}_w \gamma_\mu (1 - \gamma_5) q | N \rangle$$

- Only $C = -1$ probe;
- Local in space-time;
- No direct access to gluon d.o.f.





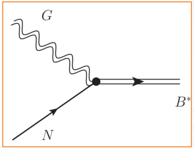
$$\text{Hadronic probe}: \quad \text{QCD structure}:$$

$$\pi, \; K \quad \Leftrightarrow \quad \langle B^* | \, ??? \, | N \rangle$$

- QCD structure of the probe unknown;

Graviton probe and QCD Energy-Momentum Tensor

- Graviproduction of resonances I. Kobzarev and L. Okun'62



SOVIET PHYSICS JETP VOLUME 16, NUMBER 5 MAY, 1963

GRAVITATIONAL INTERACTION OF FERMIONS

I. Yu. KOBZAREV and L. B. OKUN'

Institute of Theoretical and Experimental Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 14, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1904-1909 (November, 1962)

Gravitational interaction of spin-1/2 particles is considered in the linear approximation. It is shown that if gravitational interaction is taken into account, the question whether a free neutrino is two- or four-component acquires a physical meaning. The vertex part for the interaction between fermions and the gravitational field is shown to possess properties analogous to those of the electrodynamic vertex described by the Ward theorem. Observable effects due to spins are considered.

G probe : QCD structure :

$$G \quad \Leftrightarrow \quad \langle B^* | \underbrace{\bar{q}\gamma_\mu(\partial_\nu - A_\nu)q + \frac{1}{4}F^a_{\mu\alpha}F^a_{\nu\alpha}}_{\text{QCD EMT}} |N\rangle$$

- Gluon d.o.f. enter explicitly!
- No good source of G (:

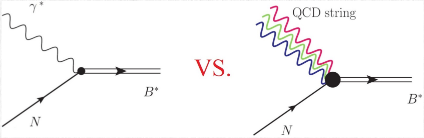
$$\frac{\text{Rate of } GN \to B^*}{\text{Rate of } \gamma N \to B^*} \simeq \frac{m_N}{M_{\rm Pl}} \frac{1}{\alpha_{\rm em}} \simeq 10^{-17}$$

Some remarks

- Short distance part of the process creates a low-energy QCD string = a tower of local probes (γ, $G, \ldots$);
- Spin J expansion of the QCD string operator:

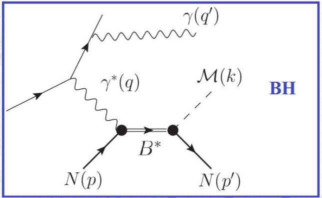
$$\bar{\Psi}(n) P \exp\left(i \int_{-n}^{n} dz^{\mu} A_{\mu}(z)\right) \Psi(-n) = \bar{\Psi} \text{---} \Psi = \sum_{J=0}^{\infty} \left[\bullet\text{---}\bullet \right]_J Y_{JM}$$

- Although non-diagonal DVCS is a hard process it probes a soft B^* excitation by low-energy QCD string;
- More analogous to B^* photoexcitation rather than hard electroproduction (qualitatively different physics);



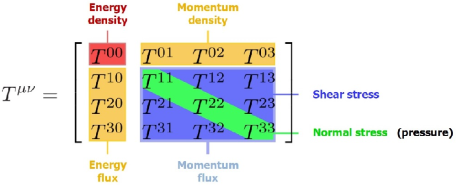
Feasibility:

- Rates are the same order as in usual DVCS/DVMP;
- In case of DVCS: interference with the Bethe-Heitler process provides enhancement of signal;

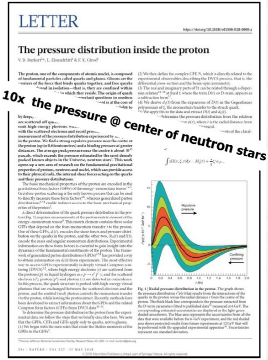


Physical contents I

Gravitational FFs of the proton, see e.g. V.D. Burkert et al. 2303.08347



Burkert, Elouadrhiri, Girod, Nature 557(2018)



M. Polyakov' 03:

$$T^{ij}(\vec{r}) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right) s(r) + \delta^{ij}\, p(r)$$

$$s(r) = -\frac{1}{4M_N}\, r \frac{d}{dr}\frac{1}{r}\frac{d}{dr}\widetilde{D}(r)$$

$$p(r) = \frac{1}{6M_N}\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\widetilde{D}(r)$$

$$\widetilde{D}(r) = \int \frac{d^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}} D(-\vec{\Delta}^2).$$

- Study of QCD EMT $N \to B^*$ transition matrix elements complements the studies of e.m. transition FFs;
- Possible access to transition spin contents (for $N \to N^*$, Δ), pressure and shear forces (for $N \to N^*$) and new insight for resonance formation;
- Studies underway. *Cf.* transition angular momentum $N \to \Delta$, J.-Y. Kim et al.'23.

Physical contents II: a unique option for baryon spectroscopy

Important advantages with respect to the usual electroproduction:

1. Excitation of resonances by non-local QCD quark light-cone operators:

$$\left\langle N^* \left| \bar{\psi}_\alpha(0) \mathrm{P} e^{ig \int_0^z dx_\mu A^\mu} \psi_\beta(z) \right| N \right\rangle$$

   ★ excitation by probes of arbitrary spin (not just $J = 1$);

2. Possible generalization to the gluon light-cone operators:

$$\left\langle B \left| G^a_{\alpha\beta}(0) \left[\mathrm{P} e^{ig \int_0^z dx_\mu A^\mu} \right]^{ab} G^b_{\mu\nu}(z) \right| A \right\rangle$$

   ★ explicit access to the gluonic DOFs.

3. Direct access to Im (spin asymmetry) and Re (charge asymmetry) of the amplitude $A^{\mathrm{DVCS}}_{N \to B^*}$. Without complicated PWA!
4. Possible access to non-usual spin-flavor configurations: *e.g.* SU(6) $[20, 1^+]$: $N = 2$ orbital excitation of the SU(6) 20-plet.
   Symmetry argument by R. Feynman'1972: *"Two quark at least must have their motion changed to get to the* $[20, 1^+]$ *from the fundamental* $[56, 0^+]$*."*
5. Large gluon components and more. Hunt for exotic.

Physical contents III: Chiral dynamics in gravitational interaction

- More general description: $N \to \pi N$ transition GPDs, M. Polyakov and S. Stratmann, arXiv:hep-ph/0609045.
- A new test ground for χPT - low energy EFT of QCD, First principle calculations!

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Chiral theory of nucleons and pions in the presence of an external gravitational field

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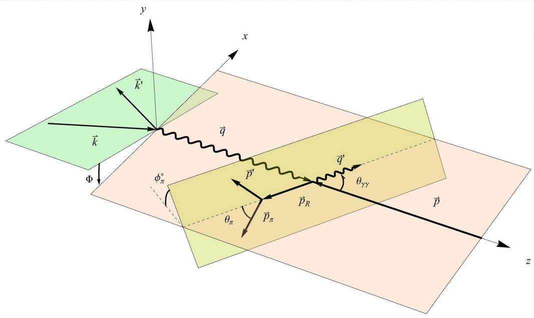
We extend the standard second order effective chiral Lagrangian of pions and nucleons by considering the coupling to an external gravitational field. As an application we calculate one-loop corrections to the one-nucleon matrix element of the energy-momentum tensor to fourth order in chiral counting, and next-to-leading order tree-level amplitude of the pion-production in an external gravitational field. We discuss the relation of the obtained results to experimentally measurable observables. Our expressions for the chiral corrections to the nucleon gravitational form factors differ from those in the literature. That might require to revisit the chiral extrapolation of the lattice data on the nucleon gravitational form factors obtained in the past.

