Lattice QCD calculations of Transverse Momentum Dependent Parton Distribution Functions (TMDs)

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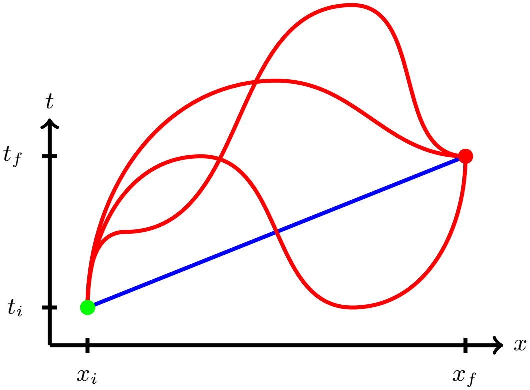
Dr. Chueng Ji's research group meeting

Path Integrals in QM



$$\mathcal{Z} = \langle x_f | \, e^{-iH(t_f - t_i)} \, | x_i \rangle = \int \mathcal{D}x \; e^{-iS[x(t)]}$$

Path Integrals in QFT

Consider scalar field $\phi(x)$ and an action $S[\psi(x)] = \int d^4x\mathcal{L}[\psi(x)]$

$$\mathcal{Z} = \int \mathcal{D}\psi(x)\ e^{-iS[\psi(x)]} \tag{1}$$

and

$$\int \mathcal{D}\psi(x) = \prod_x \int d\psi_x = \int d\psi_1 \int d\psi_2 \int d\psi_3 \int d\psi_4 \cdots \tag{2}$$

Euclidean Path Integral in QFT

Even in the discretized lattice, we have pratical problem in

$$\mathcal{Z} = \int \mathcal{D}\psi(x)\; e^{-iS[\psi(x)]} \tag{3}$$

make a Wick rotation: $\boxed{t \longrightarrow -it}$ then

$$\boxed{-iS = -i \int d^3x dt \mathcal{L}} \longrightarrow \boxed{-\int d^3x dt \mathcal{L}_E = -S_E} \tag{4}$$

Euclidean path integral

$$\boxed{\mathcal{Z}_E = \int \mathcal{D}\psi(x)\; e^{-S_E[\psi(x)]}} \tag{5}$$

then the physical observables $\mathcal{O}$ are evaluated as

$$\boxed{\langle O[\psi(x)]\rangle = \frac{\int \mathcal{D}\psi(x)\; O[\psi(x)]\; e^{-S_E[\psi(x)]}}{\int \mathcal{D}\psi(x)\; e^{-S_E[\psi(x)]}}} \tag{6}$$

QCD (Quantum ChromoDynamics)

The QCD Lagrangian density is constructed from two types of particle fields:

- Spin-$\frac{1}{2}$ Dirac fields (quarks): $\boxed{\psi_{i,f}}$
  - color $i = 1, 2, 3 = N_c$
  - flavor $f = u, d, s, c, b, t$
- Massless spin-1 vector fields (gluons): $\boxed{A_{\mu,a}}$
  - color $a = 1, 2, \cdots, 8 = N_c^2 - 1$, with $SU(3)$ local color gauge symmetry

$$\boxed{\mathcal{L}_{QCD}(\psi_f, A_\mu) = -\frac{1}{4}(F_{\mu\nu,a}[A])^2 + \sum_f \bar{\psi}_{i,f}(i\gamma^\mu (D_\mu[A])_{ij} - m_f \delta_{ij})\psi_{j,f}} \, . \tag{7}$$

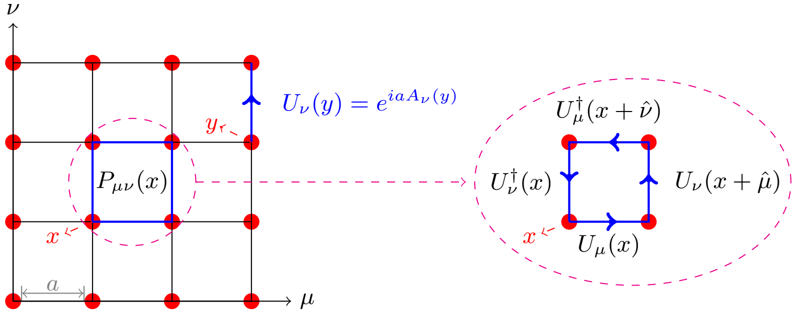
Here,

- the gluon field strength $\boxed{F_{\mu\nu,a}[A] = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - g f_{abc} A_{\mu,b} A_{\nu,c}}$
- the covariant derivative $\boxed{D_\mu[A] = \partial_\mu + igA_{\mu,a} t_a}$
- the generator t_a and structure constant f_{abc} define the $SU(3)$ color algebra: $\boxed{[t_a, t_b] = if_{abc} t_c}$
- g is the strong coupling constant.

