Novel Scaled Interpolating Basis and Operators

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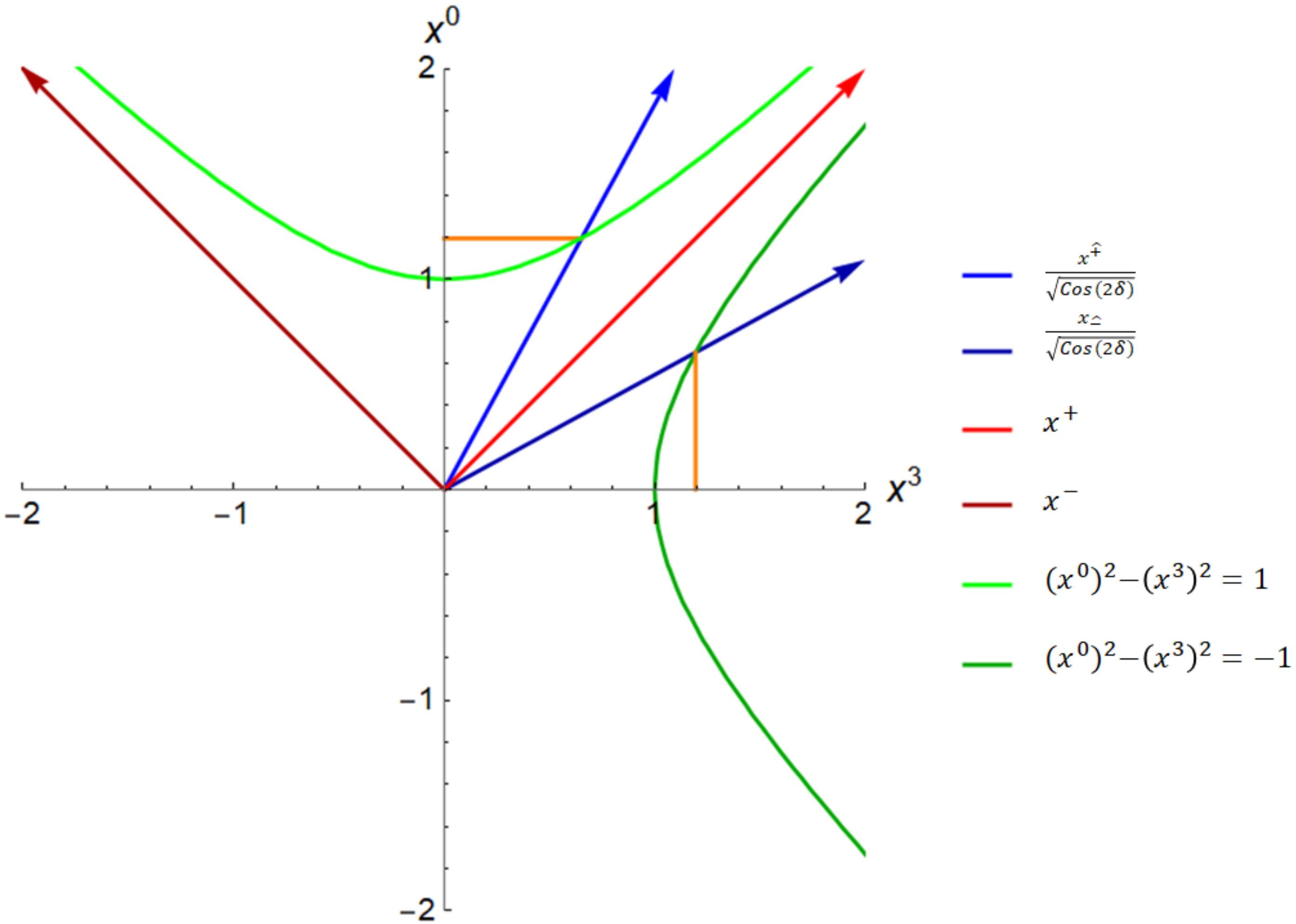
10/27/2023

