BRST Symmetry and Friends – Consistent quantization of a prototypical first-class dynamical system

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0 - Before we start . . .

In very simple terms, What is BRST symmetry?

Gauge Invariance in QED

$$\mathcal{L}_c = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu\psi \qquad D_\mu = \partial_\mu - ieA_\mu \tag{1}$$

$$\mathcal{L}_c = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\mu A_\mu\psi \tag{2}$$

gauge symmetry: $\delta A_\mu = \partial_\mu\alpha \quad \delta\psi = ie\alpha\psi \quad \delta\bar{\psi} = -ie\alpha\bar{\psi}$

$\alpha = \alpha(x) \longrightarrow$ real infinitesimal gauge parameter

$$\delta F_{\mu\nu} = 0$$

$$\delta\left[i\bar{\psi}\gamma^\mu\partial_\mu\psi\right] = -e\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi$$

$$\delta\left[e\bar{\psi}\gamma^\mu A_\mu\psi\right] = e\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi$$

Due to gauge freedom, the naive expression for the generating functional

$$Z = \int DA_\mu D\psi D\bar{\psi} e^{iS} \tag{3}$$

does not work.
Rather, we must improve it as

$$Z = \int DA_\mu D\psi D\bar{\psi} \left[\Delta_{FP} \int D\alpha \, \delta \left(F(A_\mu^\alpha) \right) \right] e^{iS} \tag{4}$$

By exponentiation techniques in the functional integration, this leads in practice to the substitution

$$\mathcal{L}_c \longrightarrow \mathcal{L}_q = \mathcal{L}_c + \mathcal{L}_{FP} + \mathcal{L}_{gf} \tag{5}$$

with

$$\mathcal{L}_c = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu \psi \tag{6}$$

$$\mathcal{L}_{FP} = \bar{c}\partial_\mu \partial^\mu c \qquad \mathcal{L}_{gf} = -\frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \tag{7}$$

The point is that $\mathcal{L}_q$ is not gauge-invariant anymore (the gauge has been fixed!!)

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad \Longrightarrow \quad \delta\mathcal{L}_{gf} = -\frac{1}{\xi}(\partial_\mu A^\mu)\Box\alpha \tag{8}$$

Actually, gauge symmetry was a very welcome, cherished and practical property (used in perturbation theory calculations, renormalization and unitarity checks, gauge-independence of physical results etc). That property is gone . . .

If we could somehow have a non-null variation for the FP term

$$\delta\mathcal{L}_{FP} = \;??$$

cancelling the variation $\delta\mathcal{L}_{gf}$ above . . .

(recall that c and $\bar{c}$ are Grassmann variables)

That can be achieved by BRST transformations!!

$$\alpha(x) \quad \longrightarrow \quad c(x)$$

$$sA_\mu = \partial_\mu c \quad s\psi = iec\psi \quad s\bar{\psi} = -iec\bar{\psi} \quad \Longrightarrow \quad s\mathcal{L}_c = 0$$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad \Longrightarrow \quad s\mathcal{L}_{gf} = -\frac{1}{\xi}(\partial_\mu A^\mu)\Box c \tag{9}$$

$$s\bar{c} = \frac{1}{\xi}\partial_\mu A^\mu \quad \Longrightarrow \quad s\mathcal{L}_{FP} = \frac{1}{\xi}(\partial_\mu A^\mu)\Box c \tag{10}$$

The BRST variations above lead to a corresponding global symmetry

$$\delta_b \Phi = \lambda\, s\Phi$$

$\lambda \rightarrow$ global anticommuting infinitesimal parameter

which can be interpreted as a quantum reminiscent symmetry remaining AFTER GAUGE FIXING.

The BRST can be significantly improved by introducing an auxiliary field b (Nakanish-Lautrup) and equivalently rewritting

$$\mathcal{L} = \mathcal{L}_c + \frac{\xi}{2} b^2 - b\partial_\mu A^\mu + \mathcal{L}_{FP} \tag{11}$$

$$sA_\mu = \partial_\mu c \quad s\psi = iec\psi \quad s\bar{\psi} = -iec\bar{\psi} \quad sc = 0 \quad s\bar{c} = b \quad sb = 0$$

$$c^2 = \bar{c}^2 = 0$$

The BRST symmetry enjoys nillpotency $s^2 = 0$ and has a consistent set of algebraic and cohomological properties which have been further explored.