Lattice QCD[1]

$\Lambda_4 = \{n_\mu = (n_1, n_2, n_3, n_4) | n_i \in a[0, 1, \cdots, L_i - 1]\}$

$$\boxed{S_{gauge} = \frac{2}{g^2} \sum_{x \in \Lambda_4} \sum_{\mu < \nu} (1 - Re\ Tr\ [P_{\mu\nu}(x)])} \tag{8}$$



where the elementary plaquette, $\boxed{P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x)}$.

[1] K. G. Wilson, Confinement of Quarks, Phys. Rev. D10 (1974) 2445.

Lattice QCD

$$S_{gauge} = \frac{2}{g^2} \sum_{x \in \Lambda_4} \sum_{\mu<\nu} (1 - Re\ Tr\ [P_{\mu\nu}(x)]) = \frac{a^4}{2g^2} \sum_{x \in \Lambda_4} \sum_{\mu<\nu} tr\ [F_{\mu\nu}(x)^2] + \mathcal{O}(a^2).$$

Physical observables $\mathcal{O}$ are evaluated as an expectation value over the relevant degrees of freedom

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \mathcal{O}\ e^{-[S_{gauge} + \int dx \bar{\psi} \mathcal{M} \psi]}}{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] e^{-[S_{gauge} + \int dx \bar{\psi} \mathcal{M} \psi]}}. \quad (9)$$

The quark fields ψ & $\bar{\psi}$ are Grassmann variables:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}[U]\ \widetilde{\mathcal{O}}\ det\mathcal{M} e^{-[S_{gauge}]}}{\int \mathcal{D}[U] det\mathcal{M} e^{-[S_{gauge}]}}. \quad (10)$$

This integration results in the "contraction" of fermion–anti-fermion pairs in all possible ways (Wick's theorem), replacing them with quark propagators $\mathcal{M}^{-1}$.

Numerical simulation for Lattice QCD

The vacuum expectation value of an observable in a Monte Carlo simulation approximation: (Sum over U_n with probability $\propto e^{-S[U_n]}$)

$$\langle O \rangle = \frac{\int \mathcal{D}[U] e^{-S_G[U]} O[U]}{\int \mathcal{D}[U] e^{-S_G[U]}} \rightarrow \langle O \rangle \approx \frac{1}{N} \sum_{U_n} O[U_n]$$

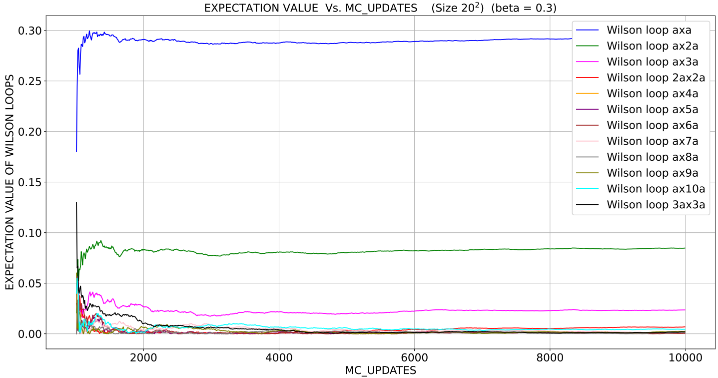


Figure 1: Markov chain - Monte Carlo simulations of $\mathbb{Z}_2$ lattice gauge theory $(1 + 1)$

Meson Two Point Function

Consider 2 ways of writing meson two point function:
Way 1: ($T-$ lattice temporal extent)

$$C_\pi(\boldsymbol{x},t) \equiv \langle O_\pi(\boldsymbol{x},t)O_\pi^\dagger(\mathbf{0},0)\rangle = \frac{Tr[e^{-\hat{H}(T-t)}O_\pi(\boldsymbol{x})e^{-\hat{H}t}\ \ O_\pi^\dagger(\mathbf{0})]}{Tr[e^{-\hat{H}T}]} \tag{11}$$

$$= \frac{\sum_{\rho,\sigma}\langle\rho|\,e^{-\hat{H}(T-t)}O_\pi(\boldsymbol{x})\ \ |\sigma\rangle\langle\sigma|\ \ e^{-\hat{H}t}\ \ O_\pi^\dagger(\mathbf{0})\,|\rho\rangle}{\sum_{\rho'}\langle\rho'|\,e^{-\hat{H}T}\,|\rho'\rangle} \tag{12}$$