Outline

- Novel scaled interpolating basis and the transformation
- Scaled interpolating Poincare operators
- Infinitesimal transformation matrices and algebra
- Scaled interpolating dynamic at the light-front
- Scaled interpolating basis in five-dimensional space –time
- Scaled interpolating de sitter operators and algebra
- Conclusion

Novel Scaled Interpolating Basis



- The interpolating parameter space $0 \le \delta < \pi/4$ can be covered by the unified scheme that introduces the independent scaled interpolating coordinates $x^{\hat{+}}/\sqrt{\mathbb{C}}$ and $x_{\hat{-}}/\sqrt{\mathbb{C}}$.

- Note that the coordinates $x^{\hat{+}}/\sqrt{\mathbb{C}}$ and $x_{\hat{-}}/\sqrt{\mathbb{C}}$ are not orthogonal but independent from each other.

- The lightcone defined by $x^\mu x_\mu = 0$ corresponds to $(x^{\hat{+}}/\sqrt{\mathbb{C}})^2 - (x_{\hat{-}}/\sqrt{\mathbb{C}})^2 = \mathbf{x}_\perp{}^2$. In the light-front limit, not only $\mathbb{C} \to 0$ but also $(x^{\hat{+}})^2 - (x_{\hat{-}})^2 \to 0$ to result in finite $\mathbf{x}_\perp{}^2$.

$$(x^0)^2 - (x^3)^2 = \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 = 2\,x^+x^-$$

Novel Scaled Interpolating Transformation

$$x^N = H.x$$

$$\begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\cos 2\delta}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\cos 2\delta}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta}{\sqrt{\cos 2\delta}} & 0 & 0 & \frac{\sin\delta}{\sqrt{\cos 2\delta}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin\delta}{\sqrt{\cos 2\delta}} & 0 & 0 & \frac{\cos\delta}{\sqrt{\cos 2\delta}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

In the IFD $(\delta \to 0)$, $\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \to x^0, x^{\hat{1}} \to x^1, x^{\hat{2}} \to x^2, \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \to x^3$, In the LFD $(\delta \to \frac{\pi}{4})$, $x^{\hat{+}} \to x^+$, $x_{\hat{-}} \to x^+$ and $\sqrt{\mathbb{C}} \to 0$. Therefore $\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}$ and $\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}$ become indeterminate unless we consider $x^+ = 0$.

$$\left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)_{\delta \to \frac{\pi}{4}-\epsilon} = \frac{x^+}{\sqrt{2\epsilon}} + \frac{x^-\epsilon}{\sqrt{2}} - \frac{x^+\epsilon^{3/2}}{6\sqrt{2}} \ldots\ldots$$

$$\left(\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)_{\delta \to \frac{\pi}{4}-\epsilon} = \frac{x^+}{\sqrt{2\epsilon}} - \frac{x^-\epsilon}{\sqrt{2}} - \frac{x^+\epsilon^{3/2}}{6\sqrt{2}} \ldots\ldots$$

- Reduction of degrees of freedom in the LF end.

Note: In ordinarily Interpolation transformation in IFD, invert the direction of z direction in the IFD

$(x^{\hat{-}} \to -x^3)$.

: Orthogonal basis set for all interpolation angle

Space-Time Interval

$$s^2 = x^{\mu} x_{\mu} = x^{\hat{\mu}} x_{\hat{\mu}} = \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (x^1)^2 - (x^2)^2 - \left(\frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2$$

Momentum space for the particle mass M,$P^{\mu} P_{\mu}$ on the mass shell is equal to M^2

$$M^2 = P^{\mu} P_{\mu} = P^{\hat{\mu}} P_{\hat{\mu}} = \left(\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - (P^1)^2 - (P^2)^2 - \left(\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2$$

- Space-time matrix tensor in new scaled interpolating dynamic

$$g^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- g^N is equal to the Makowski space-time matrix

