2024, 2/23

Introduction to relativistic thermodynamics based on action principle

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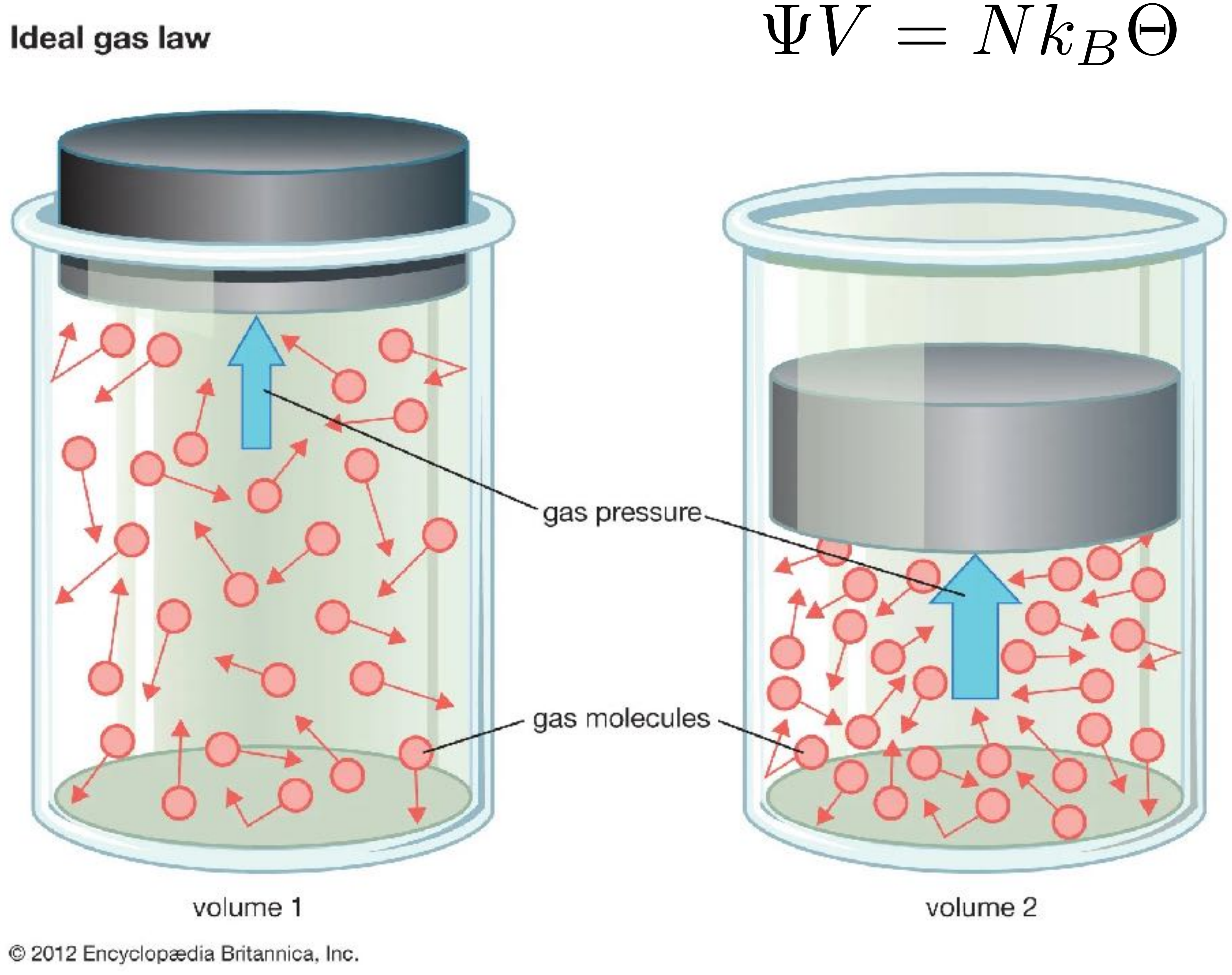
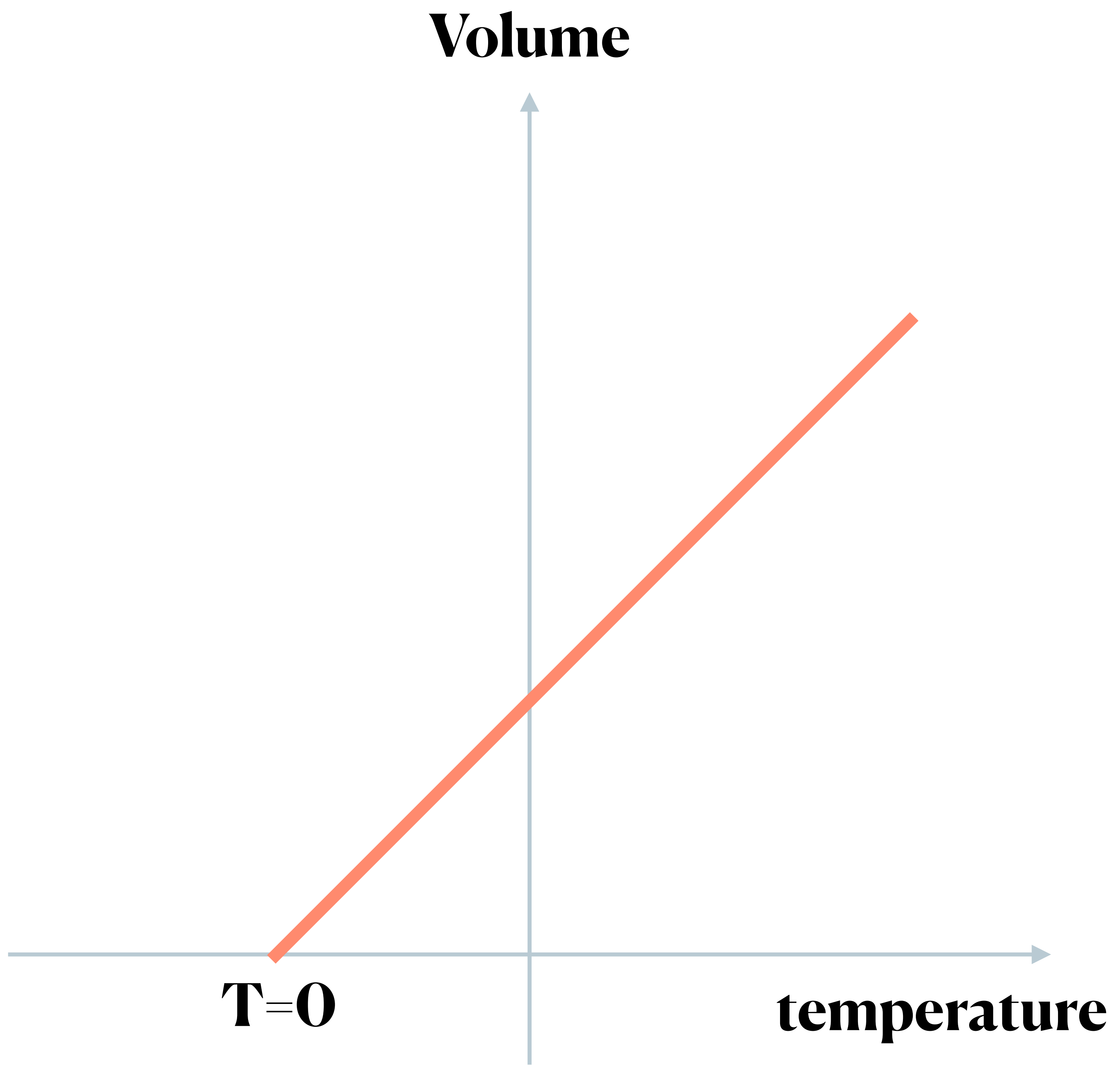
Thermodynamics of ideal gas (example)

Pressure

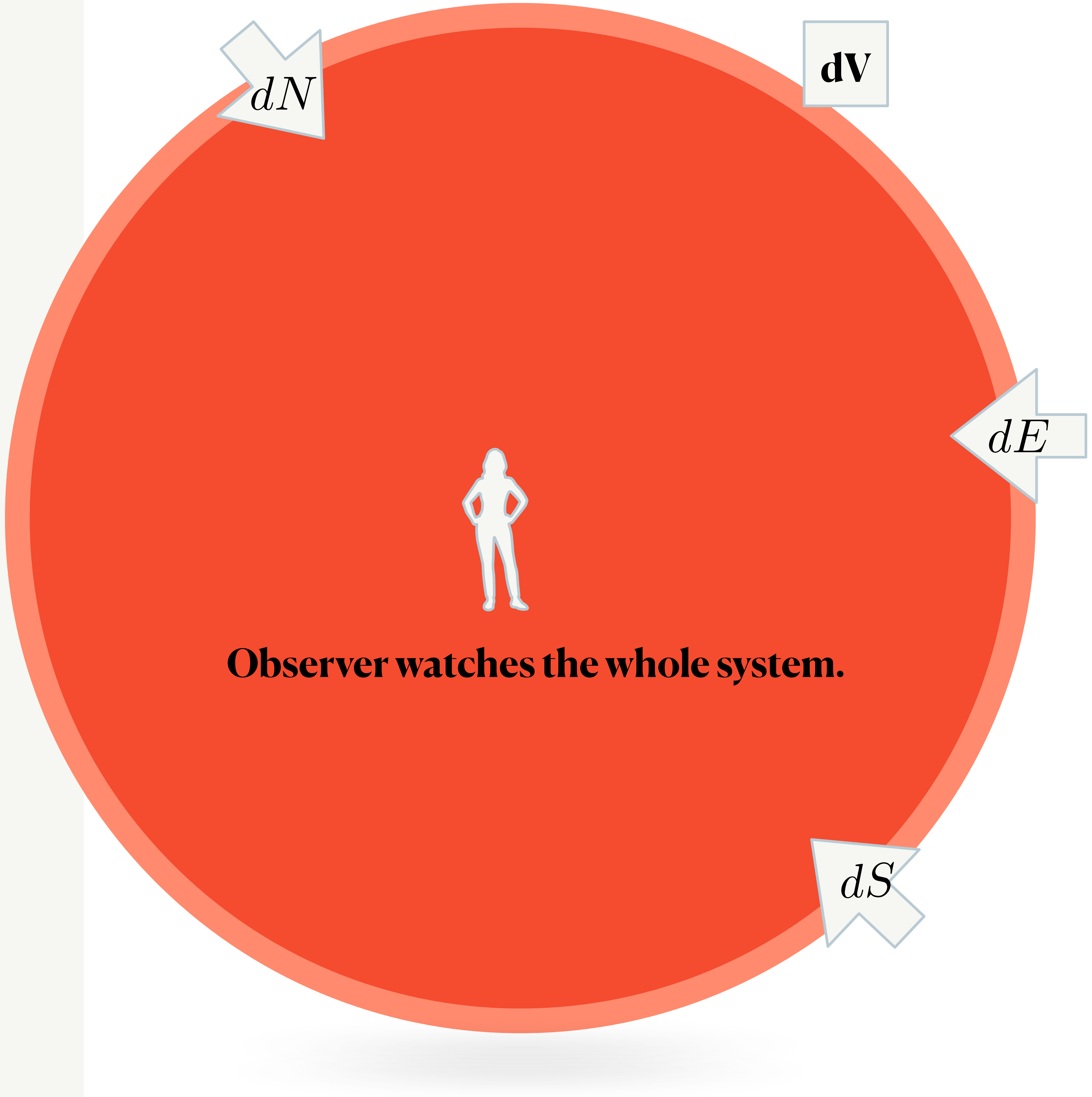
Temperature

$$\Psi V = N k_B \Theta$$





Thermal equilibrium



Temperature is homogeneous.