Kinematics and decay angular distribution

$$e(k) + N(p_N) \to e'(k') + \gamma^*(q) + N(p_N) \to e'(k') + \gamma(q') + \pi(p_\pi) + N'(p'_N)$$

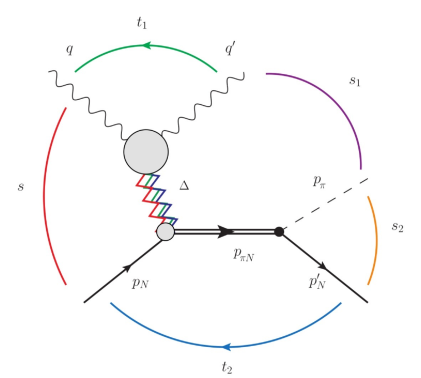


- $\gamma^* N \to B^* \gamma$: $\gamma^* N$ CMS;
- $B^* \to \pi N'$: $\pi N'$ CMS $\equiv (\pi N')$ at rest;

$$\frac{d^7\sigma}{\underbrace{dQ^2 dx_B}_{\text{lepton side}} \underbrace{dt\, d\Phi}_{\gamma^* N \to \gamma B^*} \underbrace{dW^2_{\pi N} d\Omega^*_\pi}_{B^* \to \pi N}}$$

Kinematics: invariants

- Invariant variables for $\gamma^* N \to \gamma \pi N'$



In addition to $s = (p_N + q)^2 \equiv W^2$ and $t_1 = (q - q')^2 \equiv \Delta^2$:

- $\gamma\pi$ invariant mass: $s_1 = (p_\pi + q)^2$;
- πN invariant mass: $s_2 = (p_\pi + p'_N)^2 \equiv W^2_{\pi N}$;
- $t_2 = (p'_N - p_N)^2$;

A test ground: $N \to \Delta(1232)$ DVCS

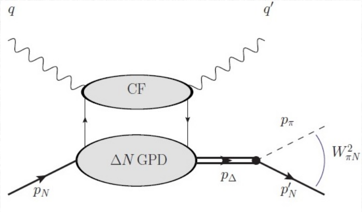
$$\gamma^*(q) + N^p(p_N) \to \gamma(q') + \Delta^+(p_\Delta) \to \gamma(q') + \pi^0(p_\pi) + N^p(p'_N)$$

K. Goeke, M.Polyakov and M. Vanderhaeghen'01:

- 3 +1 unpolarized+4 polarized leading twist $N \to \Delta$ GPDs;
- $1 + 2$ relevant in the large N_c limit;
- Early analysis: P. Guichon, L. Mossé and M. Vanderhaeghen'03;

A. Belitsky and A. Radyushkin'05:

- 4 unpolarized+4 polarized leading twist $N \to \Delta$ GPDs;



K.S. and M. Vanderhaeghen, 2303.00119

- Important goal: work out of angular dependencies of $|\text{DVCS}|^2$, $|\text{BH}|^2$ and interference term.
- Implications for experiment: necessary coverage in the cm angle of the final πN state.

$N \to \Delta$ GPDs I

- Leading twist-2: 4 unpolarized and 4 polarized GPDs;
- Unpolarized isovector $N \to \Delta$ GPDs (K. Goeke et al.2001):

$$\frac{1}{2\pi}\int dy^- e^{ixP^+y^-} \langle \Delta(p_\Delta)|\bar{\psi}(-y/2)\gamma\cdot n\tau_3\psi(y/2)|N(p_N)\rangle\Big|_{y^+=\vec{y}_\perp=0}$$

$$= \sqrt{\frac{2}{3}}\bar{\mathcal{U}}^\beta(p_\Delta)\Big\{H_M(x,\xi,t)\left(-\mathcal{K}^M_{\beta\mu}\right)n^\mu + H_E(x,\xi,t)\left(-\mathcal{K}^E_{\beta\mu}\right)n^\mu$$

$$+H_C(x,\xi,t)\left(-\mathcal{K}^C_{\beta\mu}\right)n^\mu + H_4(x,\xi,t)\underbrace{\left(\Gamma^4_{\beta\mu}\right)}_{\text{omitted structure}}n^\mu\Big\}u(p_N),$$

Jones-Scadron covariants ($\bar{P}=\frac{p_N+p_\Delta}{2}=p_\Delta-\frac{\Delta}{2}$, $\Delta=p_\Delta-p_N$, $t\equiv\Delta^2$):

$$\mathcal{K}^M_{\beta\mu} = -i\frac{3(m_\Delta+m_N)}{2m_N\left((m_\Delta+m_N)^2-t\right)}\varepsilon_{\beta\mu\lambda\sigma}\bar{P}^\lambda\Delta^\sigma;$$

$$\mathcal{K}^E_{\beta\mu} = -\mathcal{K}^M_{\beta\mu} - \frac{6(m_\Delta+m_N)}{m_N Z(t)}\varepsilon_{\beta\sigma\lambda\rho}\bar{P}^\lambda\Delta^\rho\varepsilon^\sigma_{\mu\kappa\delta}\bar{P}^\kappa\Delta^\delta\gamma^5;$$

$$\mathcal{K}^C_{\beta\mu} = \not{-}\not{i}\frac{3(m_\Delta+m_N)}{m_N Z(t)}\Delta_\beta\left(t\bar{P}_\mu-\Delta\cdot\bar{P}\Delta_\mu\right)\gamma^5;$$

$$\Gamma^4_{\beta\mu} = \frac{1}{m_N m_\Delta}\left[\Delta_\beta - \frac{(\Delta\cdot p_\Delta)}{p_\Delta^2}p_{\Delta\beta}\right]\Delta_\mu\gamma_5.$$

$N \to \Delta$ GPDs II

- Polarized $N \to \Delta$ GPDs:

$$\frac{1}{2\pi}\int dy^- e^{ixP^+y^-}\left\langle \Delta(p_\Delta)\left|\bar{\psi}(-y/2)\gamma\cdot n\gamma^5\tau^3\psi(y/2)\right|N(p_N)\right\rangle =$$

$$\sqrt{\frac{2}{3}}\bar{\mathcal{U}}^\beta(p_\Delta)\left[C_1(x,\xi,t)g_{\beta\mu}n^\mu + C_2(x,\xi,t)\frac{\Delta_\beta\Delta_\mu}{m_N^2}n^\mu + C_3(x,\xi,t)\frac{1}{m_N}\left[g_{\beta\mu}\not{\Delta} - \Delta_\beta\gamma_\mu\right]n^\mu\right.$$

$$\left.+C_4(x,\xi,t)\frac{2}{m_N^2}\left[\bar{P}\cdot\Delta g_{\beta\mu} - \Delta_\beta\bar{P}_\mu\right]n^\mu\right]u(p_N).$$

Relation to form factors

- Unpolarized GPDs are related to e.m. form factors Jones and Scadron'73:

$$\int_{-1}^{1} dx H_{M,E,C}(x,\xi,t) = 2G^*_{M,E,C}(t); \quad \int_{-1}^{1} dx H_4(x,\xi,t) = 0;$$

- Polarized transition GPDs are related to axial form factors Adler'75;
- These FFs can be accessed in neutrino-production reactions;

$$\int_{-1}^{1} dx C_{1,2,3,4}(x,\xi,t) = 2C^A_{5,6,3,4}(t).$$

Large N_c relations and sum rule

- Large N_c relations for octet-to-decuplet transition GPDs, Goeke et al.'01:

$$H_M(x,\xi,t) = \frac{2}{\sqrt{3}}\left[E^u(x,\xi,t) - E^d(x,\xi,t)\right];$$
$$C_1(x,\xi,t) = \sqrt{3}\left[\tilde{H}^u(x,\xi,t) - \tilde{H}^d(x,\xi,t)\right];$$
$$C_2(x,\xi,t) = \frac{\sqrt{3}}{4}\left[\tilde{E}^u(x,\xi,t) - \tilde{E}^d(x,\xi,t)\right];$$

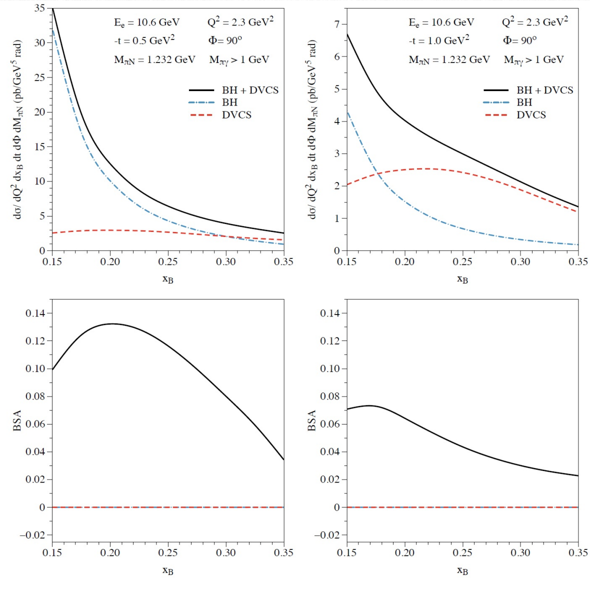
- Pion pole contribution into C_2:

$$\lim_{t\to m_\pi^2} C_2(x,\xi,t) = \sqrt{3}\frac{g_A m_N^2}{m_\pi^2 - t}\theta[\xi-|x|]\frac{1}{\xi}\Phi_\pi\left(\frac{x}{\xi}\right);$$

- Angular momentum sum rule:

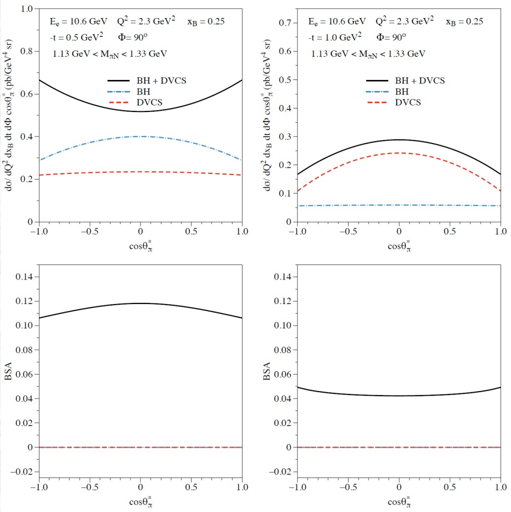
$$\lim_{t\to 0, N_c\to\infty}\int_{-1}^{1} dx x H_M(x,\xi,t) = \frac{2}{\sqrt{3}}\left[2\left(J^u - J^d\right) - M_2^u + M_2^d\right].$$

Cross sections and BSA for JLab@12 GeV I

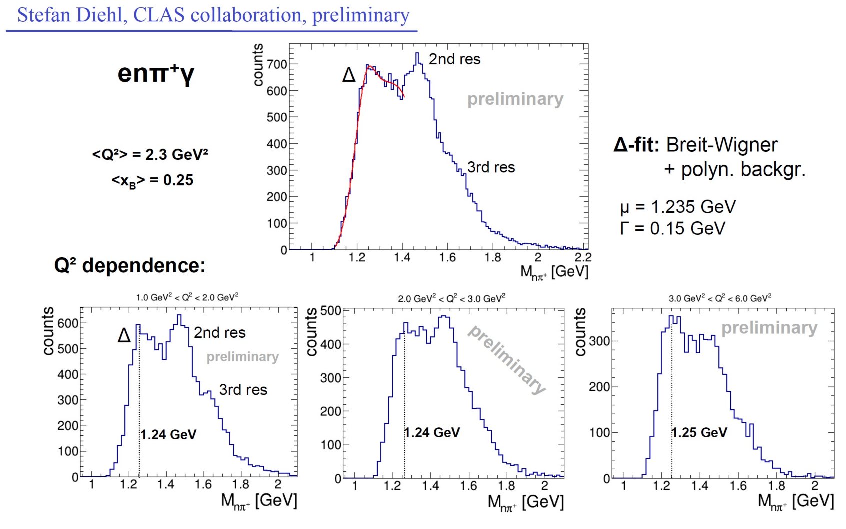


Cross sections and BSA for JLab@12 GeV II

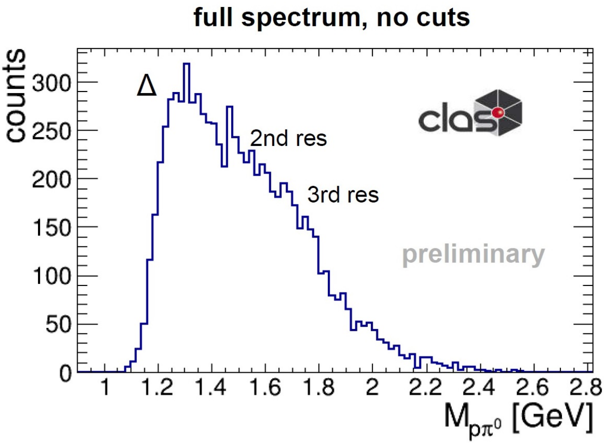
- Δ in helicity $\pm 1/2$ state: $\frac{1}{4}\left(1+3\cos^2\theta^*_\pi\right)$
- Δ in helicity $\pm 3/2$ state: $\frac{3}{4}\sin^2\theta^*_\pi$



Experimental status I: resonance spectrum for $N^* \to n\pi^+$

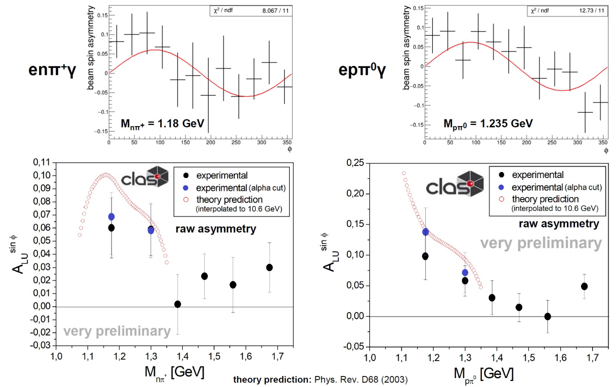


Experimental status II: resonance spectrum for $N^* \to p\pi^0$



Experimental status III: Beam Spin Asymmetry

$$\mathrm{A} = \frac{1}{P}\frac{N^+ - N^-}{N^+ + N^-} \approx A_{LU}^{\sin\phi} \sin\phi$$

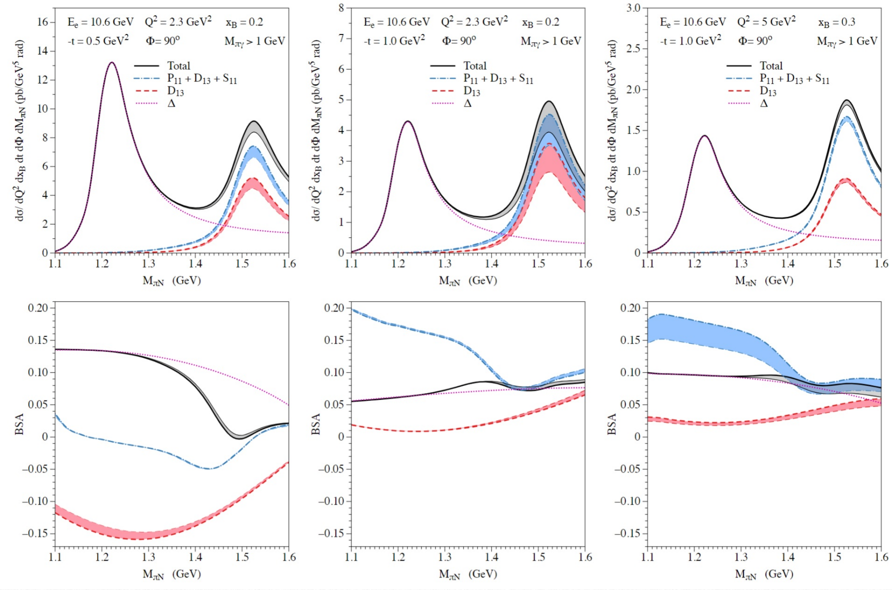


- BSA $\sim T^{\mathrm{BH}} \times \mathrm{Im}\, T^{N\Delta\,\mathrm{DVCS}}$.

Going to the 2nd resonance region

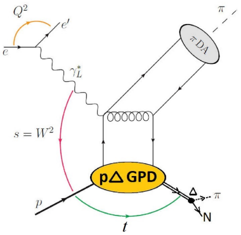
- Formalism extended to $N \to N^*$ DVCS for $N^* = \mathrm{P}_{11}(1440)$, $\mathrm{D}_{13}(1520)$, $\mathrm{S}_{11}(1535)$:
  - for spin-$\frac{1}{2}$ resonances at twist-2: 2 unpolarized GPDs (vector operator), 2 polarized GPDs (axial-vector operator);
  - for spin-$\frac{3}{2}$ resonances at twist-2: 4 unpolarized GPDs (vector operator), 4 polarized GPDs (axial-vector operator);
- t-dependence of GPDs (first moments):
  - \- unpolarized GPDs: first moments constrained by data on e.m. transition FFs (CLAS@6 GeV)
  - \- polarized GPDs: 2 dominant axial FFs constrained using PCAC + pion pole dominance:
    - normalization at $t = 0$ given by $(f_{\pi NN^*}/m_\pi)2f_\pi$;
    - t-dependence: dipole ($M_A = 1$ GeV) and pion-pole $\sim 1/(t - m_\pi^2)$;
    - isoscalar axial FF neglected;
- x & ξ dependence of GPDs: RDDA $b = 1$ and $b = \infty$ with $q(x) \sim x^{-0.5}(1-x)^3$

Cross section and BSA



Hard exclusive $\Delta\pi$ production

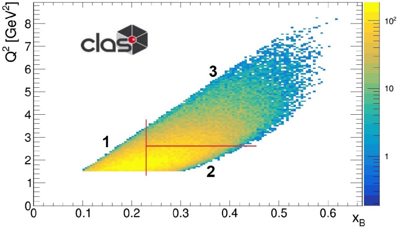
S. Diehl et al. '23

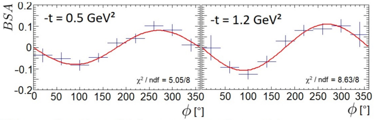


$$ep \to e\Delta^0\pi^+ \to e(p\pi^-)\pi^+$$
$$\to e(n\pi^0)\pi^+$$

$$ep \to e\Delta^+\pi^0 \to e(n\pi^+)\pi^0$$
$$\to e(p\pi^0)\pi^0$$

$$ep \to e\Delta^{++}\pi^- \to ep\pi^+\pi^-$$





BSA as a function of ϕ for representative $-t$ bins (Q^2= 2.48 GeV^2, x_B = 0.27). The red line shows the $\sin\phi$ fit.