$$= \frac{\sum_{\rho,\sigma} e^{-E_\rho(T-t)}e^{-E_\sigma t}\langle\rho|\,O_\pi(\boldsymbol{x})\,|\sigma\rangle\langle\sigma|\,O_\pi^\dagger(\mathbf{0})\,|\rho\rangle}{e^{-E_0T}(1+e^{-\Delta E_1T}+e^{-\Delta E_2T}+\cdots)} \quad |\text{where, } \Delta E_n = E_n - E_0$$

$$\Longrightarrow C_\pi(\boldsymbol{x},t) \xrightarrow{T\longrightarrow\infty} \sum_\sigma \langle 0|\,O_\pi(\boldsymbol{x})\,|\sigma\rangle\langle\sigma|\,O_\pi^\dagger(\mathbf{0})\,|0\rangle\ \ e^{-\Delta E_\sigma t} \tag{13}$$

project to zero momentum (Fourier transformation)

$$\boxed{C_\pi(\mathbf{0},t)} = \sum_{\boldsymbol{x}} e^{-i\mathbf{0}.\boldsymbol{x}}C_\pi(\boldsymbol{x},t) = \boxed{\sum_\sigma |A|^2 e^{-M_\pi t}\ (1+\mathcal{O}(e^{-\Delta M_\pi t}))} \tag{14}$$

Meson Two Point Function

Way 2:

For pion: $O_\pi(\boldsymbol{x}) = \bar{u}(\boldsymbol{x})\gamma_5 d(\boldsymbol{x})$

$$C_\pi(\boldsymbol{x},t) \equiv \langle O_\pi(\boldsymbol{x},t) O_\pi^\dagger(\mathbf{0},0)\rangle = \frac{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ e^{-[S_{gauge}+\int dx \bar{\psi}\mathcal{M}\psi]}\ \bar{\psi}_u(\boldsymbol{x},t)\gamma_5\psi_d(\boldsymbol{x},t)\ \bar{\psi}_d(\mathbf{0},0)\gamma_5\psi_u(\mathbf{0},0)}{\int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ e^{-[S_{gauge}+\int dx \bar{\psi}\mathcal{M}\psi]}} \tag{15}$$

$$= \frac{\int \mathcal{D}[U] det(\mathcal{M}_u) det(\mathcal{M}_d) e^{-S_{gauge}}\ Tr[\mathcal{M}_u^{-1}(\boldsymbol{x} \text{ to } \mathbf{0})\gamma_5 \mathcal{M}_d^{-1}(\mathbf{0} \text{ to } \boldsymbol{x})\gamma_5]}{\int \mathcal{D}[U] det(\mathcal{M}_u) det(\mathcal{M}_d) e^{-S_{gauge}}} \tag{16}$$

project to zero momentum (Fourier transformation)

$$C_\pi(\mathbf{0},t) = \sum_{\boldsymbol{x}} e^{-i\mathbf{0}.\boldsymbol{x}} \left[\frac{\int \mathcal{D}[U] det(\mathcal{M}_u) det(\mathcal{M}_d) e^{-S_{gauge}}\ Tr[\mathcal{M}_u^{-1}(\boldsymbol{x} \text{ to } \mathbf{0})\gamma_5 \mathcal{M}_d^{-1}(\mathbf{0} \text{ to } \boldsymbol{x})\gamma_5]}{\int \mathcal{D}[U] det(\mathcal{M}_u) det(\mathcal{M}_d) e^{-S_{gauge}}}\right] \tag{17}$$

Monte Carlo approximation:

$$\boxed{C_\pi(\mathbf{0},t)} \longrightarrow \boxed{\frac{1}{N_{cfgs}} \sum_{i=0}^{N_{cfgs}} \left[\sum_{\boldsymbol{x}} Tr[\mathcal{M}_u^{-1}[U_i]\gamma_5 \mathcal{M}_d^{-1}[U_i]\gamma_5]\right]} \tag{18}$$

Meson Two Point Function

For $\mathbf{p} = 0$ we find $C_\pi(\mathbf{0}, t) \xrightarrow{t\to\infty} Ce^{-m_\pi t}$