Interpolating Poincare Operators

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$M^{\hat{\mu}\hat{\nu}} = G^{\hat{\mu}}_{\alpha} M^{\alpha\beta} G^{\hat{\nu}}_{\beta} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix}$$

$$g^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix} = g_{\hat{\mu}\hat{\nu}}$$

$$M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix} 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}$$

$$E^{\hat{1}} = J^2 \sin\delta + K^1 \cos\delta, \qquad \mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$
$$E^{\hat{2}} = K^2 \cos\delta - J^1 \sin\delta, \qquad \mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$
$$F^{\hat{1}} = K^1 \sin\delta - J^2 \cos\delta, \qquad \mathcal{D}^{\hat{1}} = -K^1 \cos\delta + J^2 \sin\delta,$$
$$F^{\hat{2}} = K^2 \sin\delta + J^1 \cos\delta, \qquad \mathcal{D}^{\hat{2}} = -J^1 \sin\delta - K^2 \cos\delta.$$

Scaled Interpolating Poincare Operators

$$E^{\hat{1}} = J^2 \sin\delta + K^1 \cos\delta, \qquad \mathcal{K}^{\hat{1}} = -K^1 \sin\delta - J^2 \cos\delta,$$

$$E^{\hat{2}} = K^2 \cos\delta - J^1 \sin\delta, \qquad \mathcal{K}^{\hat{2}} = J^1 \cos\delta - K^2 \sin\delta,$$

$$M^N = \begin{pmatrix} 0 & \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & K^3 \\ -\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & 0 & J^3 & \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} \\ -\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & -J^3 & 0 & \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} \\ -K^3 & -\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & 0 \end{pmatrix}$$

$$\delta \to 0, \qquad E^{\hat{1}} \to K^1, E^{\hat{2}} \to K^2, \mathcal{K}^{\hat{1}} \to -J^2, \mathcal{K}^{\hat{2}} \to J^1$$

Number of operators = 6 , Kinematic operator => 3, Dynamic operators =>3

$$\delta \to \pi/4, \qquad \mathcal{K}^{\hat{1}} \to -E^1, \mathcal{K}^{\hat{2}} \to -E^2$$

Number of operators = 4 , Kinematic operator => 4, Dynamic operators =>0

$$M^{\tilde{\mu}\tilde{\nu}} = H^{\tilde{\mu}}_{\lambda} M^{\lambda\rho} H^{\tilde{\nu}}_{\rho} = M^N$$

- Scaled interpolating basis does not produce $F^{\hat{1}}, F^{\hat{2}}, \mathcal{D}^{\hat{1}}$ and $\mathcal{D}^{\hat{2}}$

- We don’t use superscript and subscript indices
  - Scaled interpolating Coordinates are not orthogonal
  - To maintain the identity of the original interpolating coordinates

Infinitesimal Transformation Matrix

- Infinitesimal transformation matrix of scaled interpolating operators in the scaled interpolating basis

e.g : $H.\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}.H^{-1}$

$$\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Valid for all interpolation angle

- Infinitesimal transformation matrix of scaled interpolating operators in the scaled interpolating basis coincide with the Infinitesimal transformation operators of rotation and boost operators in the standard basis

$$H.\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}.H^{-1} = K^1 \qquad H.\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}}.H^{-1} = -J^2$$

$$H.\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}}.H^{-1} = K^2 \qquad H.\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}}.H^{-1} = J^1$$

$$H.K^3.H^{-1} = K^3 \qquad H.J^3.H^{-1} = J^3$$

$$M^N = \begin{pmatrix} 0 & \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & K^3 \\ -\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & 0 & J^3 & \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} \\ -\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & -J^3 & 0 & \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} \\ -K^3 & -\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & 0 \end{pmatrix}$$

- Satisfy same Lie algebra

$$[M^{\mu\nu}, M^{\rho\lambda}] = i(g^{\nu\rho}M^{\mu\lambda} - g^{\nu\lambda}M^{\mu\rho} - g^{\mu\rho}M^{\nu\lambda} + g^{\mu\lambda}M^{\nu\rho})$$

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

Note:

Space-time matrix tensors are same

Form Invariant Operators

Example : Boost in x^1-direction $e^{-i\beta_1 K^1}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Corresponding operator $e^{-i\beta_1 \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}}$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

Example : Rotation around x^1 axis $e^{-i\theta_1 J^1}$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Corresponding operator $e^{-i\theta_1 \frac{K^{\hat{2}}}{\sqrt{\mathbb{C}}}}$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- All other scaled interpolating operators in scaled interpolating Poincare matrix act similar way.