- Amplitude involves polarized GPDs $C_{1,2,3,4}(x,\xi,\Delta^2)$;
- BSA is a twist-3 effect;

Experimental perspectives

- $N\Delta$ DVCS and $\pi\Delta$ can be measured at CLAS. Analysis underway.
- Present status: 3-4 bins in $-t$. With extra angular variables 2-3 bins in each variable;
- Statistics increase by a factor 3 in 3-4 years;
- BSA $\pi^-\Delta^{++}$ extracted;
- Possible JLab@20 upgrade: statistics may increase by a factor 100 - 1000;

arXiv:2306.09360v1 [nucl-ex]

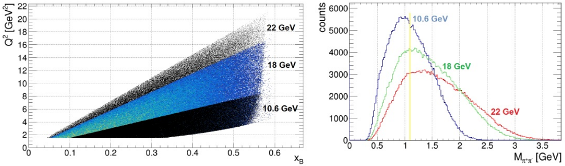


Figure 40: Comparison of the available phase space, accessible with the present CLAS12 setup, in $Q^2 - x_B$ or the $\pi^-\Delta^{++}$ process under forward kinematics ($-t < 1.5$ GeV^2) (left) and for the $\pi^+\pi^-$ invariant mass of the same process, which is used to suppress the dominant ρ production background by the cut on $M(\pi^+\pi^-) > 1.1$ GeV, indicated by the yellow line (right) for a 10.6 GeV, 18 GeV and 22 GeV electron beam.

- Can we get access to the complete angular distribution of $N\Delta$ DVCS/DVMP and $\pi\Delta$ production cross section?
- A sizable $\pi^-\Delta^{++}$ BSA a challenge for theory: twist-3 observable;
- Extension to small-x_B and studied for the EIC conditions necessary;

$N \to \pi N$ transition GPDs

M. Polyakov and S. Stratmann, arXiv:hep-ph/0609045

- Unpolarized $N \to \pi N$ GPDs:

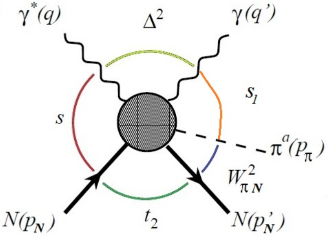
$$\int \frac{d\lambda}{2\pi} e^{i\lambda x \bar{P}\cdot n} \langle N(p'_N)\, \pi^a(p_\pi) | \bar{\psi}(-\lambda n/2) \not{n} \psi(\lambda n/2) | N(p_N) \rangle = \frac{i g_A}{m_N f_\pi} \sum_{i=1}^{4} \bar{U}(p'_N)\, \Gamma_i \tau^a H_i^{(0)} U(p_N)$$

$$\Gamma_1 = \gamma_5; \quad \Gamma_2 = \frac{m_N \not{n}}{n \cdot \bar{P}} \gamma_5; \quad \Gamma_3 = \frac{\not{k}}{m_N} \gamma_5; \quad \Gamma_4 = \frac{\not{k}\not{n}}{m_N} \gamma_5; \quad \left(\bar{P} = \frac{p'_N + p_N + p_\pi}{2}\right)$$

A guide to the kinematical variables of $H_i^{(0)}(x, \xi, \Delta^2; W^2_{\pi N}, \alpha, t_2)$:

- πN invariant mass $W^2_{\pi N} = (p' + p_\pi)^2$
- $t_1 = (p'_N + p_\pi - p_N)^2 = (q - q')^2 \equiv \Delta^2$
- $t_2 = (p'_N - p_N)^2$
- Skewness $\xi = -\frac{n \cdot \Delta}{2n \cdot \bar{P}}$
- Relative pion longitudinal momentum of the πN system:

$$\alpha = \frac{n \cdot p_\pi}{n \cdot (p'_N + p_\pi)}$$



On physical meaning of α

★ Related to πN decay angle θ^*_π defined in the πN CMS $\equiv B^*$ rest frame:

$$\alpha = \frac{W^2_{\pi N} - m^2_N + m^2_\pi + \Lambda(W^2_{\pi N}, m^2_N, m^2_\pi)\cos\theta^*_\pi}{2W^2_{\pi N}} + O(1/\mathcal{Q}^2),$$

where Λ is the Mandelstam function

$$\Lambda(x,y,z) = \sqrt{x^2 - 2xy - 2xz + y^2 - 2yz + z^2}.$$

- On the pion threshold $W_{\pi N} = m_N + m_\pi$:

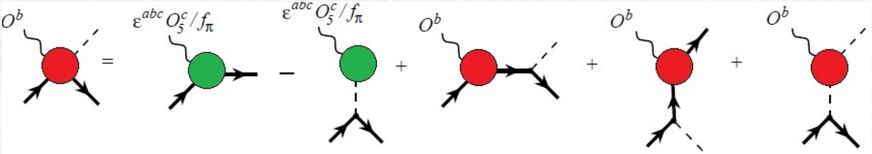
$$\alpha\Big|_{\text{threshold}} = \frac{m_\pi}{m_N + m_\pi}.$$

Some properties of $N \to \pi N$ transition GPDs

- Soft pion theorems P. Pobylitsa, M. Polyakov, and M. Strikman'01 fix $N \to \pi N$ GPDs at the threshold $W = (M_N + m_\pi)$ in terms of nucleon GPDs and pion DA;
- *E.g.* soft pion theorem for $N \to \pi N$ transition matrix element M. Polyakov and S. Stratmann, arXiv:hep-ph/0609045

$$\left\langle N\left(p'\right) \pi^a(k) \left| O^b(\lambda) \right| N(p) \right\rangle$$

of the isovector light cone operator $O^b = \bar{\psi}(-\lambda n/2) \not{n} \tau^b \psi(\lambda n/2)$:



- $N \to \pi N$ transition GPDs are real at the threshold but generally not necessarily real functions;
- $N \to \pi N$ transition GPDs contain information on πN resonance spectrum. Can we take it out?

$N \to \pi N$ GPDs and PW analysis of the πN system

- M. Polyakov'98: $H_i\left(x, \xi, \alpha, t, W^2\right) \to H^{I,L,J}\left(x, \xi, \Delta^2; W^2, t_2\right)$ PW expansion in α

$$I: \text{ isospin}; \quad L: \text{ PW in } \alpha; \quad i \to J = L \pm 1/2 \text{ (total angular momentum)}.$$

- N.B. $N \to \pi N$ GPDs develop Im part above πN threshold. Relation to πN scattering amplitude (Watson theorem):

$$\mathrm{Im} H^{I,L,J}(x, \xi, \Delta^2; W^2, t_2) = \tan\left[\delta_{\pi N}^{I,L,J}(W^2)\right] \mathrm{Re} H^{I,L,J}(x, \xi, \Delta^2; W^2, t_2);$$

$$\delta_{\pi N}^{I,L,J}(W^2) \; - \pi N \text{ phase shifts.}$$

- A solution R. Omnes'1958:

$$H^{I,L,J}\left(x, \xi, W^2\right) = H^{I,L,J}\left(x, \xi, W_{\mathrm{th}}^2\right) \exp\left\{\sum_{k=1}^{N-1} c_k W^{2k} + \frac{W^{2N}}{\pi} \int_{W_{\mathrm{th}}^2}^{\infty} ds \frac{\delta_{\pi N}^{I,L,J}(s)}{s^N\left(s - W^2 - i0\right)}\right\}.$$

  - $H^{I,L,J}\left(x, \xi, W_{\mathrm{th}}^2\right)$ and c_k fixed by near threshold behavior & chiral physics.
  - Known πN phase shifts $\delta_{\pi N}^{I,L,J}(s)$ from πN scattering.
  - N^* resonances built in the solution! How to get them out?