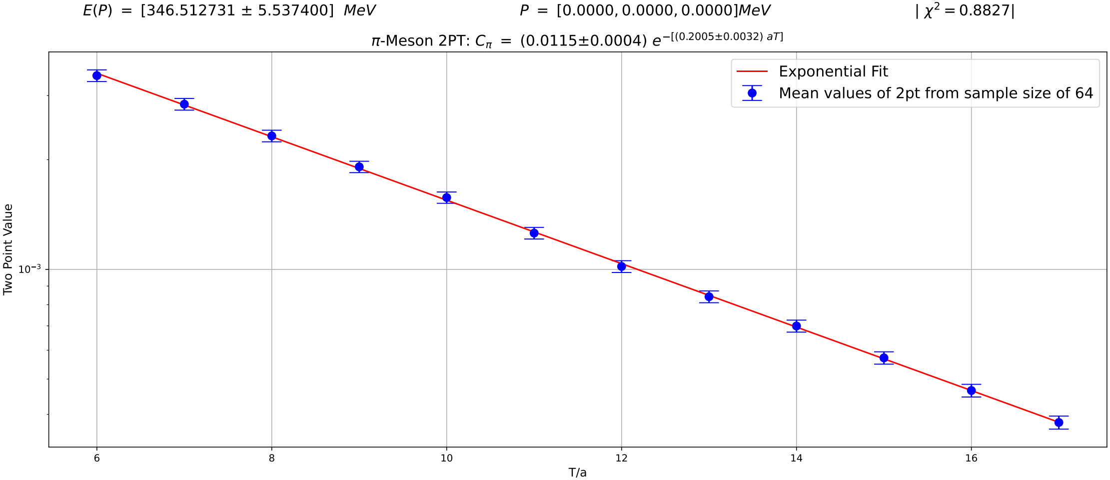
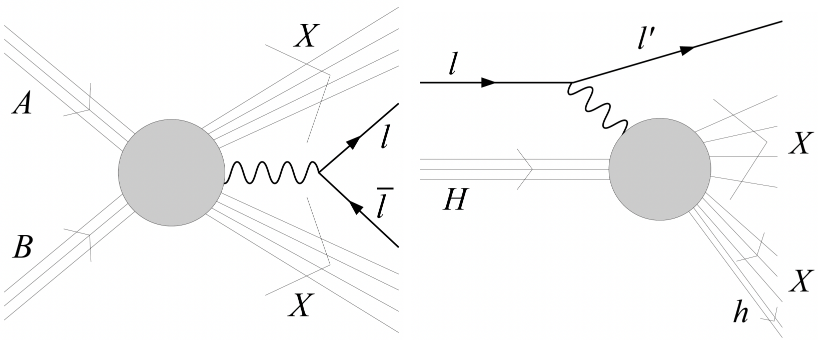


Figure 2: Meson two-point function $C_\pi(t, \mathbf{0})$ from lattice size:$(32^3, 96)$ with $a = 0.11967$ fm

Introduction: TMDs

The **intrinsic motion of quarks and gluons** inside the proton or neutron, specifically **with respect to the transverse momentum**, can be described in terms of Transverse Momentum Dependent Parton Distribution Functions (TMDs)



(a) The Drell-Yan process. (b) The SIDIS process.

Figure 3: Two examples of processes sensitive to TMD PDFs. We draw the leading contributions, in which a single electroweak gauge boson (wiggled lines) is exchanged.

Introduction: TMDs

In the SIDIS cross section

$$\frac{d\sigma}{d^3P_h d^3P_{l'}} \propto L_{\mu\nu}W^{\mu\nu} \tag{19}$$

$$\Longrightarrow \quad W^{\mu\nu}(P,q,P_h) = \int \frac{d^4l}{(2\pi)^4} e^{iq\cdot l} \sum_X \langle N(P,S)|\, J^\mu(-b)\, |Xh(P_h,S_h)\rangle \langle Xh(P_h,S_h)|\, J^\nu(0)\, |N(P,S)\rangle$$

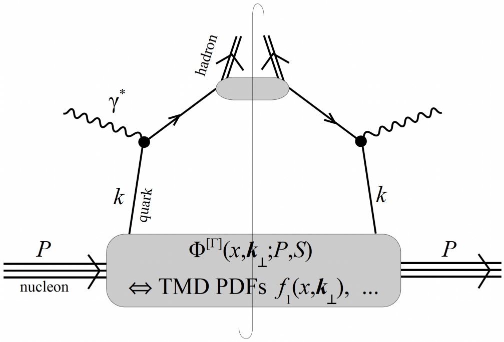


Figure 4: Simplifed factorized tree level diagram of the hadron tensor in SIDIS. *arXiv:0907.2381*

Definition of TMDs

- Consider a frame where the nucleon has large momentum in z-direction, i.e., $P^+ \gg m_N$, $\boldsymbol{P}_\mathrm{T} = 0$. In light cone coordinates, the components $\mathrm{k}^+ \colon \boldsymbol{k}_\mathrm{T} \colon k^- \sim P^+/m_N \colon 1 \colon m_N/P^+$, under boosts along the z-axis.