A recent generalization of the above ideas for higher-order derivative terms contextualized in the Bopp-Podolsky electrodynamics can be found in

C. R. Ji, A. T. Suzuki, J. H. O. Sales and R. Thibes, Eur. Phys. J. C **79**, no.10, 871 (2019).

1 - Motivation

Becchi-Rouet-Stora-Tyutin (BRST)

C. Becchi, A. Rouet and R. Stora, "The Abelian Higgs-Kibble Model. Unitarity of the S Operator," Phys. Lett. **52B**, 344 (1974).

C. Becchi, A. Rouet and R. Stora, "Renormalization of the Abelian Higgs-Kibble Model," Commun. Math. Phys. **42**, 127 (1975).

I. V. Tyutin, P. N. Lebedev Physical Institute preprint, FIAN n.39, LEBEDEV-75, "Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism," arXiv:0812.0580 [hep-th] (1975).

C. Becchi, A. Rouet and R. Stora, "Renormalization of Gauge Theories," Annals Phys. **98**, 287 (1976).

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THE ABELIAN HIGGS KIBBLE MODEL, UNITARITY OF THE *S*-OPERATOR

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Received 15 July 1974

Results concerning the renormalization of the abelian Higgs Kibble model in the 't Hooft gauges are presented. A direct combinatorial proof of the unitarity of the physical *S*-operator is described.

Commun. math. Phys. 42, 127—162 (1975)

Renormalization of the Abelian Higgs-Kibble Model

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Received November 25, 1974

Abstract. This article is devoted to the perturbative renormalization of the abelian Higgs-Kibble model, within the class of renormalizable gauges which are odd under charge conjugation. The Bogoliubov Parasiuk Hepp-Zimmermann renormalization scheme is used throughout, including the renormalized action principle proved by Lowenstein and Lam. The whole study is based on the fulfillment to all orders of perturbation theory of the Slavnov identities which express the invariance of the Lagrangian under a supergauge type family of non-linear transformations involving the Faddeev-Popov ghosts. Direct combinatorial proofs are given of the gauge independence and unitarity of the physical *S* operator. Their simplicity relies both on a systematic use of the Slavnov identities as well as suitable normalization conditions which allow to perform all mass renormalizations,

ANNALS OF PHYSICS **98**, 287–321 (1976)

Renormalization of Gauge Theories

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Gauge theories are characterized by the Slavnov identities which express their invariance under a family of transformations of the supergauge type which involve the Faddeev Popov ghosts. These identities are proved to all orders of renormalized perturbation theory, within the BPHZ framework, when the underlying Lie algebra is semi-simple and the gauge function is chosen to be linear in the fields in such a way that all fields are massive. An example, the SU2 Higgs Kibble model is analyzed in detail: the

Batalin-Fradkin-Vilkovisky (BFV)
(Hamiltonian Approach)

E. S. Fradkin and G. A. Vilkovisky, "Quantization of Relativistic Systems with Constraints," Phys. Lett. B **55**, 224 (1975).

I. A. Batalin and G. A. Vilkovisky, "Relativistic S Matrix of Dynamical Systems with Boson and Fermion Constraints," Phys. Lett. B **69**, 309 (1977).

E. S. Fradkin and T. E. Fradkina, "Quantization of Relativistic Systems with Boson and Fermion First and Second Class Constraints,"
Phys. Lett. B **72**, 343 (1978).