First law of thermodynamics:

$$dE = \Theta dS - \Psi dV + \chi dN$$

Temperature **Pressure** **Chemical potential**

The Newtonian theory of heat propagation

Heat equation: $$\vec{q} = -\kappa \nabla \Theta,$$

Evolution of temperature distribution: $$\frac{\partial \Theta}{\partial t} = -\frac{1}{c_V} \nabla \cdot \vec{q},$$

Heat

Temperature

Heat conductivity

specific heat for constant volume

Combining the two, the resulting heat equation takes parabolic form, making causality problem(?).

In Newtonian theory of thermodynamics, information propagates with infinite speed.

To rectify this deficiency, Cattaneo and others introduced a small positive time parameter:

Cattaneo equation: $$\vec{q} + \tau_R \frac{\partial \vec{q}}{\partial t} = -\kappa \nabla \Theta,$$

relaxation time scale of a medium

This equation restores causality at the cost of introducing a term that does not come from underlying microphysics.

Note that gravity is a local theory.

In general, there is no general relativistic rules on how the local thermodynamic parameters connects with some notion of global parameters.

However, with various assumptions, we can proceed.

Process for relativistic thermodynamics

1) Write in terms of densities (localize)

2) Rewrite the densities into vector fluxes

3) Construct action and find variational relation describing thermodynamics

4) Apply the 2nd law of thermodynamics

Going to relativistic thermodynamics: 1) writing in terms of densities

1) Write in terms of densities (localize)

eg, ideal gas law,

$$\Psi V = N k_B \Theta$$

Set k_B =1.

$$\Rightarrow \quad \Psi = n\Theta$$

$$dE = \Theta dS - \Psi dV + \chi dN$$

Here, S, V, N are extensive quantities.

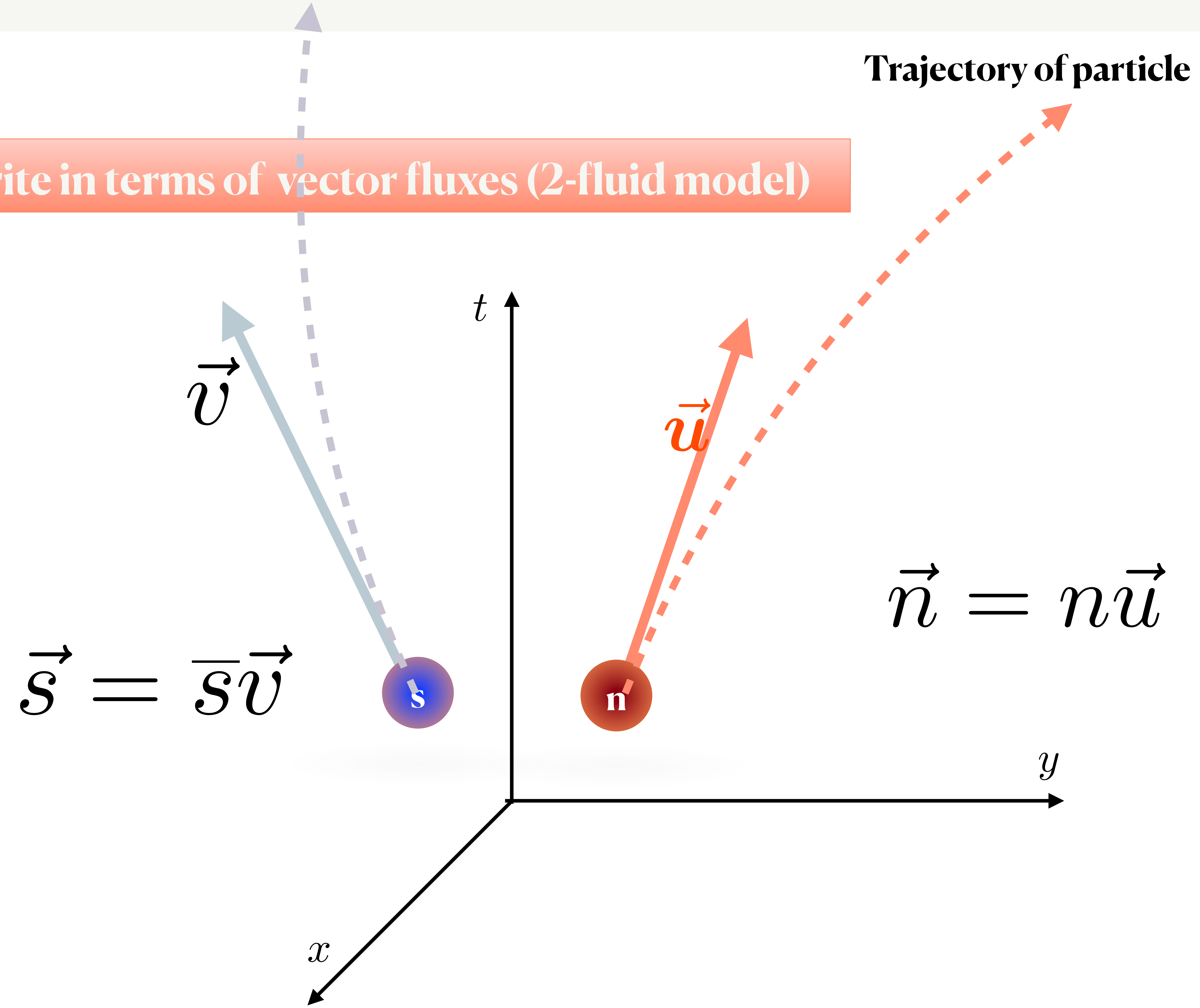
We can use scaling symmetry $Q \to \lambda Q$ for extensive quantity to get

$$\lambda(dE - \Theta dS + \Psi dV - \chi dN) + d\lambda(E + \Psi V - \chi N - \Theta S) = 0$$

$$\Rightarrow \quad \boxed{\rho + \Psi = \Theta s + \chi n} \quad \Rightarrow \quad d\rho = \Theta ds + \chi dn$$

Then, the densities must be a scalar density measured by a comoving observer with the fluid element.

2) Write in terms of vector fluxes (2-fluid model)



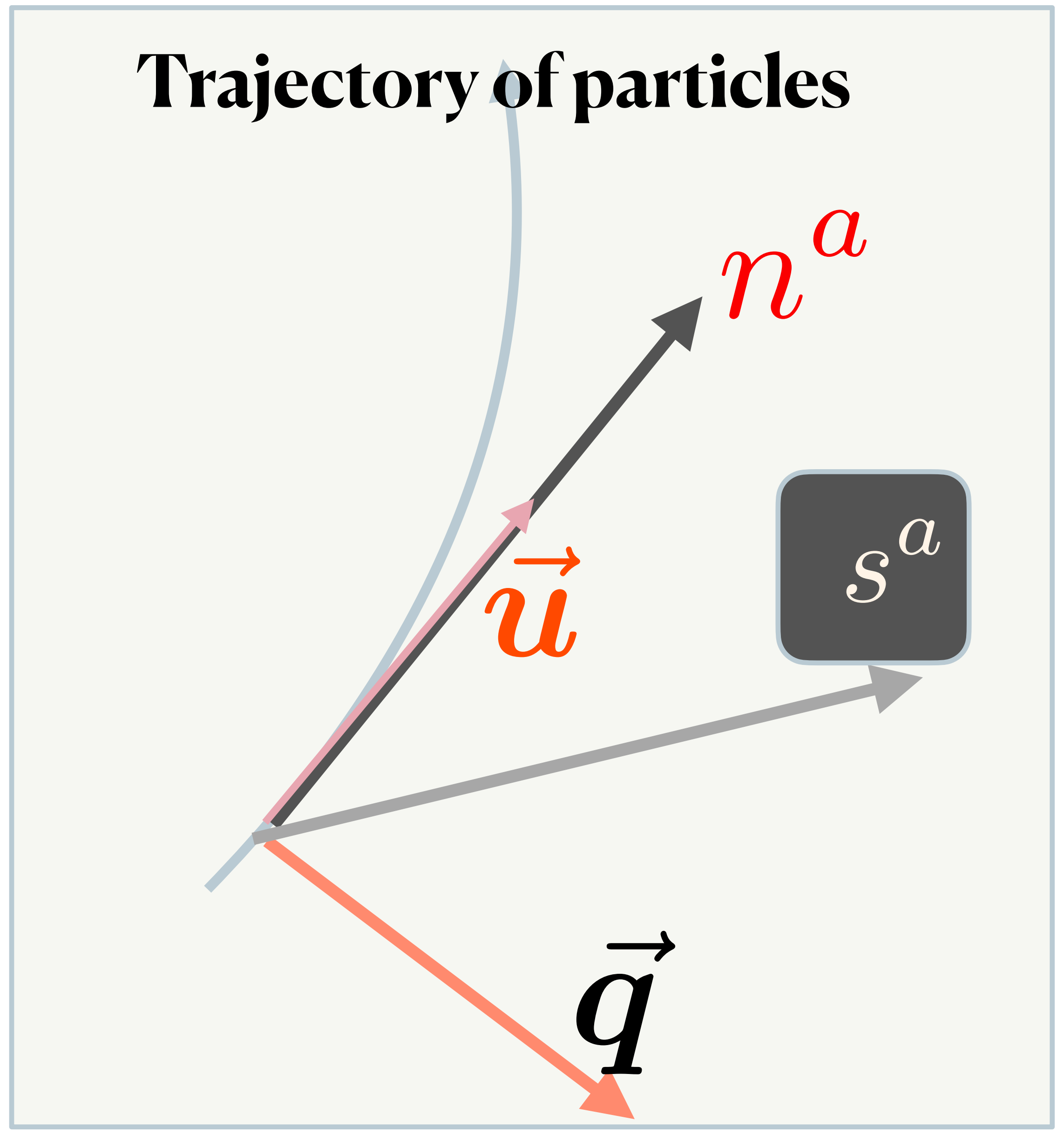
• N. Andersson and G. L. Comer, Living Rev Relativ 24, 3 (2021)

Eckart frame: Usually, one choose the number flux direction to be parallel to the time direction of comoving observer.

Heat flux

$$u^a = \frac{n^a}{n} \qquad s^a = s u^a + \frac{q^a}{\Theta}, \qquad q^a u_a = 0.$$

Entropy wrt a comoving observer



Why this 'q' is heat flux?

Later when we write the stress tensor, we find that the T_{0i} component is expressed by the `q' part.