The starting point for our discussion of TMDs are the correlator of the general form

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4 b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \; \bar{q}(0)\, \Gamma\, \mathcal{U}[\mathcal{C}_b] \; q(b) \; |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \tag{20}$$

- The gauge link $\mathcal{U}[\mathcal{C}_b]$ brings divergences; so we divide it by soft factor $\tilde{\mathcal{S}}$

Definition of TMDs

Integrating the correlator over the suppressed momentum component k^- yields

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_\mathrm{T}; P, S; \ldots) \equiv \int dk^- \Phi^{[\Gamma]}(k, P, S; \ldots)$$
$$= \int \frac{d^2 \boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi) P^+} \; e^{ix(b\cdot P) - i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}} \; \frac{\frac{1}{2} \langle P,S| \; \bar{q}(0)\, \Gamma \; \mathcal{U}[\mathcal{C}_b] \; q(b) \; |P,S\rangle}{\widetilde{\mathcal{S}}(-\boldsymbol{b}_\mathrm{T}^2;\ldots)}\Bigg|_{b^+=0} . \quad (21)$$

The above correlator can be decomposed into TMDs.

$$\Phi^{[\gamma^+]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = f_1 - \left[\frac{\epsilon_{ij}\, \boldsymbol{k}_i\, \boldsymbol{S}_j}{m_N} \; f_{1T}^\perp\right]_\mathrm{odd} , \quad (22)$$

$$\Phi^{[\gamma^+\gamma^5]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = \Lambda\, g_1 + \frac{\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{S}_\mathrm{T}}{m_N} \; g_{1T} , \quad (23)$$

$$\Phi^{[i\sigma^{i+}\gamma^5]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = \boldsymbol{S}_i \; h_1 + \frac{(2\boldsymbol{k}_i\boldsymbol{k}_j - \boldsymbol{k}_\mathrm{T}^2\delta_{ij})\boldsymbol{S}_j}{2m_N^2}\, h_{1T}^\perp$$
$$+ \frac{\Lambda \boldsymbol{k}_i}{m_N} h_{1L}^\perp + \left[\frac{\epsilon_{ij}\boldsymbol{k}_j}{m_N} h_1^\perp\right]_\mathrm{odd} . \quad (24)$$

Parametrization of the correlator in position space

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\hat{\zeta},\mu) \equiv \frac{1}{2}\,\langle P,S|\;\bar{q}(0)\,\Gamma\,\mathcal{U}[0,\eta v,\eta v+b,b]\;q(b)\;|P,S\rangle \tag{25}$$

For the Γ-structures at leading twist, the correlator can be written in the form

$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} = \widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\,\widetilde{A}_{12B} \tag{26}$$

$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} = -\Lambda\,\widetilde{A}_{6B} + i\,\{(b{\cdot}P)\Lambda - m_N(\boldsymbol{b}_{\mathrm{T}}{\cdot}\boldsymbol{S}_{\mathrm{T}})\}\,\widetilde{A}_{7B} \tag{27}$$

$$\begin{aligned}\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} &= im_N\epsilon_{ij}\boldsymbol{b}_j\,\widetilde{A}_{4B} - \boldsymbol{S}_i\,\widetilde{A}_{9B} - im_N\Lambda\boldsymbol{b}_i\,\widetilde{A}_{10B}\\ &\quad + m_N\,\{(b{\cdot}P)\Lambda - m_N(\boldsymbol{b}_{\mathrm{T}}{\cdot}\boldsymbol{S}_{\mathrm{T}})\}\,\boldsymbol{b}_i\,\widetilde{A}_{11B}\end{aligned} \tag{28}$$

(Decompositions analogous to work by Metz et al. Phys. Lett. **B618** (2005) 90-96. in momentum space)

Strategy

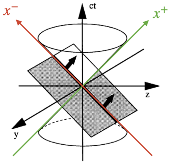


Figure 5: Light-front coordinates

- The separation b of the quark field operators has a transverse component, $\boxed{b = nb^- + b_\perp}$. So, this separation is space-like

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4 b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma \, \mathcal{U}[\mathcal{C}_b] \ q(b) \ |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \tag{29}$$

- We parametrized this correlator in terms of Lorentz-invariant amplitudes
- We choose the Lorentz frame in which this nonlocal operator is defined at one single time
- The computation of the nonlocal matrix element can be cast in terms of a Euclidean path integral and performed employing the standard methods of lattice QCD

Link geometry

The gauge link employed in this work reads

$$\mathcal{U}[\mathcal{C}_b^{(\eta v)}] = \mathcal{U}[0, \eta v, \eta v + b, b], \tag{30}$$

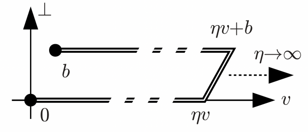


Figure 6: Staple-shaped gauge connection. The four-vectors v and P give the direction of the staple and the momentum, while b defines the separation between the quark operators. *(arXiv:1111.4249v2 [hep-lat])*

The Lorentz-invariant quantity characterizing the direction of v is the Collins-Soper type parameter

$$\boxed{\hat{\zeta} \equiv \zeta/2m_N = \frac{v \cdot P}{\sqrt{|v^2|}\sqrt{P^2}}.} \tag{31}$$

The light-like direction $v = n$ can be approached in the limit $\zeta \to \infty$.