Conformal PW expansion for GPDs I

- Idea: expand GPDs over the conformal basis (factorization of functional dependencies)
- Main advantage: trivial solution of the LO evolution equations.

- Conformal moments of quark GPDs are defined with respect to $c_n(x,\eta) = N_n \times \eta^n C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right)$; Normalization: $\lim_{\eta\to 0} c_n(x,\eta) = x^n$.

$$H_n(\eta,t) = \int_{-1}^{1} dx\, c_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right) H(x,\eta,t)\,.$$

- $c_n(x,\eta)$ form a complete basis in $[-\eta,\ \eta]$ with the weight $\left(1-\frac{x^2}{\eta^2}\right)$.
- $p_n(x,\eta)$ include the weight and θ to ensure the support:

$$p_n(x,\eta) = \eta^{-n-1}\theta\left(1-\frac{x^2}{\eta^2}\right)\left(1-\frac{x^2}{\eta^2}\right) N_n^{-1}\frac{(n+1)(n+2)}{2n+3} C_n^{\frac{3}{2}}\left(-\frac{x}{\eta}\right)\,.$$

- Orthogonality of the basis: $$\int_{-1}^{1} dx\, p_n(x,\eta)c_n(x,\eta) = (-1)^n\delta_{mn}$$

Conformal PW expansion for GPDs II

Conformal PW expansion for GPDs:

$$H(x,\eta,t)=\sum_{n=0}^{\infty} p_n(x,\eta)H_n(\eta,t)\,.$$

- Allows to factorize x, η and t dependence of GPDs.
- Scale dependence of the conformal moments is simply multiplicative:

$$H_n(\eta,t,\mu)=\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_{0n}}{2\beta_0}} H_n(\eta,t,\mu_0)\,.$$

- Conformal moments are reproduced by this series.
- Restricted support property $\nRightarrow$ GPD vanishes in the outer region.
- The expansion is to be understand as an ill-defined sum of generalized functions.

Different ways to assign meaning to conformal PW expansion

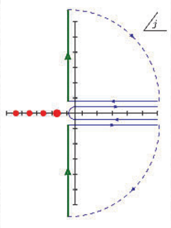
1. Sommerfeld-Watson transform + Mellin-Barnes integral techniques D. Müller and A. Schäfer'05; A. Manashov, M. Kirch and A. Schafer'05;
2. - Shuvaev transform A. Shuvaev'99, J. Noritzsch'00;
   - Dual parametrization of GPDs M. Polyakov and A. Shuvaev'02;

Mellin-Barnes techniques in simple words

- Sommerfeld-Watson transform:

$$H(x,\xi,t) = \frac{1}{2i}\oint_{(0)}^{(\infty)} dj\, \frac{(-1)^j}{\sin\pi j}\, p_j(x,\xi)\, m_j(\xi,t)\,.$$

- Residue theorem leads to conformal P.W. expansion ($\mathrm{Res}_{j=n}\frac{1}{\sin\pi j} = \frac{(-1)^j}{\pi}$).



- For $\xi = 0$ p_j form the integral kernel for the inverse Mellin transform
- In general, $p_j(x,\xi)$ are expressed through ${}_2F_1$ hypergeometric function. Asymptotic behavior of $p_j(x,\xi)$ for $j \to \infty$ is known.
- Asymptotic behavior of m_j -?
- Integral over the large arc must vanish.

- Mellin-Barnes integral representation for GPDs:

$$H(x,\xi,t) = \frac{i}{2}\int_{c-i\infty}^{c+i\infty} dj\, \frac{(-1)^j}{\sin\pi j} p_j(x,\xi) m_j(\xi,t)\,.$$

The basis for the Shuvaev transform & the dual parametrization

- How to restore $f(x)$ from its Mellin moments $M_n = \int dx x^n f(x)$?

- Formal solution:

$$f(x) = \sum_{n=0}^{\infty} M_n \delta^{(n)}(x) \frac{(-1)^n}{n!} .$$

✓ A trick: $\delta^{(n)}(x) = \frac{(-1)^n n!}{2\pi i} \left[\frac{1}{(x-i\epsilon)^{n+1}} - \frac{1}{(x+i\epsilon)^{n+1}} \right].$

Define $F(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}$; then $f(x) = \frac{1}{2\pi i} \left[F(x-i\epsilon) - F(x+i\epsilon) \right]$.

Idea of the Shuvaev transform (see A. Shuvaev'99, J. Noritzsch'00):

- Introduce $f_\xi(y)$ whose Mellin moments generate Gegenbauer moments of GPD:

$$\int_0^1 dy y^n f_\xi(y) = m_n(\xi)$$

- One can explicitly construct the kernel $K(x, \xi; y)$ such that

$$H(x, \xi) = \int_0^1 dy\, K(x, \xi; y)\, f_\xi(y) .$$

Dual Parametrization: basic facts

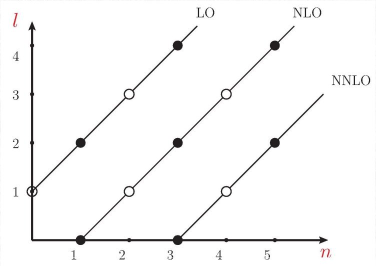
Dual Parametrization (M. Polyakov, A. Shuvaev'02, D. Müller, M. Polyakov and K.S.'15):

- Mellin moments expanded in a set of suitable orthogonal polynomials. E.g. partial waves of the t-channel (t-channel refers to $\bar{h}h \to \gamma^* \gamma$):

$$N_n^{-1} \frac{(n+1)(n+2)}{2n+3} H_n(\eta, t) = \eta^{n+1} \sum_{l=0}^{n+1} B_{nl}(t) P_l\left(\frac{1}{\eta}\right)$$

Conformal PW expansion is then rewritten as:

$$H(x,\eta,t) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{nl}(t)\, \theta\left(1 - \frac{x^2}{\eta^2}\right) \left(1 - \frac{x^2}{\eta^2}\right) C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right) P_l\left(\frac{1}{\eta}\right)$$



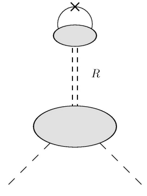
- Polynomiality implemented via Wigner-Ekkart theorem ($l \leq n+1$).
- Discrete symmetries (C, T) through the selection rules for l^{PC} (X. Ji, R. Lebed'01).
- Generalized FFs $B_{nl}(t)$ are renormalized multiplicatively.

t-channel point of view and duality

- Conformal PW expansion converges for $\eta > 1$.
- By means of the crossing relation one gets conformal PW expansion for two particle GDAs.

$$\frac{x}{\eta} \leftrightarrow 1-2z; \quad \frac{1}{\eta} \leftrightarrow 1-2\zeta; \quad t \leftrightarrow W^2$$

- Duality in the spirit of R. Dolen, D. Horn, C. Schmid'67. GPDs are presented as infinite series of t-channel Regge exchanges M. Polyakov'98:



$$\langle \pi(p')|\, \hat{O}\, |\pi(p)\rangle \sim \text{ Crossing of } \sum_{R_J} \sum_{\substack{\text{polarization}\\ \text{of } R_J}} \frac{1}{t-M_{R_J}^2}$$
$$\times \underbrace{\langle \pi(p')\pi(-p)|R_J\rangle}_{R_J\pi\pi \text{ effective vertex}} \underbrace{\langle R_J|\, \hat{O}\, |0\rangle}_{\text{F.T. of DA of } R_J} .$$

- Expansion in the t-channel PW:

$$\cos\theta_t = \frac{s-u}{\sqrt{1-\frac{4\,m^2}{t}}\,(Q^2+t)} = -\frac{1}{\eta\sqrt{1-\frac{4m^2}{t}}} + O(\frac{1}{Q^2})\,.$$

Dual parametrization: summing up the formal series I

- Same idea as the Shuvaev transform: Mellin moments of $Q_k(y,t)$ generate the generalized F.Fs. B_{nl}:

$$B_{n\ n+1-2\nu}(t) = \int_0^1 dy y^n Q_{2\nu}(y,t)\,.$$

Then $\quad H(x>-\eta,\eta,t) = \sum_{\nu=0}^{\infty} \int_0^1 dy\, K_{2\nu}(x,\eta,y) y^{2\nu} Q_{2\nu}(y,t)\,.$

How to construct the convolution kernels?