- HCK, Y. Lee (2022), HCK (2023) etc.
- N. Andersson and G. L. Comer, Living Rev Relativ 24, 3 (2021)

Process for relativistic thermodynamics

3) Construct action and find variational relation

(later pages)

One of the main results is

First law of thermodynamics:

$$\boxed{d\rho(n, s, \vartheta) = \chi dn + \Theta ds + \varsigma d\vartheta,}$$

$$\vartheta \equiv \beta q, \qquad \varsigma = \frac{q}{\Theta}$$

Heat

(conjecture) Extended irreversible thermodynamics (Jou et.al., 1993)

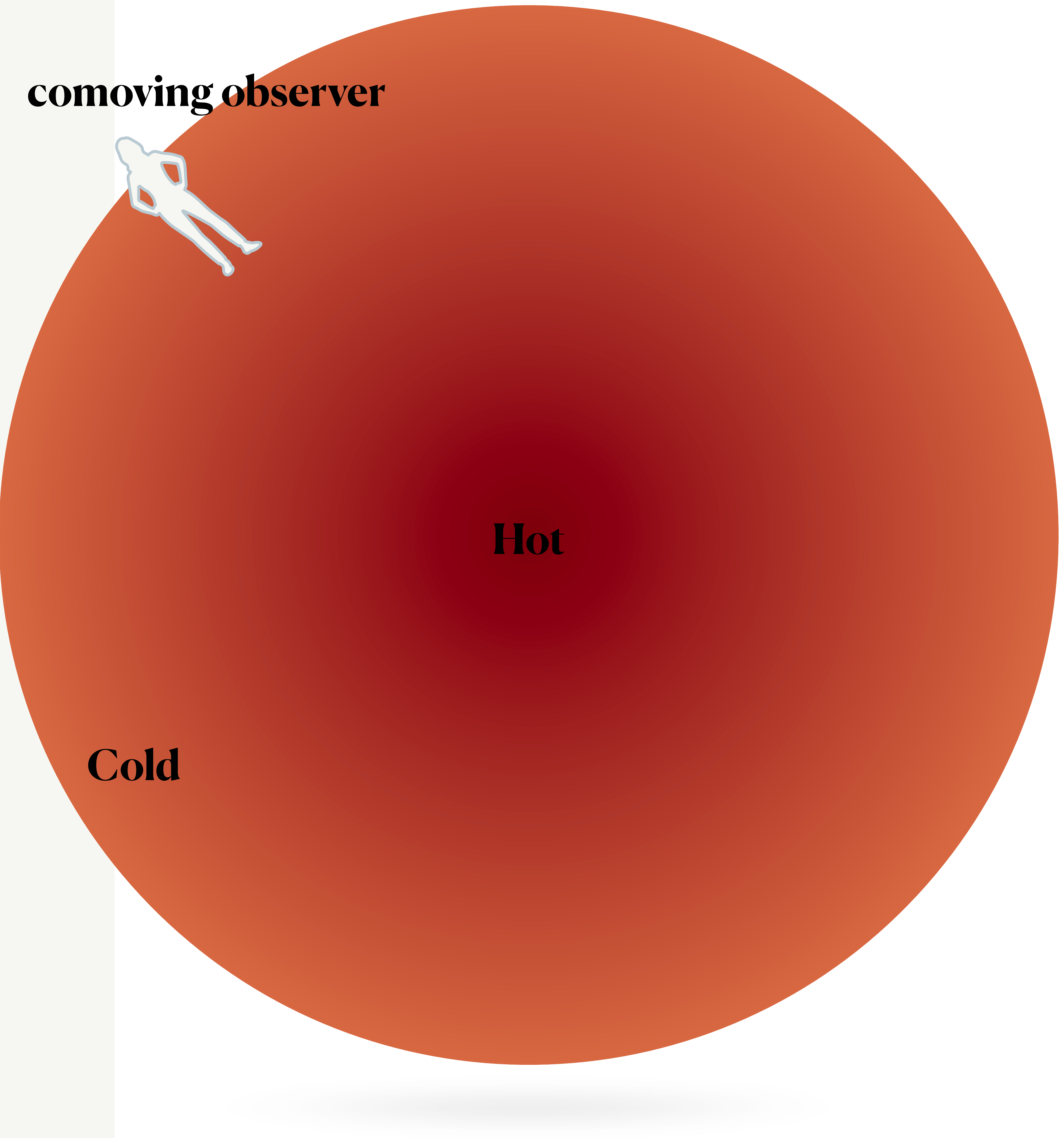
Correction from thermal equilibrium due to heat

This result naturally signifies the heat dependent energy density!

$\rho(n, s, \vartheta)$

- arXiv:2311.06994 (HCK)

An interesting result of relativistic thermodynamics:



Thermal equilibrium

No heat etc

Static system

Geometry is static.

Temperature, energy density

etc are time-independent.

Tolman temperature

In the presence of gravity, the zeroth law of thermodynamics will be violated!

DECEMBER 15, 1930 *PHYSICAL REVIEW* *VOLUM*

Tolman TEMPERATURE EQUILIBRIUM IN A STATIC GRAVITATIONAL FIELD

BY RICHARD C. TOLMAN AND PAUL EHRENFEST
NORMAN BRIDGE LABORATORY OF PHYSICS, PASADENA, CALIFORNIA

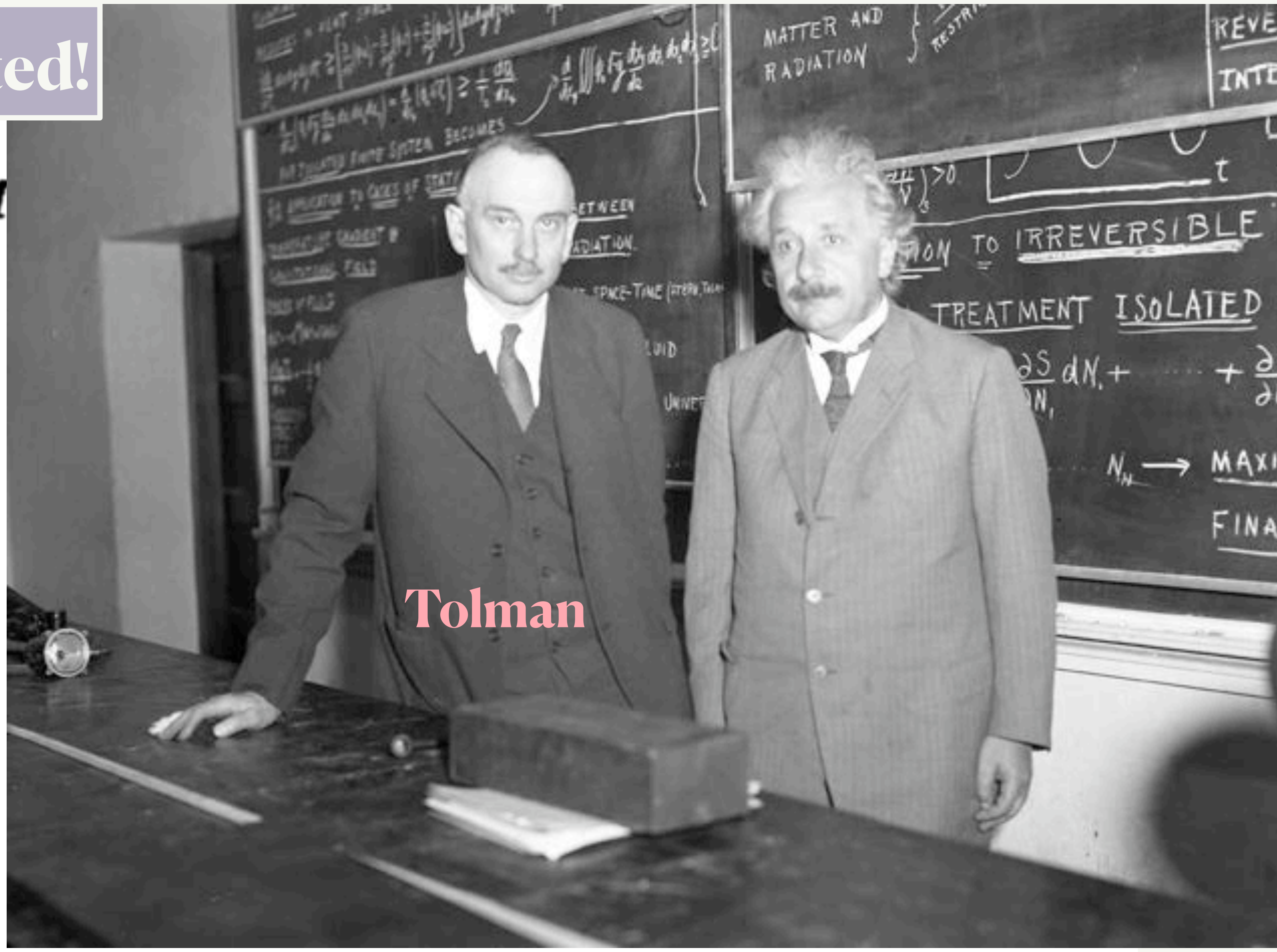
(Received October 27, 1930)

In the case of a gravita o- dynamic equilibrium, it has T_0 as measured by a local obser a- tional potential at the point le the conditions of thermal equ ic gravitational field which coul id parts. Writing the line element for the general static field in the form

$$\Theta(x^i) = \frac{T_\infty}{\sqrt{-g_{00}(x^i)}},$$

$$ds^2 = g_{ij}dx_idx_j + g_{44}dt^2 \qquad i, j = 1, 2, 3,$$

where the g_{ij} and g_{44} are independent of the time t, it is shown that the dependence of proper temperature on position at thermal equilibrium is such as to make the quantity $T_0\sqrt{g_{44}}$ a constant throughout the system.



Thermal equilibrium, Static geometry.

A main supporting argument is

European Journal of Physics

PAPER

Tolman temperature gradients in a g

Jessica Santiago[2,1] and Matt Visser[1]

Published 18 February 2019 • © 2019 European Physical Soc
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Citation Jessica Santiago and Matt Visser 2019 Eur. J. Phys.

+ Article information

Abstract

Tolman's relation for the temperature gradient in an ec
is broadly accepted within the general relativity comm
gradients in thermal equilibrium continues to cause cc
contradicts *naive* versions of the laws of classical ther

Gravity's universality: The physics underlying Tolman temperature gradients

May, 2018, Int. J. Mod. Phys. D

Jessica Santiago* and **Matt Visser**

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ABSTRACT:

We provide a simple and clear verification of the physical need for temperature gradients in equilibrium states when gravitational fields are present. Our argument will be built in a completely *kinematic* manner, in terms of the *gravitational red-shift/blue-shift* of light, together with a relativistic extension of Maxwell's two column argument. We conclude by showing that it is the *universality* of the gravitational interaction (the uniqueness of free-fall) that ultimately permits Tolman's equilibrium temperature gradients without any violation of the laws of thermodynamics.

If **the temperature gradient is dependent on the matter kinds**, the radiated photons from the above and from the bottom may not have the same temperature.

Then, By putting some photo-tube which connect the top and the bottom one can construct a permanent engine.

To avoid a permanent engine, the Tolman temperature gradient must be independent of the matter contents.