TMDs in Fourier space and x-integrations (Mellin moments)

$$\tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \equiv \int d^2\boldsymbol{k}_\mathrm{T}\, e^{i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}}\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \tag{32}$$

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_\mathrm{T}^2 \ldots) \equiv n! \left(-\frac{2}{m_N^2}\partial_{\boldsymbol{b}_\mathrm{T}^2}\right)^n\; \tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \tag{33}$$

In the limit $|\boldsymbol{b}_\mathrm{T}| \to 0$, one recovers conventional $\boldsymbol{k}_\mathrm{T}$-moments of TMDs:

$$\tilde{f}^{(n)}(x, 0; \ldots) = \int d^2\boldsymbol{k}_\mathrm{T} \left(\frac{\boldsymbol{k}_\mathrm{T}^2}{2m_N^2}\right)^n f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv f^{(n)}(x)\ . \tag{34}$$

$\boldsymbol{k}_\mathrm{T}$-moments like $f_1^{(0)}(x)$ and $f_{1T}^{\perp(1)}(x)$ are ill-defined without further regularization, we therefore do not attempt to extrapolate to $\boldsymbol{b}_\mathrm{T} = 0$, but rather state our results at finite $|\boldsymbol{b}_\mathrm{T}|$.
In our studies so far, we only considered the first x-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$

$$f^{[1]}(\boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv \int_{-1}^{1} dx\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots)\ . \tag{35}$$

where, $x = \frac{k^+}{P^+}$

TMDs in Fourier space and invariant amplitudes

$\widetilde{A}_i(b^2, b\cdot P, (b\cdot P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)$

Certain x-integrated TMDs in Fourier space directly correspond to the amplitudes $\widetilde{A}_{iB}$ evaluated at $b\cdot P = 0$:

$$\tilde{f}_1^{[1](0)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v\cdot P) = 2\, \widetilde{A}_{2B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,,$$

$$\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v\cdot P) = -2\, \widetilde{A}_{12B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,,$$

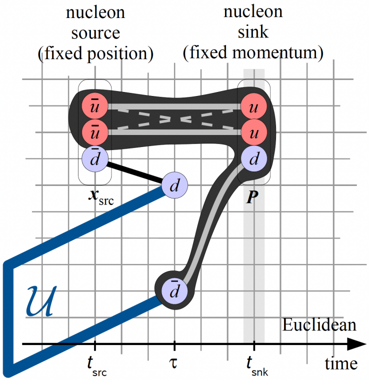
Generalized Sivers shifts from amplitudes

All other renormalization and soft factor related dependences cancel out in the ratio.

- $\langle \boldsymbol{k}_y \rangle^{\text{Sivers}} = \langle \boldsymbol{k}_y \rangle_{TU}$ is T-odd, it describes a feature of the transverse momentum distribution of (unpolarized) quarks in a transversely polarized proton.

$$\begin{aligned}
\langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_{\text{T}}^2; \hat{\zeta}, \eta v\cdot P) \;&\equiv\; m_N \frac{\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_{\text{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}{\tilde{f}_{1}^{[1](0)}(\boldsymbol{b}_{\text{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)} \\
&= -m_N \frac{\widetilde{A}_{12B}(-\boldsymbol{b}_{\text{T}}^2, 0, 0, -1/(m_N \hat{\zeta})^2, \eta v\cdot P)}{\widetilde{A}_{2B}(-\boldsymbol{b}_{\text{T}}^2, 0, 0, -1/(m_N \hat{\zeta})^2, \eta v\cdot P)} \\
&\xrightarrow{\boldsymbol{b}_{\text{T}}^2=0} \left. \frac{\int dx \int d^2\boldsymbol{k}_{\text{T}}\, \boldsymbol{k}_y \;\Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\text{T}}, P, S; \ldots)}{\int dx \int d^2\boldsymbol{k}_{\text{T}} \;\Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\text{T}}, P, S; \ldots)} \right|_{\boldsymbol{S}_{\text{T}} = (1,0)}
\end{aligned} \tag{36}$$

Lattice Setup [2]



- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma \, \mathcal{U}[0, \eta v, \eta v + b, b] \ q(b) \ |P, S\rangle$
- Euclidean time: Place entire operator at one time slice, i.e., b, ηv purely spatial
- Extrapolate $\eta \longrightarrow \infty$, $\hat{\zeta} \longrightarrow \infty$ numerically

[2] Figure Credits: Dr. Engelhardt (NMSU)