Original interpolating formalism produce $\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, F^{\hat{1}}$ and $F^{\hat{2}}$ operators. To see the difference we can scaled those operators as well.

Example:

$$e^{i\rho_1 \frac{\mathcal{D}^{\hat{1}}}{\sqrt{\mathbb{C}}}}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{(-1+\cosh\rho_1)}{\mathbb{C}^2} & \frac{\sinh\rho_1}{\mathbb{C}} & 0 & -\frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} \\ \frac{\sinh\rho_1}{\mathbb{C}} & \cosh\rho_1 & 0 & -\frac{\sinh\rho_1 \mathbb{S}}{\mathbb{C}} \\ 0 & 0 & 1 & 0 \\ \frac{(-1+\cosh\rho_1)\mathbb{S}}{\mathbb{C}^2} & \frac{\sinh\rho_1 \mathbb{S}}{\mathbb{C}} & 0 & 1 - \frac{(-1+\cosh\rho_1)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$$e^{i\rho_2 \frac{\mathcal{D}^{\hat{2}}}{\sqrt{\mathbb{C}}}}$$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{(-1+\cosh\rho_2)}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2}{\mathbb{C}} & -\frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} \\ 0 & 1 & 0 & 0 \\ \frac{\sinh\rho_2}{\mathbb{C}} & 0 & \cosh\rho_2 & -\frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} \\ \frac{(-1+\cosh\rho_2)\mathbb{S}}{\mathbb{C}^2} & 0 & \frac{\sinh\rho_2 \mathbb{S}}{\mathbb{C}} & 1 - \frac{(-1+\cosh\rho_2)\mathbb{S}^2}{\mathbb{C}^2} \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

- The operators are not produced by the scaled interpolating Poincare matrix , do not exhibit form invariant and they depend on the interpolation angle.

- Scaled interpolating basis can define for all interpolation angle without any problem except the exact LF end or light-like region.

- Scaled interpolating basis produce scaled interpolating space time matrix tensor and scaled interpolating operators , they exhibit same form invariant with standard basis space-time matrix and Poincare operators including the Poincare algebra

↓

Scaling process extract region of interpolating dynamic that has form invariant with standard instant form dynamic.

Form invariant => Mathematical model is invariant , but physical system can be different in the region

What will happen exactly at the LF-end ?

Original interpolating dynamic

$$e^{-i\beta_1 E^{\hat{1}}}$$

Example:

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cosh(\beta_1\sqrt{\mathbb{C}}) & \sqrt{\mathbb{C}}\sinh(\beta_1\sqrt{\mathbb{C}}) & 0 & 0 \\ \frac{\sinh(\beta_1\sqrt{\mathbb{C}})}{\sqrt{\mathbb{C}}} & \cosh(\beta_1\sqrt{\mathbb{C}}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2\sinh(\frac{\beta_1}{2}\sqrt{\mathbb{C}})^2\mathbb{S}}{\mathbb{C}} & \frac{\mathbb{S}\sinh(\beta_1\sqrt{\mathbb{C}})}{\sqrt{\mathbb{C}}} & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix}$$

$\delta \to 0$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Dynamic operator

$\delta \to \frac{\pi}{4}$

Kinematic operator

$$\begin{pmatrix} x'^{+} \\ x'^{1} \\ x'^{2} \\ x'^{-} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta_1^2}{2} & \beta_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{+} \\ x^{1} \\ x^{2} \\ x^{-} \end{pmatrix}$$

$$x'^+ = x^+$$

$$x'^1 = \beta_1 x^+ + x^1$$

$$x'^2 = x^2$$

$$x'^- = \frac{(\beta_1)^2}{2} x^+ + \beta_1 x^1 + x^-$$

Scaled interpolating dynamic at the Light-front ($\delta \to \pi/4$)

Example: $e^{-i\beta_1 \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}}$

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix}$$

$\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}$ is still a kinematic operator in the scaled light-front reduced coordinate system .