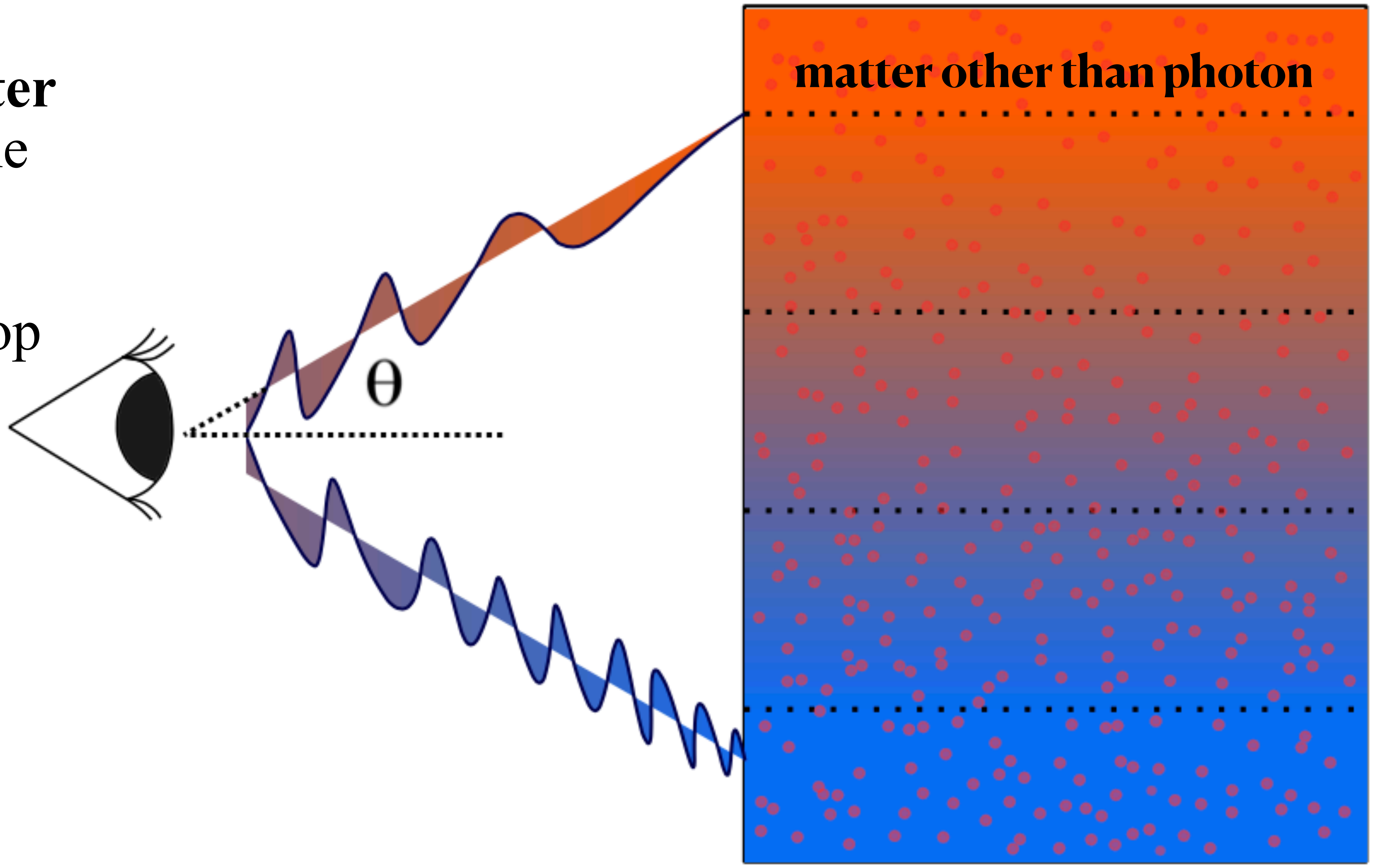


Figure 2. External observer looking at photons leaking from the box containing the photon gas, with the photons arriving at some angle to the vertical.

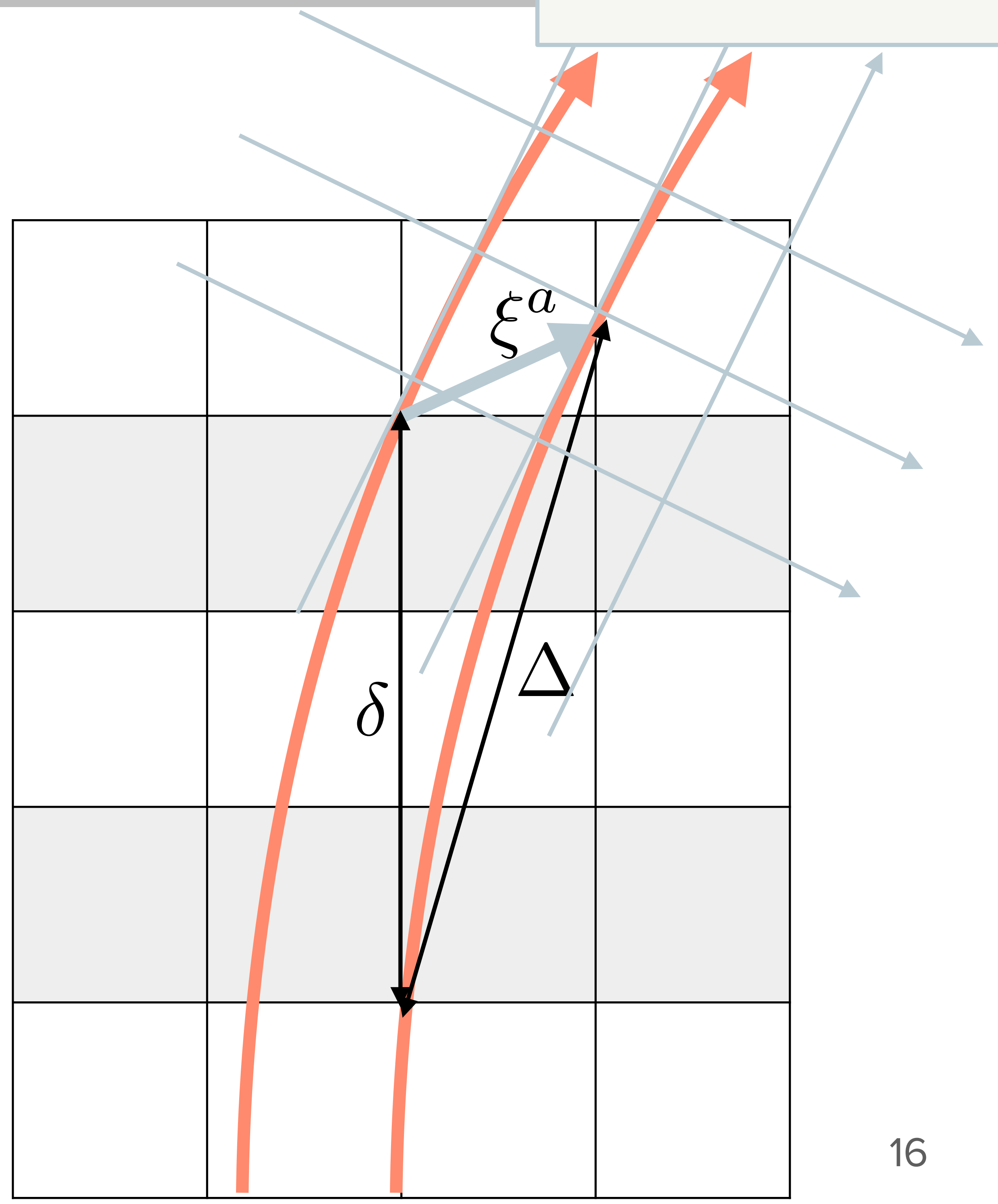
Variational formulation for Fluid dynamics: One fluid case:

Picture from pixabay

Eulerian variation and Lagrangian variation

- **Eulerian** (δ): An army of observers at rest with respect to a generic frame of reference make notes of the evolution as the various fluid elements intersect their worldlines. Therefore, δQ is a change in Q at fixed spacetime point.
- **Lagrangian** (Δ): Each observer attaches him to a particular fluid element and monitors how that element changes. Therefore, ΔQ is a variation of the field wrt to a frame dragged along by ζ^a ($x^a \to x^a + \zeta^a$).

$$\Delta = \delta + \pounds_\xi$$



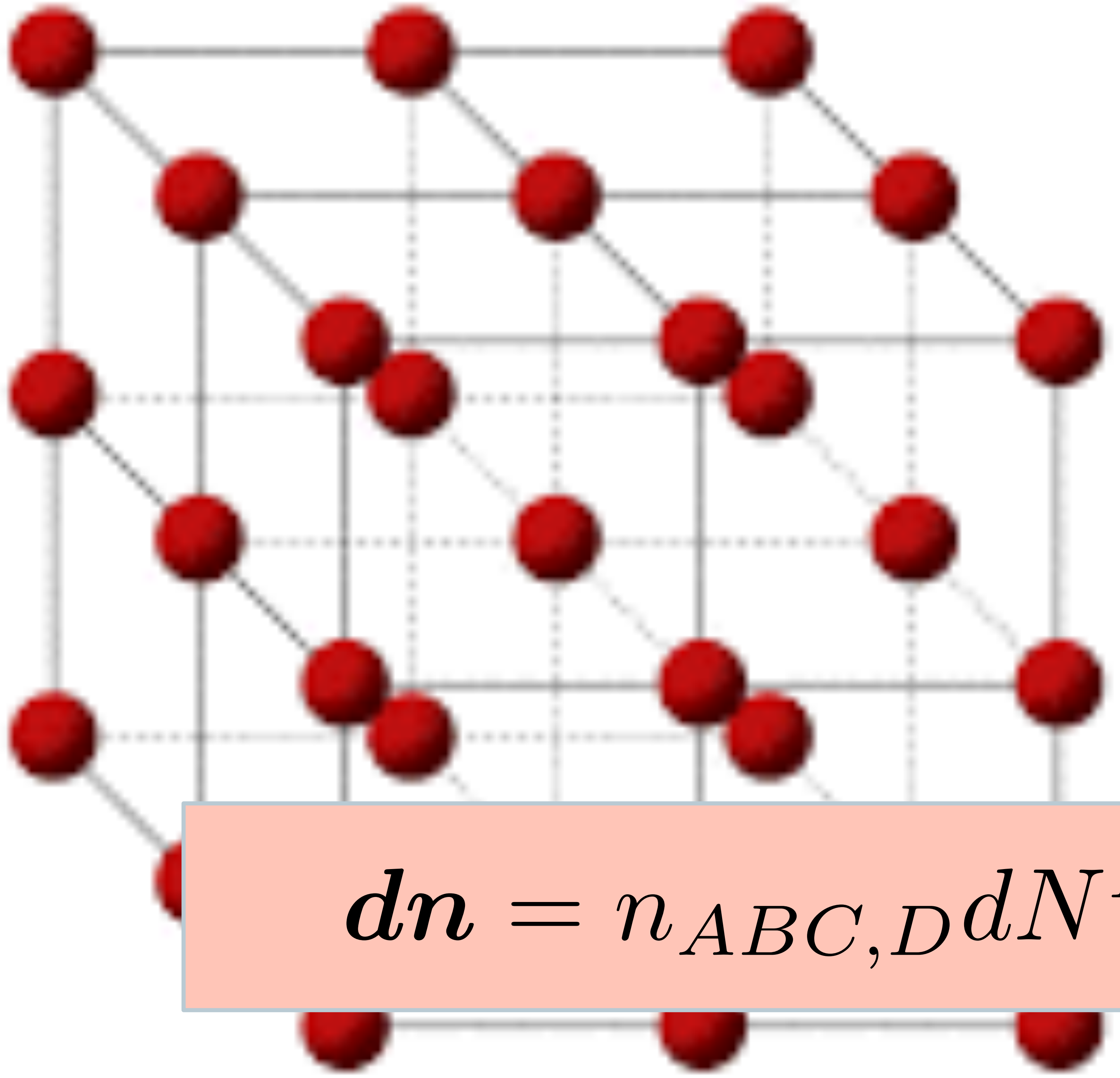
See Ref. Covariant thermodynamics & Relativity

by Cesar Simon Lopez-Monsalvo (2011)