- M. Polyakov and A. Shuvaev'02 (see also M. Polyakov and KS'08):

$$K_{2\nu}(x,\eta,y) = \mathrm{disc}_{z=x} F^{(2\nu)}(z,\eta,y)\,, \quad \text{where}$$

$$F^{(2\nu)}(z,\eta,y) = \frac{1}{y^{2\nu+1}}\left(1+y\frac{\partial}{\partial y}\right)\int_{-1}^{1} ds \eta^k \frac{z_s^{1-k}}{\sqrt{z_s^2-2z_s+\eta^2}}\,, \quad z_s \equiv \frac{2(z-\eta s)}{(1-s^2)y}.$$

Dual parametrization: summing up the formal series II

Two ways to compute the discontinuity:

1. Expand in powers of $\frac{1}{z_s}$ and employ Rodriguez formula for Gegenbauer polynomials $\Rightarrow$ formally recover conformal PWE for GPD.
2. Consider the discontinuity due to the cut $1-\sqrt{1-\eta^2} < z_s < 1+\sqrt{1-\eta^2}$ (and from poles at $z_s = 0$ for $k \geq 2$) $\Rightarrow$ analytical expressions for the convolution kernels $K_{2\nu}(x, \eta, y)$ in terms of elliptic integrals.

Basic properties

- GPDs satisfy polynomiality property and the support property.
- The D-term is the natural ingredient of the dual parametrization.
- Scale dependence of $Q_k(x)$ is given by DGLAP equations.
- $Q_0(x)$ is fixed in terms of (t-dependent) PDFs:

$$Q_0(x) = q(x) + \bar{q}(x) - \frac{x}{2} \int_x^1 \frac{dy}{y^2} \left(q(y) + \bar{q}(y) \right);$$

- $Q_2(x)$ contains FFs of the EMT (J^q, shear forces);
- x-dependence of forward like functions should implement the insight from the Regge theory;
- A principle allowing to take into account only a finite number of conformal PWs (i.e. Q_ks)?

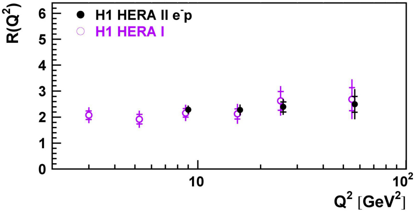
Minimalist model and skewness effect

- Consider the "minimalist model": include just $Q_0(x)$
- Assume that $q(x) \sim 1/x^{\alpha^q}$.

Skewness effect in the "minimalist" dual model equals conformal ratio (K. Kumericki, D. Mueller and K. Passek-Kumericki'08, 09)

$$r^q_{Q_0} \equiv \frac{H^q(\xi,\xi)}{H^q(\xi,0)}\Big|_{\xi\sim 0} \simeq \frac{2^{\alpha^q}\Gamma(\alpha^q+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\alpha^q)} \approx 3/2 \quad \text{for} \quad \alpha^q \approx 1\,;$$

Skewness effect from H1:



$$\mathbf{R} = 2^{\alpha_q} r^q \sim \frac{\sqrt{\sigma_{DVCS}}}{\sigma_{DIS}};$$

The observable ratio $R(\mathrm{Q}^2)$ for fixed $W = 82$ GeV. The Figure is taken from H1'07.

N.B. A. Shuvaev, Martin et al.'99.

Some lessons for us

- In order to describe the data the dual parametrization model must include some additional forward like functions $Q_{2\nu}$ with $\nu > 0$.
- These functions must be singular enough in order to make influence on the small ξ asymptotic behavior of $\mathrm{Im} A(\xi)$:

$$Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}} .$$

Seems to be a problem:

- This leads to divergencies of generalized form factors require regularization

$$B_{2\nu-1\,0} = \mathrm{Reg} \int_0^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x) = \int_{(0)}^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x) + B^{\mathrm{f.p.}}_{2\nu-1\,0}$$

Analytical regularization

- Compute for large positive j. Then analytically continue to $j = -1$
- This is precisely a so-called analytic (or canonical) regularization ($1 < \alpha < 2$):

$$\int_{(0)}^1 dx \frac{f(x)}{x^{1+\alpha}} = \int_0^1 dx \frac{1}{x^{1+\alpha}} \left[f(x) - f(0) - x f'(0) \right] - \frac{f(0)}{\alpha} - \frac{f'(0)}{\alpha - 1} .$$

Convolutions with hard kernels

- Extraction of the information on GPDs from the Compton F.Fs is the problem of deconvolution.
- Consider the elementary amplitude:

$$\mathcal{H}^{(+)}(\xi,t)=\int_0^1 dx H(x,\xi,t)\left[\frac{1}{\xi-x-i0}-\frac{1}{\xi+x-i0}\right]=4\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1}B_{n\,l}(t)P_l\left(\frac{1}{\xi}\right);$$

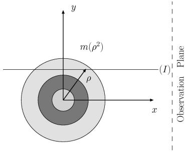
$$\mathrm{Im}\mathcal{H}^{(+)}(\xi,t)=2\int_{\frac{1-\sqrt{1-\xi^2}}{\xi}}^{1}\frac{dx}{x}\,N(x,t)\frac{1}{\sqrt{\frac{2x}{\xi}-x^2-1}}.$$

- Explicit expression also exists for $\mathrm{Re}\mathcal{H}^{(+)}(\xi,t)$.
- GPD quintessence: $N(x,t)=\sum_{\nu=0}^{\infty}x^{2\nu}Q_{2\nu}(x,t)=Q_0(x)+x^2Q_2(x)+x^4Q_4(x)+\dots$
- The amplitude automatically satisfies the dispersion relation in $\omega=\frac{1}{\xi}$ (O. Teryaev'05) with the subtraction constant given by the D-FF:

$$D(t)=\int_0^1\frac{dx}{x}\left(\frac{1}{\sqrt{1+x^2}}-1\right)Q_0(x,t)+\int_0^1\frac{dx}{x}[N(x,t)-Q_0(x,t)]\frac{1}{\sqrt{1+x^2}}$$

- GPD quintessence and D-FF is the maximal amount of info one can obtain about GPDs from the amplitude.

Abel transform tomography



The observer at ∞ looking along a line parallel to the x-axis a distance y above the origin sees the projection:

$$a(y^2) = \int_{-\infty}^{\infty} dx\, m(\rho^2) = \int_{y^2}^{\infty} d\rho^2 \frac{m(\rho^2)}{\sqrt{\rho^2 - y^2}}$$

- M. Polyakov'07: with the help of Joukowski conformal map $\frac{1}{w} = \frac{1}{2}\left(x + \frac{1}{x}\right)$ it is possible to present the relation between $\mathrm{Im}\mathcal{H}(\xi)$ and GPD quintessence $N(x)$ in the form of the Abel integral equation.
- The inverse transform for $N(x)$:

$$N(x) = \frac{1}{\pi} \frac{x(1-x^2)}{(1+x)^{\frac{3}{2}}} \int_{\frac{2x}{1+x^2}}^{1} \frac{d\xi}{\xi^{\frac{3}{2}}} \frac{1}{\sqrt{\xi - \frac{2x}{1+x^2}}} \left\{ \frac{1}{2} \mathrm{Im}\mathcal{H}^{(+)}(\xi) - \xi \frac{d}{d\xi} \mathrm{Im}\mathcal{H}^{(+)}(\xi) \right\}.$$

- $N(x,t) = \underbrace{Q_0(x,t)}_{\text{PDFs}} + x^2 \underbrace{Q_2(x,t)}_{\text{FFs of EMT tensor}} + x^4 Q_4(x,t) + \dots$

Froissart- Gribov projection I

Gribov'61, Froissart'61

DR for the elementary amplitude analytically continued to the t-channel:

$$\mathcal{H}^{(+)}(\cos\theta_t, t) = \int_0^1 dz\, \frac{2z}{1-z^2} \Phi^{(+)}(z, \cos\theta_t, t) = \int_0^1 dx\, \frac{2x\cos^2\theta_t}{1-x^2\cos^2\theta_t} H^{(+)}(x,x,t) + 4D(t)\,,$$

where $\Phi^{(+)}(z,\omega,t) = H^{(+)}\left(\frac{z}{\omega}, \eta = \frac{1}{\omega}, t\right)$.

Let us define

- SO(3) PWAs

$$a_J(t) \equiv \frac{1}{2}\int_{-1}^{1} d(\cos\theta_t)\, P_J(\cos\theta_t)\mathcal{H}^{(+)}(\cos\theta_t, t)$$

- GDAs with a definite angular momentum J

$$\Phi_J^{(+)}(z,t) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta_t)\, P_J(\cos\theta_t)\, \Phi^{(+)}(z, \cos\theta_t, t)$$

Neumann's integral representation for the Legendre functions $\mathcal{Q}_J$:

$$\frac{1}{2}\int_{-1}^{1} dz\, P_J(z)\frac{1}{z'-z} = \mathcal{Q}_J(z') \quad J \geq 0, \quad \text{integer.}$$

Froissart- Gribov projection II

- For even positive J

$$a_{J>0}(t) = \int_0^1 dz \frac{2z}{1-z^2} \Phi_J^{(+)}(z,t) = 2\int_0^1 dx \frac{\mathcal{Q}_J(1/x)}{x^2} H^{(+)}(x,x,t)\,.$$

- For $J=0$ we get

$$a_{J=0}(t) = 2\int_0^1 dx \left[\frac{\mathcal{Q}_0(1/x)}{x^2} - \frac{1}{x}\right] H^{(+)}(x,x,t) + 4D(t)\,.$$

- N.B. $\frac{\mathcal{Q}_J(1/x)}{x^2} \sim x^{J-1}$ for small x.