- We can build similar argument for the rest of the scaled interpolating operators

Time evolution parameter: $\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}$

At the light front, time evolution parameter coincide with one of the space parameter. Reduced degrees of freedom

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- Scenario 1 $\Rightarrow$ $x^+ = 0$ $\frac{x^+}{\sqrt{Cos(2\delta)}} \to x_f^+$ (Finite light-front time)

$x'^+_f = x_f^+$

$x'^1 = (Cosh\ \beta_1 - 1)x^1$

$x'^2 = x^2$

- Scenario 2 $\Rightarrow$ $x^+ \neq 0$ at LF-end

$\frac{x^+}{\sqrt{Cos(2\delta)}} \to \infty$ (Infinite Light-front time)

$\beta_1 = 0$	$\beta_1 \neq 0$
$x'^1 = x^1$	$x'^1 \to \infty$
$x'^2 = x^2$	$x'^2 = x^2$

❑ Writing scaled interpolating basis in terms of light front coordinates

$$x^N = Q.\,x^{LFD}$$

$$(x^0)^2 - (x^3)^2 = \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 = 2\,x^+ x^-$$

$$\begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta+\sin\delta}{\sqrt{2\mathbb{C}}} & 0 & 0 & \frac{\cos\delta-\sin\delta}{\sqrt{2\mathbb{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\cos\delta+\sin\delta}{\sqrt{2\mathbb{C}}} & 0 & 0 & \frac{-\cos\delta+\sin\delta}{\sqrt{2\mathbb{C}}} \end{pmatrix} \begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix}$$

$$x'^N = Q.\,x'^{LFD} = (\mathrm{H}.\,\Lambda_{\hat{s}}.\,H^{-1})x^N$$

$$x'^{LFD} = Q^{-1}(\mathrm{H}.\,\Lambda_{\hat{s}}.\,H^{-1})\,Q.\,x^{LFD}$$

$\Lambda_{\hat{s}}$ =Scaled interpolating operator
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \theta 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\theta 2^2}{2} & \theta 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: $e^{-i\beta_1 \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}}$ In the light-front basis

$$\begin{pmatrix} x'^+ \\ x'^1 \\ x'^2 \\ x'^- \end{pmatrix} = \begin{pmatrix} \cosh(\frac{\beta_1}{2})^2 & \frac{\sqrt{\mathbb{C}}\sinh\beta_1}{\sqrt{2}(\cos\delta+\sin\delta)} & 0 & \frac{(\cos\delta-\sin\delta)\sinh(\frac{\beta_1}{2})^2}{(\cos\delta+\sin\delta)} \\ \frac{(\cos\delta+\sin\delta)\sin\beta_1}{\sqrt{2\mathbb{C}}} & \cos\beta_1 & 0 & \frac{(\cos\delta-\sin\delta)\sin\beta_1}{\sqrt{2\mathbb{C}}} \\ 0 & 0 & 1 & 0 \\ \frac{(\cos\delta+\sin\delta)\sinh(\frac{\beta_1}{2})^2}{(\cos\delta-\sin\delta)} & \frac{\sqrt{\mathbb{C}}\sinh\beta_1}{\sqrt{2}(\cos\delta-\sin\delta)} & 0 & \cosh(\frac{\beta_1}{2})^2 \end{pmatrix} \begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix}$$

$$\begin{pmatrix} x'^+ \\ x'^1 \\ x'^2 \\ x'^- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\beta_1^2}{2} & \beta_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix}$$