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QUANTIZATION OF RELATIVISTIC SYSTEMS WITH CONSTRAINTS

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Received 23 October 1974

The general dynamical system with constraints is quantized, and the *S*-matrix is constructed in the most general class of gauges including relativistic ones. In the case when constraints do not form a group a new type of additional diagrams arises securing unitarity of the theory: the four-fermion interaction of ghost fields.

upon the appeared structure coefficients is that the absolutely antisymmetrical (with respect to any pair of equally placed indices parts of expressions

$$\{U^c_{ab}, U^e_{fg}\}, \ \{V^a_b, U^c_{fg}\}$$

vanish. Let us supplement this system with fermion canonical variables $(\eta^a, \mathcal{P}_a)$ consistently considered as elements of Grassman algebra. Finally let us introduce the arbitrary functions $\chi^a(q^A, \pi_A)$. Our theorem says: the following functional integral does not depend on the choice of $\chi^a(q^A, \pi_A)$:

$$Z = \int \mathrm{d}q\, \mathrm{d}\pi\, \mathrm{d}\eta\, \mathrm{d}\mathcal{P} \exp\left[i\int \mathrm{d}t(\pi_A \dot{q}^A + \mathcal{P}_a \dot{\eta}^a - H)\right], \quad (8)$$

$$H = H_0 - G_a\chi^a - \mathcal{P}_a\{\chi^a, G_b\}\eta^b - \mathcal{P}_a U^a_{bc}\chi^b\eta^c + \mathcal{P}_a V^a_b \eta^b - \tfrac{1}{2}\mathcal{P}_a\eta^b\{\chi^a, U^n_{bm}\}\mathcal{P}_n\eta^m. \quad (9)$$

The functions χ^α may depend also on π_α, then eq. (8) gives the *S*-matrix in a nondegenerate gauge. Moreover the complete Hamiltonian (9) may be identically rewritten as

$$H = H_0 + \mathcal{P}_a V^a_b \eta^b - \{\mathcal{P}_a\chi^a, G_b\eta^b - \tfrac{1}{2}U^b_{mn}\mathcal{P}_b\eta^m\eta^n\} \quad (10)$$

in terms of the generalized boson – fermion brackets:

$$\{A, B\} = \frac{\delta A}{\delta q}\bigg|_{\mathrm{r}} \frac{\delta B}{\delta \pi}\bigg|_{\mathrm{l}} - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}\bigg|_{\mathrm{r}} \frac{\delta A}{\delta \pi}\bigg|_{\mathrm{l}},$$

η_A denoting the number of fermions in A, and subscripts "r" and "l" right and left derivatives. Eq. (8) with the Hamiltonian H in the form (10) is also correct for the gauge functions χ^α depending on η, $\mathcal{P}$.

Discussion. In the general case $U^\alpha_{\beta\gamma}$ are the func-

Anti-BRST

G. Curci and R. Ferrari, "On a Class of Lagrangian Models for Massive and Massless Yang-Mills Fields," Nuovo Cim. A **32**, 151 (1976).

G. Curci and R. Ferrari, "Slavnov Transformations and Supersymmetry," Phys. Lett. B **63**, 91 (1976).

I. Ojima, "Another BRS Transformation," Prog. Theor. Phys. **64**, 625 (1980).

625

Progress of Theoretical Physics, Vol. 64, No. 2, August 1980

Another BRS Transformation

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In the covariant canonical formalism of Yang-Mills theory and of the internal-Lorentz part of vierbein formalism, a new invariance is found under "another BRS transformation" satisfying the nilpotency property. In this transformation, the roles of the Faddeev-Popov ghost c in the BRS transformation are played by the Faddeev-Popov anti-ghost $\bar{c}$. Some implications of this new symmetry including the Ward-Takahashi identities following from it are discussed.

New Forms of BRST Symmetries

M. Lavelle and D. McMullan, “A new symmetry for QED,” Phys. Rev. Lett. **71**, 3758 (1993).

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V. O. Rivelles, “On New forms of the BRST transformations,” Class. Quant. Grav. **19**, 2525 (2002).

S. K. Rai and B. P. Mandal, “New forms of BRST symmetry in rigid rotor,” Mod. Phys. Lett. A **25**, 2281 (2010).

S. K. Rai and B. P. Mandal, “Various forms of BRST symmetry in Abelian 2-form gauge theory,” Mod. Phys. Lett. A **26**, 913 (2011).