Numerical Results

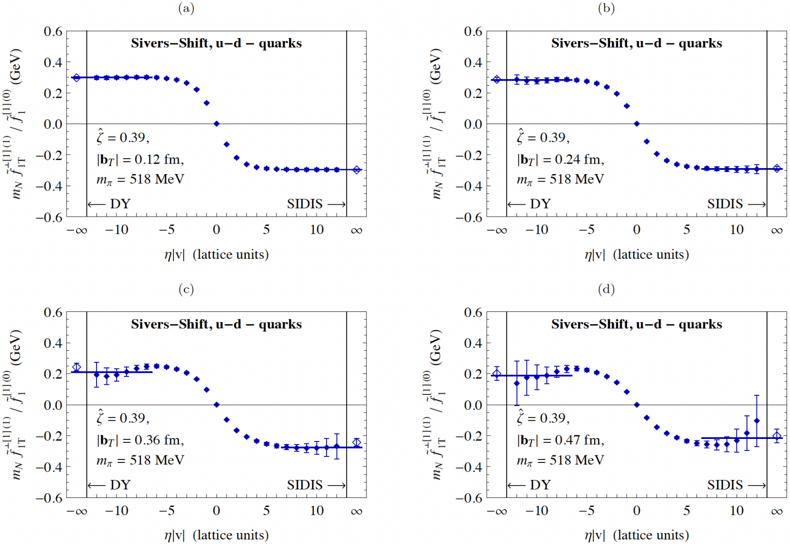


Figure 7: Extraction of the generalized Sivers shift on the lattice with $m_\pi = 518$MeV *(arXiv:1111.4249v2 [hep-lat])*

Numerical Results

Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$

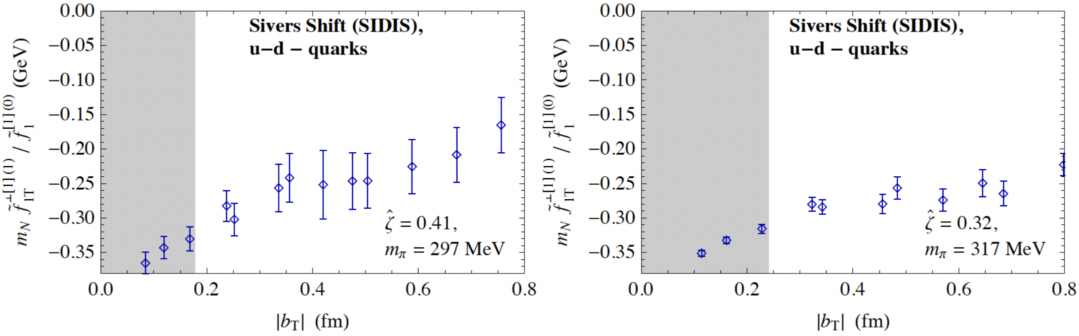


Figure 8: Generalized Sivers shift as a function of the quark separation $|\boldsymbol{b}_T|$ for the SIDIS case ($|\eta v| = \infty$).
arXiv:2301.06118 [hep-lat]

Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$

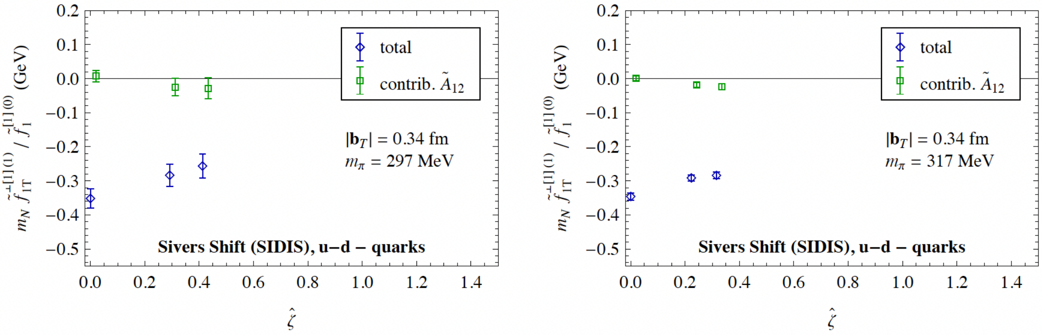


Figure 9: we show the $\hat{\zeta}$-dependence of the generalized Sivers shift, depicting both the full result and the result obtained with just $\tilde{A}_{12}$ in the numerator. *arXiv:2301.06118 [hep-lat]*

Few More Numerical Results

- M. Engelhardt, *et al.*, PoS **LATTICE2022**, 103 (2023), [arXiv:2301.06118 [hep-lat]].
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My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

$$\frac{1}{2}\langle P,S|\ \bar{q}(0)\ \gamma^+\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle = 2P^+\left(\widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\ \widetilde{A}_{12B}\right) \tag{37}$$