Mellin moments of GPD quintessence $\Leftrightarrow$ Froissart- Gribov projection

$$\int_0^1 dy\, y^{J-1} N(y,t) = \int_0^1 dx \left[\frac{1}{\sqrt{x}} \frac{d}{dx} R_J(x)\right] H^{(+)}(x,x,t)\,,$$

where the auxiliary functions

$$\frac{1}{\sqrt{x}} \frac{d}{dx} R_J(x) = (\frac{1}{2} + J) \frac{\mathcal{Q}_J(1/x)}{x^2}\,.$$

Froissart- Gribov projection III

- For even $J > 0$ we get

$$a_{J>0}(t) = \frac{4}{2J+1} \sum_{\substack{n=J-1 \\ \text{odd}}}^{\infty} B_{nJ}(t) = \frac{4}{2J+1} \int_0^1 dy y^{J-1} N(y,t).$$

- For $J = 0$ it reads

$$\begin{aligned} a_{J=0}(t) &= 4 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} B_{n0}(t) = 4\,\mathrm{Reg} \int_0^1 \frac{dy}{y} \left(N(y,t) - Q_0(y,t)\right) \\ &= 4 \int_{(0)}^1 \frac{dy}{y} \left(N(y,t) - Q_0(y,t)\right) + 4D^{\mathrm{f.p.}}(t). \end{aligned}$$

Non-analytic contribution into $a_{J=0}(t)$:

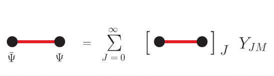
$$-4 \int_{(0)}^1 \frac{dy}{y} Q_0(y,t) + 4D^{\mathrm{f.p.}}(t) \equiv -2 \int_{(0)}^1 \frac{dx}{x} H^{(+)}(x,0,t) + 4D^{\mathrm{f.p.}}(t).$$

Interpretation of GPD quintessence

$$N(x,t) = \underbrace{Q_0(x,t)}_{\text{PDFs}} + x^2 \underbrace{Q_2(x,t)}_{\text{FFs of EMT tensor}} + x^4 Q_4(x,t) + \ldots$$

- Only a principle possibility to separate Q_ks via logarithmic scaling violation.
- Spin J expansion of the QCD string operator:

$$\bar{\Psi}(n) P \exp\left(i \int_{-n}^{n} dz^\mu A_\mu(z) \right) \Psi(-n) =$$



- For massless hadrons:

$$\int_0^1 dx x^{J-1} N(x,t) = B_{J-1\,J}(t) + B_{J+1\,J}(t) + B_{J+3\,J}(t) + \ldots \equiv F_J(t).$$

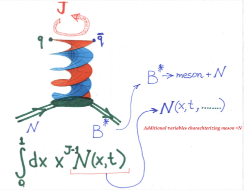
- Spoiled a bit by threshold corrections for $\beta \neq 1$. Some resummation needed?
- GPD quintessence is a new tool to study QCD strings. Also new possibilities for studies of nucleon excitations.

Can we handle with QCD string for the non-diagonal case?

- Hard part of DVCS creates a soft QCD string.

$$
\begin{aligned}
&(\bar{q}(z)\gamma_\mu \mathrm{Pexp}\left\{ i\int_0^1 dx^\mu A_\mu(x) \right\} q(0)\Big|_{z\to 0} \\
&= z^\nu \underbrace{\bar{q}\gamma_\mu \nabla_\nu q}_{\text{Spin}-2:\, q-\text{part of EMT}} + z^\nu z^\rho \underbrace{\bar{q}\gamma_\mu \nabla_\nu \nabla_\rho q}_{\text{Spin}-3} + \ldots
\end{aligned}
$$

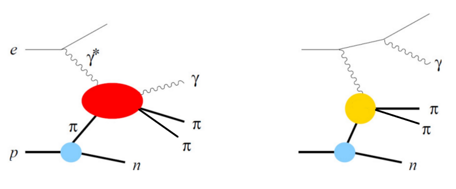
- How to decompose QCD string into probes of different spin? A tool is provided by the Abel tomography.
- $N(x, t, W, t', \alpha)$ is a complex function;
- The Abel tomography machinery is general and be applied for $N(x, t, W, t', \alpha)$;
- x-dependence is inherited from the x_B dependence of the DVCS amplitude.



A test ground for the formalism: $\pi \to \pi\pi$ ND DVCS

$$e(l) + p(p) \to e(l') + \gamma(q') + \pi^+(p'_\pi) + n(p')$$

- Can be studied through the Sullivan-type process:



- No complications due to spin-$\frac{1}{2}$.
- Access to the meson spectrum: $\rho(770)$, $f_2(1270)$ *etc.*
- An option for the EIC?

Some experimental prospects?

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Generalized Parton Distributions of Pions at the Forthcoming Electron-Ion Collider

- **N.B.** $\gamma^* N \to \rho N' \to \pi\gamma N'$ a background for $N \to \Delta$ DVCS.

$\pi \to \pi\pi$ transition GPDs

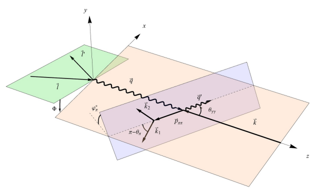
- $\pi \to \pi\pi$ unpolarized transition GPD ($\bar{P} \equiv \frac{k+k_1+k_2}{2}$):

$$\frac{1}{2}\int \frac{d\lambda}{2\pi} e^{i\lambda x \bar{P}\cdot n} \langle \pi(k_1)\pi(k_2)| \bar{\psi}\left(-\frac{\lambda n}{2}\right) \not{n} \psi\left(\frac{\lambda n}{2}\right) |\pi(p_\pi)\rangle$$
$$= \frac{1}{2\bar{P}\cdot n} i\varepsilon(n, \bar{P}, \Delta, k_1) \frac{1}{f_\pi^3} H_{\pi\to\pi\pi}(x, \xi, \Delta^2, W_{\pi\pi}^2, \theta_\pi^*, \varphi_\pi^*);$$

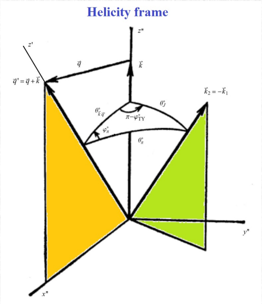
- $\pi \to \pi\pi$ polarized transition GPD:

$$\frac{1}{2}\int \frac{d\lambda}{2\pi} e^{i\lambda x \bar{P}\cdot n} \langle \pi(k_1)\pi(k_2)| \bar{\psi}\left(-\frac{\lambda n}{2}\right) \not{n}\gamma_5 \psi\left(\frac{\lambda n}{2}\right) |\pi(p_\pi)\rangle$$
$$= \frac{1}{2\bar{P}\cdot n} (\bar{P}\cdot n) \frac{1}{f_\pi} \tilde{H}_{\pi\to\pi\pi}(x, \xi, \Delta^2, W_{\pi\pi}^2, \theta_\pi^*, \varphi_\pi^*);$$

- Transition GPD arguments: x, ξ, $\Delta^2 = t$ and of the invariant mass of $\pi\pi$ system $W_{\pi\pi}^2$ and the helicity frame pion decay angles θ_π^*, φ_π^*.