Construction of matter space

Carter(1989)

The matter space (3D)



Particles

$$N^A,$$

$$A \wedge B \equiv \frac{1}{2!}(A \otimes B - B \otimes A)$$

$$n_{[ABC]} \equiv \frac{1}{3!}(n_{ABC} - n_{ACB} + n_{CAB} - n_{CBA} + n_{BCA} - n_{BAC})$$

define a 3-form field, which depends only on the matter space coordinates.

$$\boldsymbol{n} \equiv n_{ABC} dN^A \wedge dN^B \wedge dN^C$$

$$\boldsymbol{dn} = n_{ABC,D} dN^D \wedge dN^A \wedge dN^B \wedge dN^C = n_{[ABC,D]} dN^D \wedge dN^A \wedge dN^B \wedge dN^C$$

Because, n_ABC is a completely antisymmetric on ABC,

$$\boldsymbol{dn} = 0$$

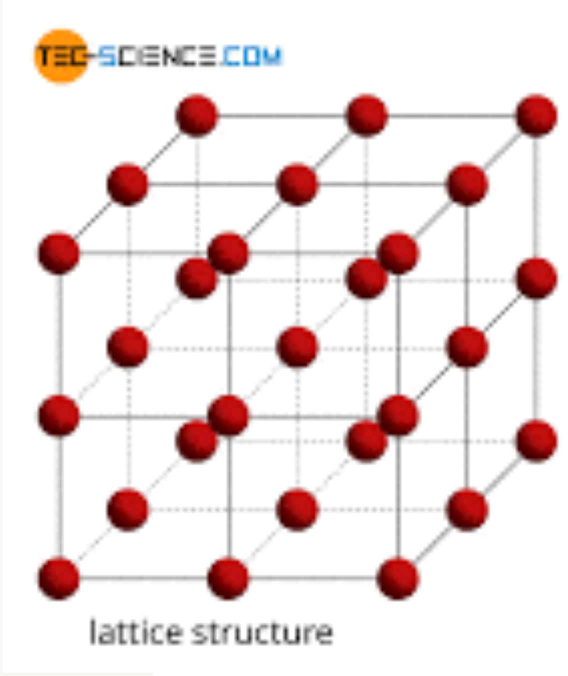
The variational formulation: One fluid

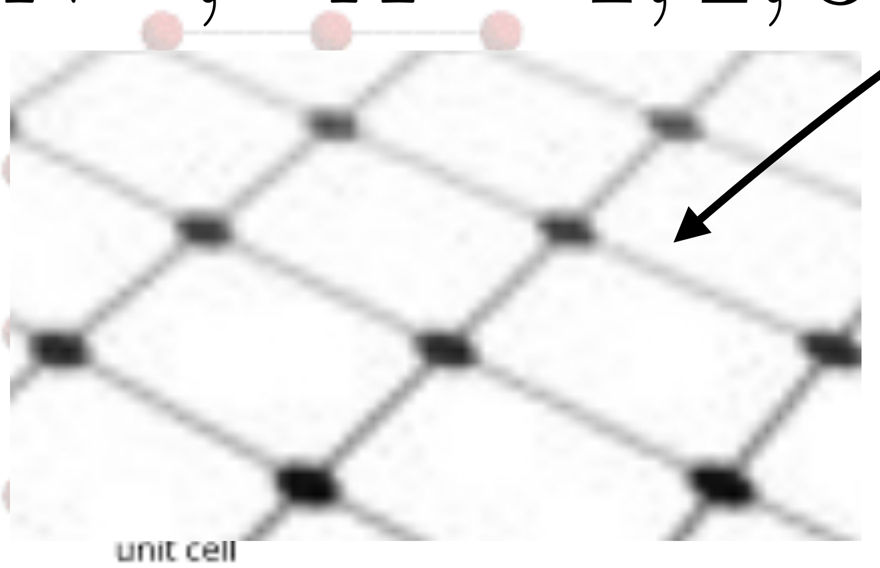
Carter(1989)

The particle number conservation:

The matter space (3D)

$N^A, \quad A = 1, 2, 3$



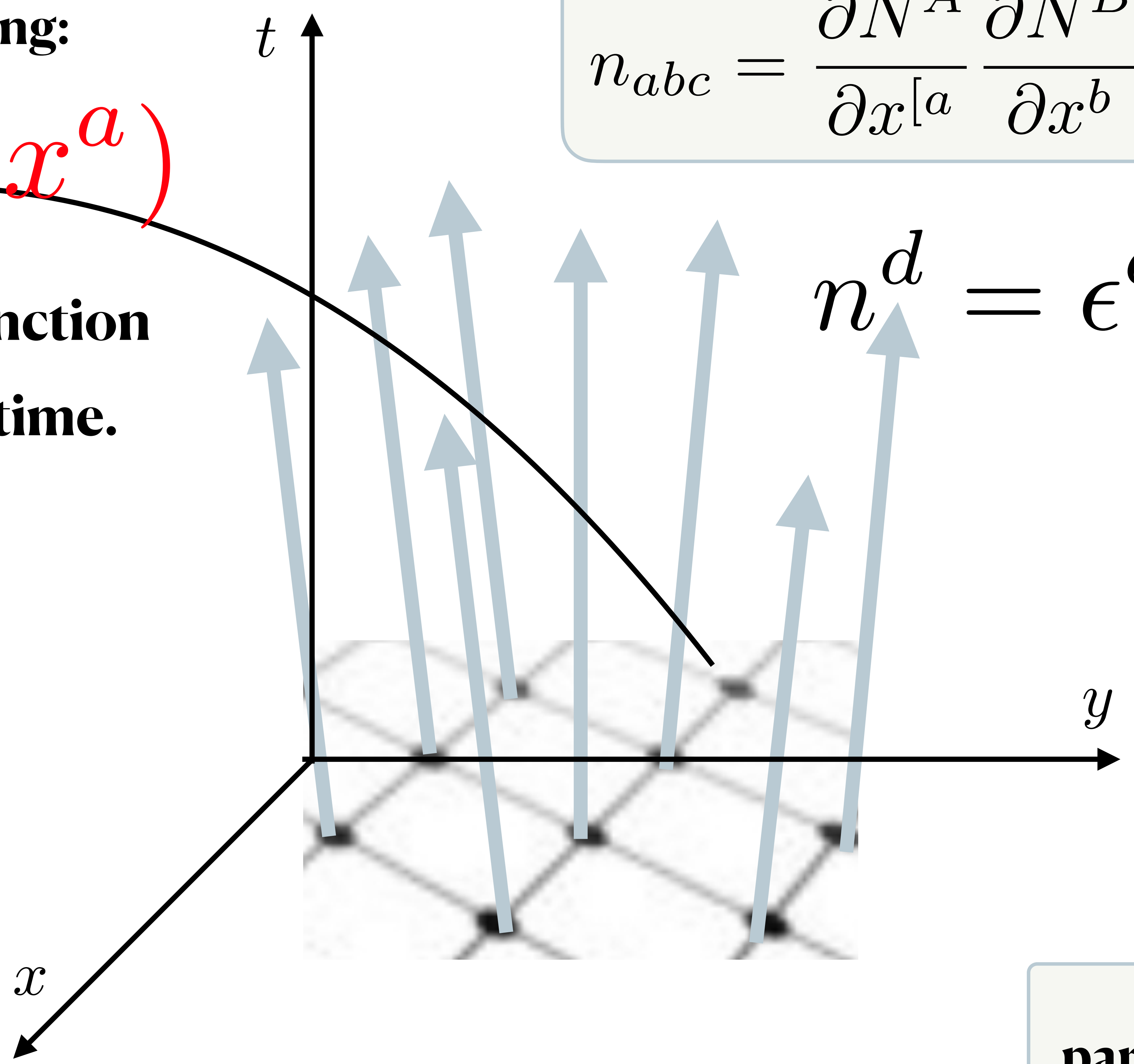


$$n_{ABC}(N^D)$$

Mapping:

$N^A(x^a)$

a scalar function on spacetime.



$$n_{abc} = \frac{\partial N^A}{\partial x^{[a}} \frac{\partial N^B}{\partial x^b} \frac{\partial N^C}{\partial x^{c]}} n_{ABC}$$

$$n^d = \epsilon^{dabc} n_{abc}$$

particle number!

$$\boldsymbol{dn} = 0 \longrightarrow \nabla_{[a} n_{bcd]} = 0 \quad \rightarrow \quad \nabla_a n^a = 0.$$

By construction, the particle number is conserved.

4. Number density 3-form from matter space to spacetime 3-form:

$$n_{abc} = \frac{\partial N^A}{\partial x^{[a}} \frac{\partial N^B}{\partial x^b} \frac{\partial N^C}{\partial x^{c]}} n_{ABC}$$

Here n_{ABC} is antisymmetric and provides matter space with a geometric structure.

$$\blacktriangleright \quad \boldsymbol{n} \equiv n_{ABC} dN^A \wedge dN^B \wedge dN^C = n_{ABC} \left(\frac{\partial N^A}{\partial x^a} dx^a \right) \wedge \left(\frac{\partial N^B}{\partial x^b} dx^b \right) \wedge \left(\frac{\partial N^C}{\partial x^c} dx^c \right) \tag{5}$$

This gives the above result. If integrated over a volume in matter space, it gives a measure of the number of particles in that volume. The matter space to construct the three-forms are automatically closed on spacetime:

$$\begin{aligned}
\blacktriangleright \quad \nabla_{[a} n_{bcd]} &= \nabla_{[a} \frac{\partial N^B}{\partial x^b} \frac{\partial N^C}{\partial x^c} \frac{\partial N^D}{\partial x^{d]}} n_{BCD} \\
&= \cancelto{0 \text{ by symmetry}}{\frac{\partial^2 N^B}{\partial x^{[a} \partial x^b}} \frac{\partial N^C}{\partial x^c} \frac{\partial N^D}{\partial x^{d]}} n_{BCD} + \cancelto{0 \quad N \text{ space is 3 dim}}{\frac{\partial N^A}{\partial x^{[a}} \frac{\partial N^B}{\partial x^b} \frac{\partial N^C}{\partial x^c} \frac{\partial N^D}{\partial x^{d]}}} \frac{\partial n_{BCD}}{\partial N^A} \\
&= 0 \ \blacksquare
\end{aligned} \tag{6}$$

Here, n_{ABC} is a function of N^A only. Note that

$$\frac{dN^A}{d\tau} = u^a \nabla_a N^a = n^{-1} \frac{1}{3!} \epsilon^{abcd} n_{bcd} \nabla_a N^A = \frac{n^{-1}}{3!} \epsilon^{abcd} \nabla_a N^A \nabla_b N^B \nabla_c N^C \nabla_d N^D n_{BCD} = 0. \tag{7}$$

5. Introduce Lagrangian displacements ξ^a: Then, we have

$$0 = \Delta N^A = \delta N^A + \pounds_{\boldsymbol{\xi}} N^A \quad \rightarrow \quad \delta N^A = -\pounds_{\boldsymbol{\xi}} N^A = -\xi^a \frac{\partial N^A}{\partial x^a}. \tag{8}$$