Scaled interpolating basis of five-dimensional space –time (De Sitter Space)

$$y^N = H.y$$

$$\begin{pmatrix} \frac{y^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ y^{\hat{1}} \\ y^{\hat{2}} \\ \frac{y_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ y^{\hat{4}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\sin\delta}{\sqrt{\mathbb{C}}} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\sin\delta}{\sqrt{\mathbb{C}}} & 0 & 0 & \frac{\cos\delta}{\sqrt{\mathbb{C}}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}$$

$$\frac{y^{\hat{+}}}{\sqrt{\mathbb{C}}} = \frac{l_1 \sinh(t/l_1)\cos(\delta) + l_1 \cosh(t/l_1)\sin(\rho)\cos(\theta)\sin(\delta)}{\sqrt{\mathbb{C}}}$$

$$y^{\hat{1}} = l_1 \cosh(t/l_1)\sin(\rho)\sin(\theta)\cos(\phi)$$

$$y^{\hat{2}} = l_1 \cosh(t/l_1)\sin(\rho)\sin(\theta)\sin(\phi)$$

$$\frac{y_{\hat{-}}}{\sqrt{\mathbb{C}}} = \frac{l_1 \sinh(t/l_1)\sin(\delta) + l_1 \cosh(t/l_1)\sin(\rho)\cos(\theta)\cos(\delta)}{\sqrt{\mathbb{C}}}$$

$$y^{\hat{4}} = l_1 \cosh(t/l_1)\cos(\rho)$$

De sitter radius

$$-l^2 = \left(\frac{y^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{y_{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2 - (y^{\hat{1}})^2 - (y^{\hat{2}})^2 - (y^{\hat{4}})^2$$

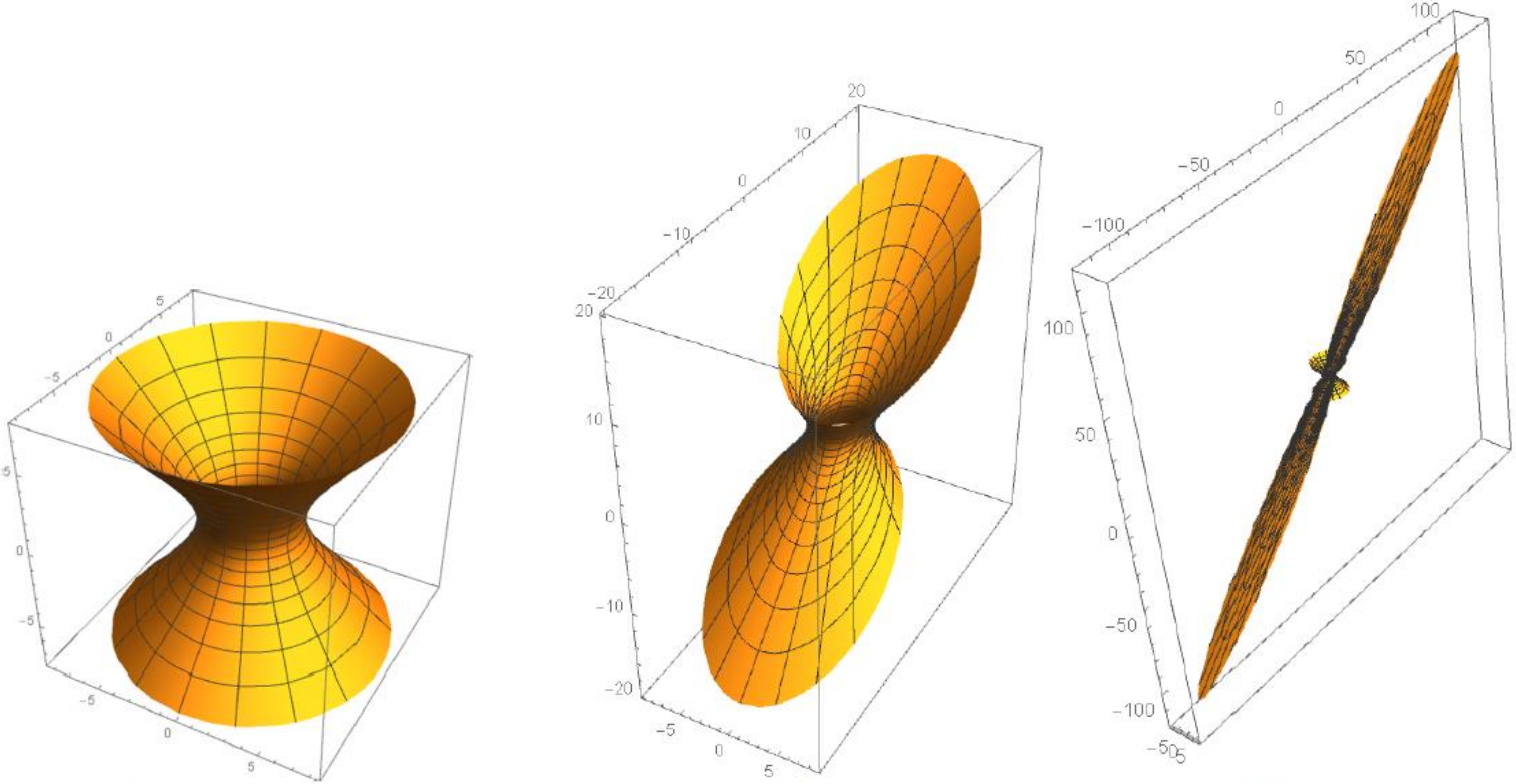
Space-time matrix tensor of scaled interpolating de-siter space