Angles in the helicity frame



- $\cos\theta^*_\pi$ is linear in $s_1 = (q' + p_\pi)^2$;
- $\cos\varphi^*_\pi$ is linear in $t_2 = (p'_N - p_N)^2$;
- Polar and azimuthal angle through the Gram determinants:

$$\cos\theta^*_\pi = \frac{G_2\begin{pmatrix} k_1 + k_2, q' \\ k_1 + k_2, k_1 \end{pmatrix}}{\{\Delta_2(k_1 + k_2, q')\,\Delta_2(k_1 + k_2, k_2)\}^{\frac{1}{2}}};$$

$$\sin^2\varphi^*_\pi = \frac{\Delta_2(k + q, q')\,\Delta_4(k + q, q', k, k_2)}{\Delta_3(k + q, q', k)\,\Delta_3(k + q, q', k_2)};$$

- Gram determinants:

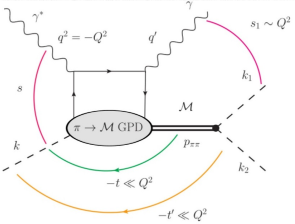
$$\mathrm{G}_n\begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix} = \det(p_i \cdot q_j);$$

- Symmetric Gram determinants:

$$\Delta_n(p_1, \dots p_n) = \mathrm{G}_n\begin{pmatrix} p_1, \dots, p_n \\ p_1, \dots, p_n \end{pmatrix} = \det(p_i \cdot p_j)$$

How to treat the angular structure? Real-valued spherical harmonics.

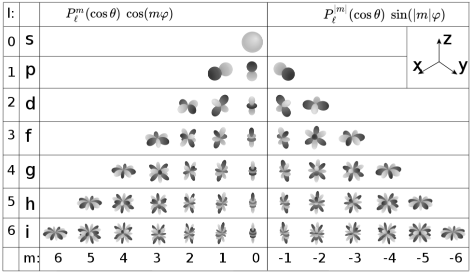
- Partial wave expansion both in $\theta^*_\pi \Leftrightarrow \alpha$ and φ^*_π.



$$Y_\ell^m(\theta^*_\pi, \varphi^*_\pi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta - \pi^*) e^{im\varphi^*_\pi}$$

the real-valued spherical harmonics read :

$$Y_\ell^m = \begin{cases} \frac{1}{\sqrt{2}}\left(Y_{\ell,|m|} - i\,Y_{\ell,-|m|}\right) & \text{if } m < 0; \\ Y_{\ell,0} & \text{if } m = 0; \\ \frac{(-1)^m}{\sqrt{2}}\left(Y_{\ell,|m|} + i\,Y_{\ell,-|m|}\right) & \text{if } m > 0; \end{cases}$$



PW expansion of $\pi \to \pi\pi$ GPDs

- PW expansion in angles θ^*_π and φ^*_π for unpolarized GPD:

$$H_{\pi\to\pi\pi}(x,\xi,t,W^2_{\pi\pi},\theta^*_\pi,\varphi^*_\pi) = \frac{1}{\sqrt{1-\cos^2\theta^*_\pi}\sin\varphi^*_\pi} \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{-1} H^{\ell m}_{\pi\to\pi\pi}(x,\xi,t,W^2_{\pi\pi})\,Y_{\ell m}(\theta^*_\pi,\varphi^*_\pi);$$

N.B. Spherical harmonics in are odd under $\varphi^*_\pi \to -\varphi^*_\pi$.

- PW expansion in angles θ^*_π and φ^*_π for polarized GPD:

$$\tilde{H}_{\pi\to\pi\pi}(x,\xi,t,W^2_{\pi\pi},\theta^*_\pi,\varphi^*_\pi) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} \tilde{H}^{\ell m}_{\pi\to\pi\pi}(x,\xi,t,W^2_{\pi\pi})\,Y_{\ell m}(\theta^*_\pi,\varphi^*_\pi);$$

N.B. Spherical harmonics in are even under $\varphi^*_\pi \to -\varphi^*_\pi$.

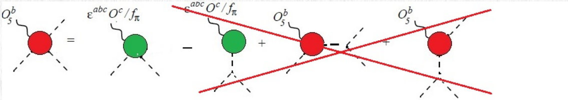
Soft pion theorems for $\pi \to \pi\pi$ GPDs

Sangyeong Son, studies under way

- Soft pion $\equiv W = 2m_\pi$;
- PCAC: trade soft pion for a chiral rotation;
- Only the chiral rotation of the operators is relevant:

$$\left[Q_5^a, \bar{\psi}(x)\gamma_\mu t^b \psi(y)\right] = \mathrm{i}\varepsilon^{abc}\bar{\psi}(x)\gamma_\mu\gamma_5 t^c\psi(y)$$
$$\left[Q_5^a, \bar{\psi}(x)\gamma_\mu\gamma_5 t^b \psi(y)\right] = \mathrm{i}\varepsilon^{abc}\bar{\psi}(x)\gamma_\mu t^c\psi(y);$$

- The structure of the soft pion theorems is simpler than in $N \to \pi N$ case.

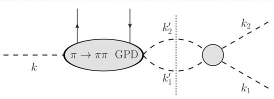


- Polarized isovector $\pi \to \pi\pi$ GPD is expressed at the threshold in terms of the usual pion isovector GPD.
- Unpolarized $\pi \to \pi\pi$ GPD is zero at the threshold.

How to go beyond the threshold? I (in collaboration with H. Son)

- The Watson'54 final state interaction theorem for $\pi \to \pi\pi$ transition GPD:

$$\text{for} \quad W_{\pi\pi}^2 < 16 m_\pi^2: \quad \text{Im}\, \tilde{H}^I_{\pi\to\pi\pi}(x,\xi,w^2,\theta^*_\pi,\varphi'_\pi)$$
$$= \frac{1}{2!}\int d(\text{phase space}) \left(\tilde{H}^I_{\pi\to\pi\pi}(x,\xi,w^2,\theta'_\pi,\varphi'_\pi)\right)^* A^I_{\pi\pi}(k_1,k_2|k'_1,k'_2)$$



- $\pi\pi$-scattering amplitude:

$$A^I_{\pi\pi} = 8\pi W_{\pi\pi} \sum_\ell (2\ell+1) a^I_\ell(W^2_{\pi\pi}) P_\ell \left[\cos\left(\theta_{\rm cm}\right)\right].$$

- Elastic unitarity condition:

$$\text{Im}\, a^I_\ell(W^2_{\pi\pi}) = |\vec{k}_1||a^I_\ell(W^2_{\pi\pi})|^2;$$

- $\delta^I_\ell\left(W^2_{\pi\pi}\right)$ are the $\pi\pi$ scattering phases:

$$a^I_\ell(W^2_{\pi\pi}) = \frac{1}{|\vec{k}_1|} \sin\left[\delta^I_\ell\left(W^2_{\pi\pi}\right)\right] e^{i\delta^I_\ell(W^2_{\pi\pi})}.$$

How to go beyond the threshold? II

- The equation for the expansion coefficients $\tilde{H}^{I}_{\ell;m}$:

$$\text{Im}\, \tilde{H}^{I}_{\ell;m}(x,\xi,w^2) = \tan\left[\delta^{I}_{\ell}\left(w^2\right)\right] \text{Re}\, \tilde{H}^{I}_{\ell;m}(x,\xi,w^2).$$

- Omnes'58: N-subtracted dispersion relation

$$\tilde{H}^{I}_{\ell;m}(x,\xi,w^2)$$
$$= \sum_{k=0}^{N-1} \frac{w^{2k}}{k!} \frac{\mathrm{d}^k}{\mathrm{d}w^{2k}} \tilde{H}^{I}_{\ell;m}(x,\xi,w^2=0) + \frac{w^{2N}}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\tan\left(\delta^{I}_{\ell}(s)\right) \text{Re}\left\{\tilde{H}^{I}_{l;m}(x,\xi,s)\right\}}{s^N\left(s-w^2-i\epsilon\right)}.$$

- The Omnes solution (for $N=0$):

$$\tilde{H}^{I}_{\ell;m}(x,\xi,W^2) = \tilde{H}^{I}_{\ell;m}(x,\xi,W^2=4m_\pi^2) \exp\left[\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\delta^{I}_{\ell}(s)}{s-m_{\pi\pi}^2-i\epsilon}\right]$$

- Transition GPDs are complex functions above threshold!

Summary and Outlook

1. New tool for baryon spectroscopy: arbitrary spin-J probe and PW analysis of excited states.
2. A new bridge between PW analysis and QCD.
3. Access to $N \to N^*$ EMT matrix elements: mechanical properties of resonances.
4. A lab for chiral perturbation theory on the light cone: soft pion theorems and chiral expansion.
5. GPD formalism worked out for $N \to \Delta(1232)$, $P_{11}(1440)$, $D_{13}(1520)$, $S_{11}(1535)$. Can be studied at JLab@12 GeV and an option for JLab@22 GeV.
6. A development for hyperons $N \to \Lambda, \Sigma$ and production of strange mesons?
7. $\pi \to \pi\pi$ and $N \to \pi N$ transition GPDs emerge as a tool to study the spectrum of hadrons.
8. First step: development of the formalism for $\pi \to \pi\pi$ transition GPDs: Abel tomography, threshold theorems and the Omnes dispersion relations.

Thank you for your attention!