Here, we use the fact that N^A is a scalar function on spacetime, x.

$$\boxed{\Delta = \delta + \pounds_\xi}$$

6. After some algebra, one find

$$\Delta n_{abc} = 0.$$

▶
$$\begin{aligned}
\delta n_{abc} &= \left(\frac{\partial n_{ABC}}{\partial N^D}\nabla_a N^A \nabla_b N^B \nabla_c N^C\right)\delta N^D \\
&\quad + n_{ABC}\left[\nabla_a \delta N^A \nabla_b N^B \nabla_c N^C + \nabla_a N^A \nabla_b \delta N^B \nabla_c N^C + \nabla_a \delta N^A \nabla_b N^B \nabla_c \delta N^C\right] \\
&= -\xi^d \left(\frac{\partial n_{ABC}}{\partial N^D}\nabla_a N^A \nabla_b N^B \nabla_c N^C \nabla_d N^D\right) \\
&\quad - n_{ABC}\left[\left(\xi^d \nabla_a \nabla_d N^A + (\nabla_d N^A)(\nabla_a \xi^d)\right)\nabla_b N^B \nabla_c N^C\right. \\
&\qquad + \nabla_a N^A \left(\xi^d \nabla_b \nabla_d N^B + (\nabla_d N^B)(\nabla_b \xi^d)\right)\nabla_c N^C \\
&\qquad \left. + \nabla_a N^A \nabla_b N^B \left(\xi^d \nabla_c \nabla_d N^C + (\nabla_d N^C)(\nabla_c \xi^d)\right)\right] \\
&= -\xi^d \left\{(\nabla_d n_{ABC})[\nabla_a N^A \nabla_b N^B \nabla_c N^C] + n_{ABC}\nabla_d\left[(\nabla_a N^A)(\nabla_b N^B)(\nabla_c N^C)\right]\right\} \\
&\quad - n_{ABC}\left[(\nabla_d N^A)\nabla_b N^B \nabla_c N^C(\nabla_a \xi^d) + \nabla_a N^A \nabla_d N^B \nabla_c N^C(\nabla_b \xi^d) + \nabla_a \delta N^A \nabla_b N^B \nabla_d N^C(\nabla_c \xi^d)\right] \\
&= -\xi^d \nabla_d n_{abc} - \left[n_{dbc}(\nabla_a \xi^d) + n_{adc}(\nabla_b \xi^d) + n_{abd}(\nabla_c \xi^d)\right] \\
&= -\pounds_\xi n_{abc}. \blacksquare
\end{aligned} \tag{9}$$

which in turn implies

$$\boxed{\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a\left(\nabla_b \xi^b + \frac{1}{2} g^{bc}\delta g_{bc}\right).}$$

Proof: next page

where in the second equality, we use, using Eq. (8),

$$\nabla_a \delta N^A = -\nabla_a[(\nabla_d N^A)\xi^d] = -(\nabla_a \nabla_d N^A)\xi^d - (\nabla_d N^A)(\nabla_a \xi^d),$$

Now,

$$\begin{aligned}
\delta n^a &= \frac{1}{3!}\delta[\epsilon^{abcd} n_{bcd}] = \frac{1}{3!}(\delta\epsilon^{abcd})n_{bcd} + \frac{1}{3!}\epsilon^{abcd}\delta n_{bcd} \\
&= -\frac{1}{3!}\epsilon^{abcd} n_{bcd}\frac{1}{2}g^{ef}\delta g_{ef} + \frac{1}{3!}\epsilon^{abcd}\{-\xi^e\nabla_e n_{bcd} - [n_{ecd}(\nabla_b\xi^e) + n_{bed}(\nabla_c\xi^e) + n_{bce}(\nabla_d\xi^e)]\} \\
&= -n^a\frac{1}{2}g^{bc}\delta g_{bc} - \xi^e\nabla_e n^a + \left(\delta^a_e n^b - n^a\delta^b_e\right)(\nabla_b\xi^e) \\
&= n^b\nabla_b\xi^a - \xi^b\nabla_b n^a - n^a(\nabla_b\xi^b + \frac{1}{2}g^{bc}\delta g_{bc}) \\
&= -\pounds_\xi n^a - n^a(\nabla_b\xi^b + \frac{1}{2}g^{bc}\delta g_{bc}).
\end{aligned} \tag{10}$$

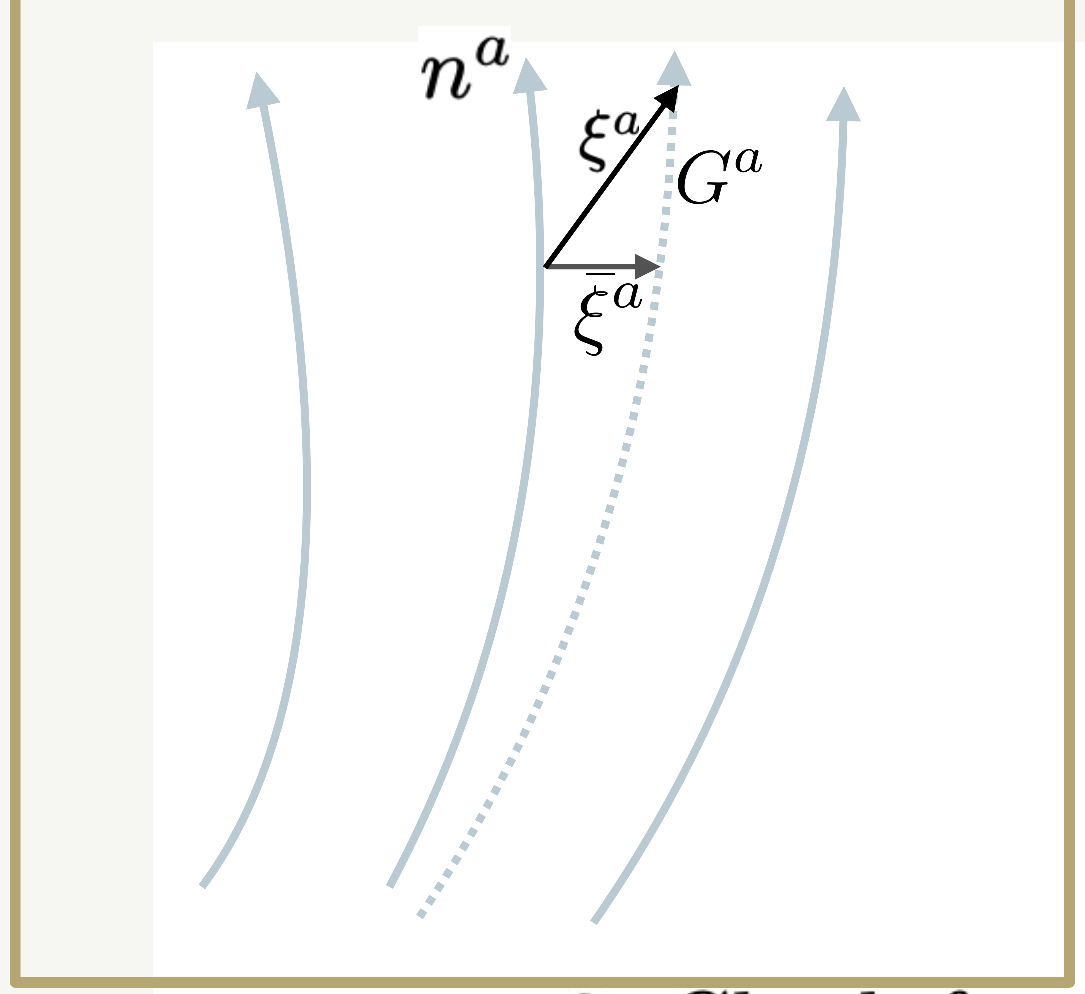
Here we use Eqs. (4), (75), and Eq. (3).

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}, \qquad \epsilon_{0123} = \sqrt{-g}, \qquad \epsilon^{0123} = -\frac{1}{\sqrt{-g}}$$

$$\epsilon_{abcd} = \sqrt{-g}[abcd] \quad \rightarrow \quad \delta\epsilon^{abcd} = -\frac{1}{2}\epsilon^{abcd}g^{ef}\delta g_{ef},$$

$$\epsilon^{abcd}\epsilon_{aefg} = -\delta^{bcd}_{efg}, \qquad \epsilon^{abcd}\epsilon_{abef} = -\delta^{bcd}_{bef} = -2\delta^{cd}_{ef}, \qquad \epsilon^{abcd}\epsilon_{abce} = -3!\delta^d_e,$$

Check for the DOF:



8. Check for the degrees of freedom: Pull-back construction has 3 dof. Lagrangian variation has 4-dof. Gauge freedom in the Lagrangian variation that can be used to reduce the # of indep comps:

$$\xi^a = \bar{\xi}^a + G^a. \tag{15}$$

From Eq. (**10**), and $\nabla_a n^a = 0$, we get

$$\delta n^a = \bar{\delta} n^a - \frac{1}{2}\epsilon^{abcd}\nabla_b(\epsilon_{cdef} n^e G^f) = \bar{\delta} n^a - \delta^{ab}_{ef}\nabla_b(n^e G^f), \tag{16}$$

where $\bar{\delta}$ denotes the δn using $\bar{\xi}^a$. If we set $G^f = u^f G$, the last term vanishes.

▶
$$\begin{aligned}(\delta - \bar{\delta})n^a &= \pounds_{\bar{\xi}-\xi} n^a + n^a \nabla_b(\bar{\xi}^a - \xi^a) = -\pounds_G n^a - n^a \nabla_b G^b \\ &= -G^b \nabla_b n^a + n^b \nabla_b G^a - n^a \nabla_b G^b = n^b \nabla_b G^a - \nabla_b(n^a G^b) = 2\nabla_b[n^{[b} G^{a]}] - G^a(\nabla_b n^b)\end{aligned}$$

Now, if we use $\nabla_a n^a = 0$, we get Eq. (16). ■

Variational formulation for Fluid mechanics (one fluid case)

The action: $$I = I_{EH} + I_M = \int_R \left(\frac{1}{2\kappa} R + L \right) \sqrt{-g} d^4 x, \qquad \kappa = \frac{8\pi G}{c^4},$$

The Geometry: $$G_{ab} = \kappa T_{ab}$$

$$L \quad \rightarrow \quad \Lambda \qquad\qquad \Lambda(n)$$

$$\frac{\partial n^2}{\partial g_{ab}} = n^a n^b, \qquad \frac{\partial n^2}{\partial n^a} = g_{ab} n^b$$

Eg, $\frac{\partial \Lambda}{\partial n^a} = \frac{1}{2n} \left(\frac{\partial \Lambda}{\partial n} \right) g_{ab} n^b$

ropic system of the fluid, the matter Lagrangian, Λ, should be a scalar.
have a single matter, n^a, it must be a function of $n^2 = -g_{ab} n^a n^b$.