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \eta^N$$

Space-time matrix is invariant

$$ds^2 = dt^2 - l_1^2 \cosh^2\left(\frac{t}{l_1}\right)\left(d\rho^2 + \sin^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right) = (2dy^+ dy^- - dy_\perp^2 - dy_4^2)$$

De Sitter Space-time



(a) IFD De Sitter Space (b) $\delta = 0.6$ De Sitter Space (c) $\delta = 0.78$ De Sitter Space

Scaled interpolating de Sitter operators

$$R^N = \begin{pmatrix} 0 & \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & K^3 & -\frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ -\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & 0 & J^3 & \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\Gamma^{\hat{1}} \\ -\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & -J^3 & 0 & \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & -\Gamma^{\hat{2}} \\ -K^3 & -\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & 0 & -\frac{\Gamma_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ \frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}} & \Gamma^{\hat{1}} & \Gamma^{\hat{2}} & \frac{\Gamma_{\hat{-}}}{\sqrt{\mathbb{C}}} & 0 \end{pmatrix}$$

$$\delta \to 0,$$

Number of homogenous operators => 10 ,
Homogenous Kinematic operator => 6,
Homogenous Dynamic operators =>4

$$\delta \to \pi/4,$$

Number of homogenous operators => 7 ,
Homogenous Kinematic operator => 7,
Homogenous Dynamic operators =>0

- Infinitesimal transformation matrix of scaled interpolating operators in the scaled interpolating basis of five-dimensional space

$$\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^{3} = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J^{3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix}, \quad \frac{\Gamma^{\hat{-}}}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}$$

- Infinitesimal transformation matrix of scaled homogenous interpolating operators in five-dimensional scaled interpolating basis coincide with the standard five-dimensional Infinitesimal transformation operators in the standard de Sitter basis

$$H.\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}}.H^{-1}=K^1 \qquad H.\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}}.H^{-1}=-J^2 \qquad H.\frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}}.H^{-1}=\Gamma^0$$

$$H.\Gamma^1.H^{-1}=\Gamma^1$$

$$H.\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}}.H^{-1}=K^2 \qquad H.\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}}.H^{-1}=J^1$$