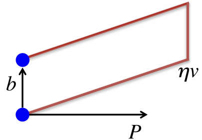
$$\Longrightarrow\ \langle\boldsymbol{k}_y\rangle_{TU}(\boldsymbol{b}_\mathrm{T}^2, x, \hat{\zeta}, \eta v{\cdot}P)\ \equiv\ m_N\frac{\tilde{f}_{1T}^{\perp(1)}(\boldsymbol{b}_\mathrm{T}^2;\hat{\zeta},\ldots,\eta v{\cdot}P)}{\tilde{f}_1^{(0)}(\boldsymbol{b}_\mathrm{T}^2;\hat{\zeta},\ldots,\eta v{\cdot}P)} \tag{38}$$

$$= -m_N\frac{\int d(b\cdot P)e^{ix(b\cdot P)}\widetilde{A}_{12B}(b^2, b{\cdot}P, (b{\cdot}P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)}{\int d(b\cdot P)e^{ix(b\cdot P)}\widetilde{A}_{2B}(b^2, b{\cdot}P, (b{\cdot}P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)} \tag{39}$$

- The range of accessible $b \cdot P$ is limited:

$$\boxed{\frac{v{\cdot}b}{v{\cdot}P} = b{\cdot}P\,\frac{R(\hat{\zeta}^2)}{m_N^2}\,,}\qquad \because (b^+ = 0 \text{ and } v_T = \mathbf{P}_T = 0) \tag{40}$$

where $R(\hat{\zeta}^2) \equiv 1 - \sqrt{1+\hat{\zeta}^{-2}} = \frac{m_N^2}{v\cdot P}\frac{v^+}{P^+}$.



My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

Extract b_L-even component of imaginary part of γ^+ correlator

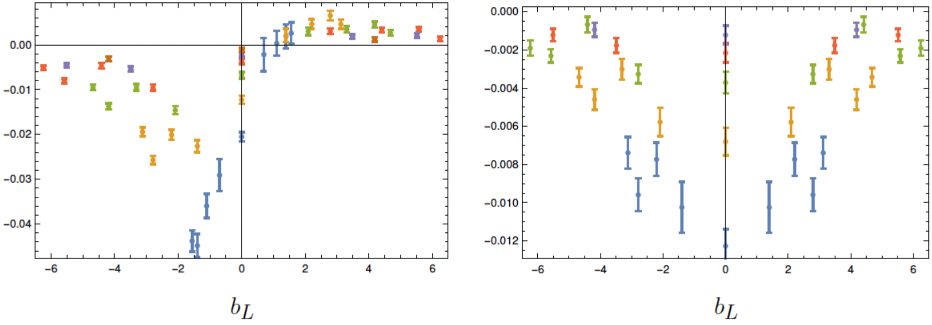


Figure 10: [3] $\left[\frac{1}{2}\langle P,S|\ \bar{q}(0)\ \gamma^+\ \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P,S\rangle = 2P^+\left(\widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\ \widetilde{A}_{12B}\right)\right]$

[3] PDFLattice 2019: M. Engelhardt

My PhD work: Extension to include the dependence on $x = \frac{k^+}{P^+}$

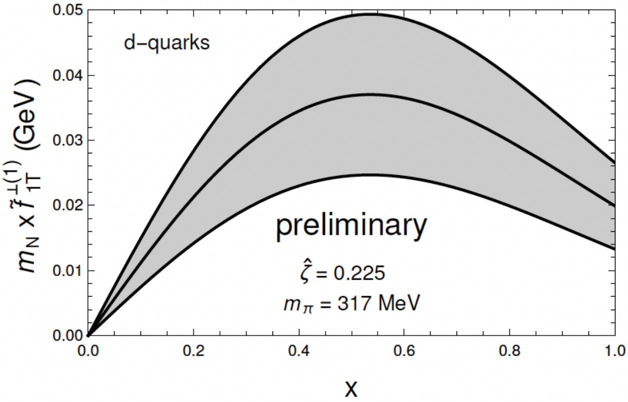


Figure 11: Nucleon SIDIS d-quark generalized Sivers shift as a function of momentum fraction x, multiplied by x [4][5]

[4] "TMD Handbook." arXiv:2304.03302 [hep-ph].

[5] M. Engelhardt, J. R. Green, S. Krieg, S. Meinel, J. Negele, A. Pochinsky et al., to be published

Conclusions

- It is feasible to obtain the x-dependence of TMD ratios: **Sivers shift**
- Inspite of constraints $\frac{v\cdot b}{v\cdot P} = b\cdot P \frac{R(\hat{\zeta}^2)}{m_N^2}$, it is possible to improve the analysis