Lagrangian density becom

Contribute to Stress tensor

$$\delta(\sqrt{-g}\Lambda) = \sqrt{-g} \left[\chi_a \delta n^a + \frac{1}{2} (\Lambda g^{ab} + n^a \chi^b) \delta g_{ab} \right]$$ **+ total derivatives**

$$\chi_a \equiv \frac{\partial \Lambda}{\partial n^a}$$ **: The canonical momentum.**

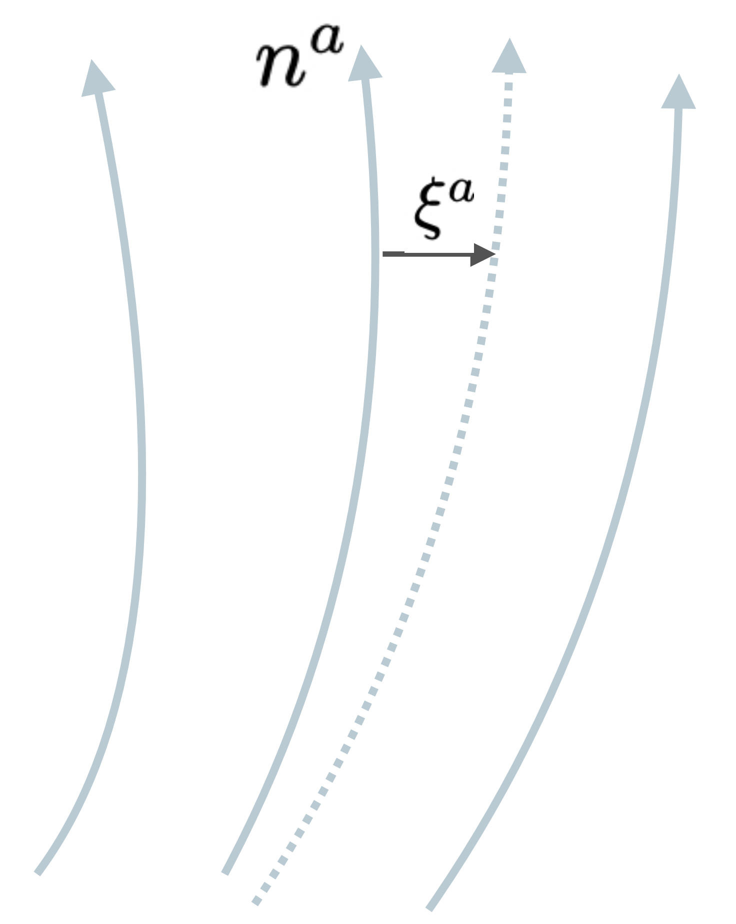
Then, the equation of motion becomes, $\chi_a = 0$. **This result is not so interesting.**

The reason is that all the variations are not free because of the particle number conservation law.

The (constrained) variational formulation: One fluid 3 Carter(1989)

Back to the variational principle, introduce the Lagrangian displacement, ξ^a
tracking the motion of the fluid element.

Congruence of world-lines



By definition, $\Delta N^A = \delta N^A + \pounds_\xi N^A = 0$ **Carter(1973)**

This naturally determines the variation of the number flux: $\delta n^a = \frac{1}{3!}\delta(\epsilon^{abcd} n_{bcd})$

$$\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bc} \delta g_{bc} \right) .$$

Then, the (constrained) variation of the Lagrangian density becomes

$$\delta(\sqrt{-g}\Lambda) = \sqrt{-g}\left[f_a \xi^a + \frac{1}{2}[(\Lambda - n^c\chi_c) g_{ab} + n_a \chi_b]\delta g^{ab} \right] + \text{total derivatives}$$

$$f_b \equiv 2 n^a \nabla_{[a}\chi_{b]} = 0$$

Stress tensor

$$\nabla_a T^{ab} = -f^b$$

Interpretation of the equation of motion

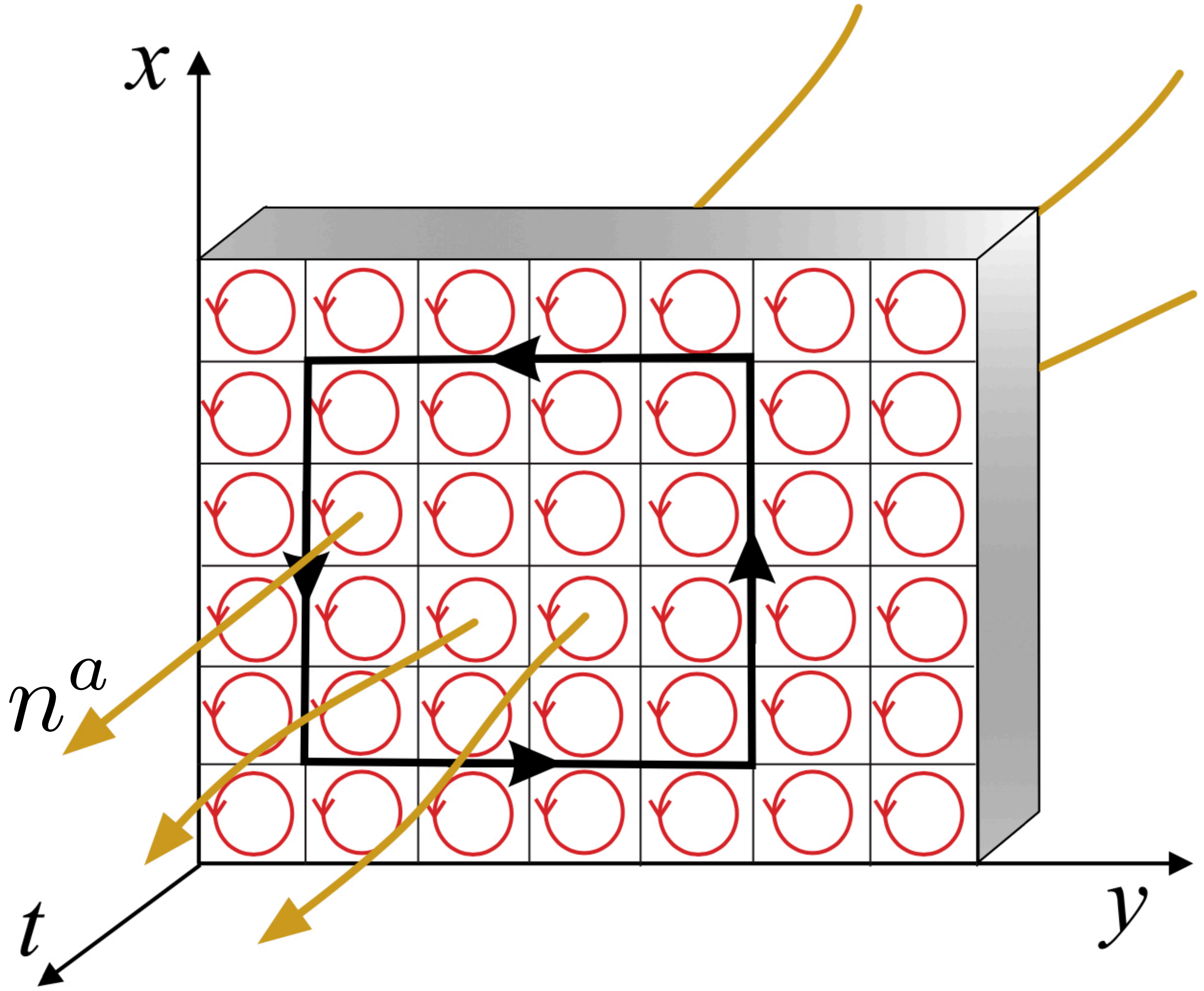
The vorticity two-form: $\omega_{ab} = \nabla_{[a}\chi_{b]}$

$$\boxed{f_b = n^a \omega_{ab} = 0}$$

This equation plays a crucial role in fluid dynamics (Carter 1989, Bekenstein 1987) in explaining turbulence (Pulling and Saffman 1998) and Kelvin-Helemholtz theorem (Landau-Lifschitz 1959).

Geometrical interpretation of the EOM

Andersson &Comer(2021)



Geometrically, the two-form is a collection of oriented world-tubes. The four-velocity of the individual fluid element lies inside the world-tube.

Consider the closed black contour.
If that contour is attached to fluid-element worldlines, then the number of world-tubes contained within the contour will not change (Kelvin-Helemholtz theorem).

Helmholtz’s theorem:

Helmholtz’s theorem (wiki):

In fluid mechanics, **Helmholtz's theorems**, named after Hermann von Helmholtz, describe the three-dimensional motion of fluid in the vicinity of vortex lines. These theorems apply to inviscid flows and flows where the influence of viscous forces are small and can be ignored.

Helmholtz's three theorems are as follows:[1]

Helmholtz's first theorem

The strength of a vortex line is constant along its length.

time independent.

Helmholtz's second theorem

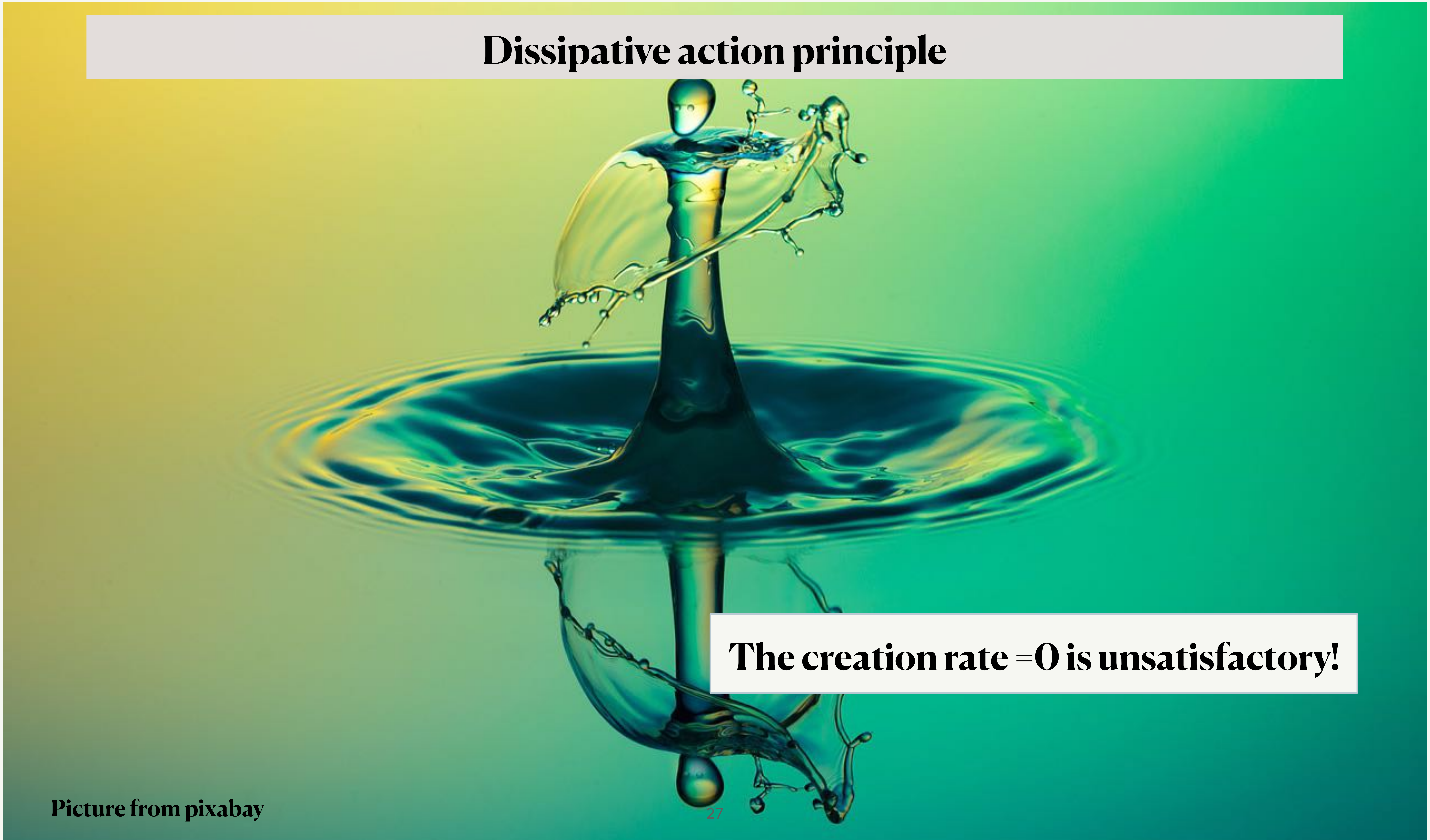
A vortex line cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.

vortex line moves with the fluid.