$$H.\Gamma^2.H^{-1}=\Gamma^2$$

$$H.K^3.H^{-1}=K^3 \qquad H.J^3.H^{-1}=J^3 \qquad H.\frac{\Gamma_{\hat{-}}}{\sqrt{\mathbb{C}}}.H^{-1}=\Gamma^3$$

$$R^N=\begin{pmatrix} 0 & \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & K^3 & -\frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ -\frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} & 0 & J^3 & \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\Gamma^{\hat{1}} \\ -\frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} & -J^3 & 0 & \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & -\Gamma^{\hat{2}} \\ -K^3 & -\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} & -\frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} & 0 & -\frac{\Gamma_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ \frac{\Gamma^{\hat{+}}}{\sqrt{\mathbb{C}}} & \Gamma^{\hat{1}} & \Gamma^{\hat{2}} & \frac{\Gamma_{\hat{-}}}{\sqrt{\mathbb{C}}} & 0 \end{pmatrix}$$

$$R^{\alpha\beta}=\begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix}$$

They all satisfy Lie algebra given below.

$$[R^{\alpha\beta}, R^{\gamma\delta}] = i(\eta^{\beta\gamma}R^{\alpha\delta} - \eta^{\beta\delta}R^{\alpha\gamma} - \eta^{\alpha\gamma}R^{\beta\delta} + \eta^{\alpha\delta}R^{\beta\gamma})$$

Where $\alpha = \beta = \gamma = \delta = 0, 1, 2, 3, 4$

$$\Gamma^{\mu} = R^{4\mu}, \qquad \mu = 0, 1, 2, 3$$

$$\begin{aligned}
[R^{\mu\nu}, R^{\rho\lambda}] &= i(\eta^{\nu\rho}R^{\mu\lambda} - \eta^{\nu\lambda}R^{\mu\rho} - \eta^{\mu\rho}R^{\nu\lambda} + \eta^{\mu\lambda}R^{\nu\rho}) \\
[R^{\mu\nu}, \Gamma^{\rho}] &= i(\eta^{\nu\rho}\Gamma^{\mu} - \eta^{\mu\rho}\Gamma^{\nu}) \\
[\Gamma^{\mu}, \Gamma^{\nu}] &= iR^{\mu\nu}
\end{aligned}$$

Where the range of the indices μ ,ν,ρ and λ is 0,1,2,3.

- Scaled interpolating operators satisfy the same de sitter Lie algebra

Scaling process extract region of interpolating dynamic that has form invariant with standard instant form dynamic even in the higher dimension.

Contracting scaled interpolating de Sitter space-time

Translation in new interpolating de Sitter space can be written as $y'^N = H.e^{i(a/l_1)^N \Gamma_N}.H^{-1}y^N$. To contract the de Sitter space we can take $l_1 \to \infty$. Then we can obtained new interpolating translation in the Minkowski space.

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{a^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ 0 & 1 & 0 & 0 & a^{\hat{1}} \\ 0 & 0 & 1 & 0 & a^{\hat{1}} \\ 0 & 0 & 0 & 1 & \frac{a_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathbb{C}}} \\ 1 \end{pmatrix}$$

Homogenous scaled translation operators in five-dimensional space S0 (4,1) can be contracted to inhomogeneous scaled translation operators in Poincare space IS0(3,1)

Infinitesimal translation operators in scaled interpolating basis

$$H.\frac{P^{\hat{+}}}{\sqrt{\mathbb{C}}}.H^{-1} = P^0 \qquad H.P^2.H^{-1} = P^2$$

$$H.P^1.H^{-1} = P^1 \qquad H.\frac{P_{\hat{-}}}{\sqrt{\mathbb{C}}}.H^{-1} = P^3$$

Conclusion

- Scaling process extract space-like region of interpolating dynamic that has form invariant with standard instant form dynamic. (Independent of the dimension)

- Exactly at the LF-end, Scaled interpolating dynamics produce only kinematic operators mainly due to the reduction of degrees of freedom at the LF-end

Application

Quantum correlation in the helicity amplitudes and magnifying zero-mode