Helmholtz's third theorem

A fluid element that is initially irrotational remains irrotational.

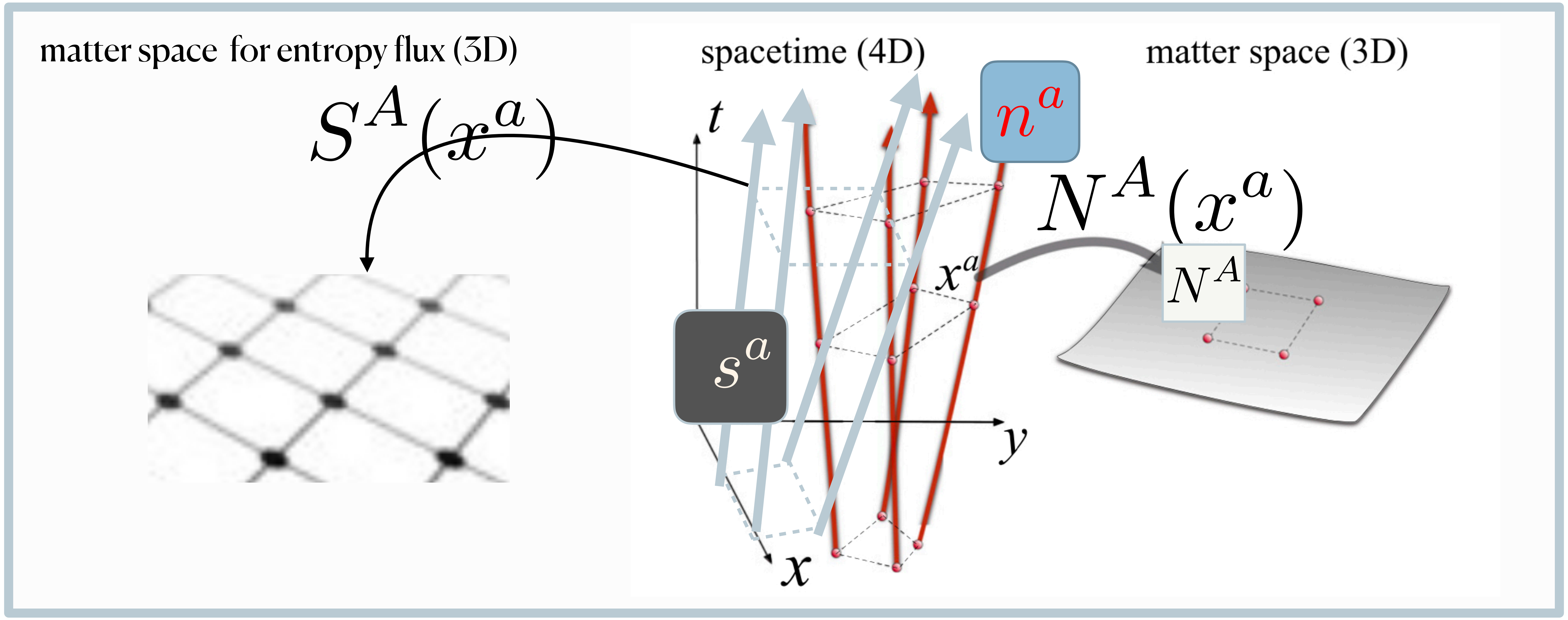
Dissipative action principle



The creation rate =0 is unsatisfactory!

Picture from pixabay

Minimal model for heat conduction: Two-fluid model



Let the three form field n residing on the matter space {N^A} depend on other matter space {S^E} too:

$$\boldsymbol{n} \equiv n_{ABC}(N^D, S^E) dN^A \wedge dN^B \wedge dN^C.$$

$$\boldsymbol{s} \equiv s_{ABC}(S^D, N^E) dS^A \wedge dS^B \wedge dS^C$$

Dissipative action principle

Let the three form field n residing on the matter space $\{N^A\}$ depend on other matter space $\{S^E\}$ too:

$$\boldsymbol{n} \equiv n_{ABC}(N^D, S^E) dN^A \wedge dN^B \wedge dN^C.$$

Then, the creation rate of n^a does not vanish because of the $\{S^E\}$ dependence:

$$\Gamma_N \equiv \nabla_a n^a = \sum_{S \neq N} \frac{1}{3!} \epsilon^{abcd} \frac{\partial S^A}{\partial x^a} \frac{\partial N^B}{\partial x^b} \frac{\partial N^C}{\partial x^c} \frac{\partial N^D}{\partial x^d} \left(\frac{\partial n_{BCD}}{\partial S^A} \right) \neq 0.$$

The creation rate is determined by how much the three form field depends on the other matter space coordinates.

Dissipative action principle

Now, we may introduce a Lagrangian displacement ξ^a_N, tracking the motion of the fluid element.
From the standard definition of Lagrangian variations, $\Delta_N \equiv \delta + \pounds_{\xi_N}$ we have

$$\Delta_N N^A = \delta N^A + \pounds_{\xi_N} N^A = 0,$$

Then, the matter space variation becomes,

$$\delta n_{bcd} = -\pounds_{\xi_N} n_{bcd} + \frac{\partial N^B}{\partial x^{[b}} \frac{\partial N^C}{\partial x^c} \frac{\partial N^D}{\partial x^{d]}} \Delta_N n_{BCD},$$

$$\Delta_N n_{BCD} = \sum_S \frac{\partial n_{BCD}}{\partial S^E} \left(\xi^a_N - \xi^a_S\right) \frac{\partial S^E}{\partial x^a}.$$

Straightforward calculation gives,

$$\chi_a \delta n^a = \chi_a \left(n^b \nabla_b \xi^a_N - \xi^b_N \nabla_b n^a - n^a \nabla_b \xi^b_N - \frac{1}{2} n^a g^{bc} \delta g_{bc} \right) - \sum_{S \neq N} R^{NS}_a (\xi^a_S - \xi^a_N).$$

$$R^{NS}_a \equiv \frac{1}{3!} \chi_N^{BCD} \frac{\partial n_{BCD}}{\partial S^A} \left(\frac{\partial S^A}{\partial x^a} \right).$$

Dissipative action principle

Based on the result, we get the variation of the Lagrangian density, (up to total derivatives)

$$\delta(\sqrt{-g}\Lambda) = -\sqrt{-g}\left\{\left(f_a^N + \chi_a\Gamma_N - R_a^N\right)\xi_N^a + \left(f_a^S + \Theta_a\Gamma_S - R_a^S\right)\xi_S^a - \frac{1}{2}T^{ab}\delta g_{ab}\right\},$$

where $T^{ab} \equiv \Psi g^{ab} + (n^a\chi^b + s^a\Theta^b)$ and the pressure is $\Psi = \Lambda - \chi_a n^a - s^a\Theta_a$ and

$$f_a^N \equiv 2n^b\nabla_{[b}\chi_{a]}, \qquad f_a^S \equiv 2s^b\nabla_{[b}\Theta_{a]}, \qquad R_a^N \equiv R_a^{SN} - R_a^{NS} = -R_a^S.$$

Equation of motions:

$$f_a^N + \Gamma_N\chi_a = R_a^N, \qquad f_a^S + \Gamma_S\Theta_a = R_a^S.$$

Entropy/particle creation rates: $\Gamma_S = -\dfrac{1}{\Theta}s^aR_a^S \qquad \Gamma_N = -\dfrac{1}{\chi}u^aR_a^N, \qquad \chi \equiv -u^a\chi_a.$

The energy-momentum conservation law is satisfied automatically from the equation of motion

$$\nabla_b T^b{}_a = f_a^N + f_a^S + \chi_a\Gamma_N + \Theta_a\Gamma_S = 0, \qquad \text{because} \qquad R_a^N + R_a^S = 0.$$

Second law of thermodynamics:

$$\Gamma_S = -\frac{1}{\Theta} s^a R_a^S$$

$$s\Theta\Gamma_s = -s^a R_a = s^a R_a^{SN} \geq 0.$$

Introduce privileged observer u^a such that

$$s^a = su^a + \varsigma^a,$$

Beacuse R_a^{NS} are normal to u^a and s^a, we may set

$$R_a^{NS} = \epsilon_{abcd}\phi_n^b u^c \varsigma^d.$$

Beacuse R_a^{SN} is normal to u^a, we may set

$$R_a^{SN} = R_\varsigma \varsigma_a + \epsilon_{abcd}\phi_s^b u^c \varsigma^d.$$

Resistivity

$$s\Theta\Gamma_S = R_\varsigma \varsigma^2$$

The second law of thermodynamics becomes $R_\varsigma \geq 0$

Proof for variational formulation is compatible with non-vanishing creation rate:

Now, we are ready to refute the proof (2). Let us consider two distinct variations generated by ξ_N^a and $\bar{\xi}_N^a = \xi_N^a - G_N^a$. We also introduce two distinct variations for the fluid S generated by ξ_S^a and $\bar{\xi}_S^a = \xi_S^a - G_S^a$. The difference between the two variations of n^a becomes

$$\begin{aligned}(\delta - \bar{\delta})n^a &= \pounds_{\bar{\xi}-\xi} n^a + n^a \nabla_b(\bar{\xi}^b - \xi^b) + \frac{n^a}{n\chi} R_e^{NS}(\bar{\xi}_N^e - \xi_N^e - \xi_S^e + \bar{\xi}_S^e) \\ &= -\delta^{ab}_{[ef]} \nabla_b(n^e G_N^f) - G_N^a \Gamma_N + \frac{n^a}{n\chi} \sum_{S \neq N} R_e^{NS}(-G_N^e + G_S^e). \end{aligned} \tag{21}$$

When the two variations are related by their flow directions so that $G_N^a = G_N n^a$ and $G_S^a = G_S s^a$, we find the difference vanishes automatically without any additional requirement:

$$(\delta - \bar{\delta})n^a = -G_N^a \Gamma_N - \frac{G_N n^a}{n\chi} R_e^{NS} n^e = 0, \tag{22}$$

where we use $s^e R_e^{NS} = 0 = n^e R_e^{SN}$ and Eq. (17). This result allows us to describe systems with $\nabla_a n^a \neq 0$ by means of the action formulation with dissipation.

Adding viscosities, etc

Allow n^x_{ABC} to depend on g_S^{AB}?

Here, $g_N^{AB} = \left(\frac{\partial N^A}{\partial x^a}\right)\left(\frac{\partial N^B}{\partial x^b}\right) g_{ab}$

is the induced metric.

This generalization develops dissipative stresses.

Thanks for listening !