2 - A Prototypical First-Class System

Given a non-degenerated symmetric two-form $f^{ij}(q^k)$, consider a prototypical dynamical system defined by

$$H(q^i, q^0, p_i, p_0) = U(q^k, p_k) + V(q^k) + q^0 T(q^k)\,, \tag{12}$$

with $V(q^k)$ and $T(q^k)$ given real functions and

$$U(q^k, p_k) \equiv \frac{R^{ijkl} T_i T_j p_k p_l}{2 f^{ij} T_i T_j}\,, \tag{13}$$

$$R^{ijkl}(q^m) \equiv f^{ij} f^{kl} - f^{ik} f^{jl}\,, \tag{14}$$

$$T_i \equiv \frac{\partial T}{\partial q^i}\,. \tag{15}$$

In this way,

$$H = H(q^i, q^0, p_i, p_0) \tag{16}$$

with $i = 1, \ldots, n$, denotes a Dirac-Bergmann dynamical system with first-class constraints $T(q^k)$ and p^0.

2 - A Prototypical First-Class System

Hamilton equations

$$\begin{cases} \dot{q}^i = \dfrac{R^{ijkl}T_kT_lp_j}{f^{ij}T_iT_j}\,, \\ \dot{p}_i = -U_i - V_i - q_0T_i\,, \\ T = 0\,, \end{cases} \tag{17}$$

with

$$U_i \equiv \frac{\partial U}{\partial q^i} = \frac{\left[\left(R^{mjkl}{}_{,i}f^{rs} - R^{mjkl}f^{rs}{}_{,i}\right)T_s + 2f^{rk}R^{msjl}T_{si}\right]T_rT_mT_jp_kp_l}{2f^{ij}f^{kl}T_iT_jT_kT_l}\,, \tag{18}$$

T_{ij} denoting the second-order derivative

$$T_{ij} \equiv \frac{\partial^2 T}{\partial q^j \partial q^i} \quad \text{and} \quad V_i \equiv \frac{\partial V}{\partial q^i}\,. \tag{19}$$

The indexes l, m, r, s in equation (18) also run from 1 to n.

2 - A Prototypical First-Class System

The Hamiltonian (12) defines a *constrained* system with $n-1$ degrees of freedom. In fact, by introducing the compact notation

$$w \equiv f^{ij} T_i T_j \,, \qquad B \equiv f^{ij} T_i p_j \,, \text{ and } \mathcal{O}_i{}^j \equiv \delta_i^j - \frac{T_i f^{jk} T_k}{w} \,, \tag{20}$$

we may easily determine q_0 in (17) as

$$q_0 = -\frac{f^{ij} T_i (\dot{p}_j + W_j)}{w} \quad \text{with} \quad W_j \equiv U_j + V_j \tag{21}$$

and rewrite the remaining $2n$ first-order differential equations as

$$\begin{cases} \dot{q}^i = f^{ij} \left(p_j - \dfrac{B}{w} T_j \right) \,, \\ \mathcal{O}_i{}^j (\dot{p}_j + W_j) = 0 \,. \end{cases} \tag{22}$$

As the operator $\mathcal{O}_i{}^j$ is not invertible, due to the zero-mode $\mathcal{O}_i{}^j T_j = 0$, the momenta p_i cannot be univocally determined.

2 - A Prototypical First-Class System

The Hamiltonian (12) characterizes a *gauge-invariant* system described by the clasical action

$$S_c = \int_{t_1}^{t_2} dt \, \left[\dot{q}^i p_i + \dot{q}_0 p_0 - W - q_0 T - \lambda p_0 \right] \, , \tag{23}$$

which is left invariant by

$$\begin{cases} \delta q_0 = \epsilon_0 + \dot{\epsilon} \, , \\ \delta \lambda = \dot{\epsilon}_0 + \ddot{\epsilon} \, , \\ \delta p_i = \dfrac{\epsilon_0 T T_i}{\dot{T}} - \epsilon T_i \, , \end{cases} \tag{24}$$

representing gauge transformations generated by the two first-class constraints p_0 and T.

3 - BFV Quantization - Ordinary BRST Symmetry

Corresponding to the first class constraints $T(q^k)$ and p^0, we introduce the ghosts variables $(\mathcal{C}, \bar{\mathcal{C}})$, $(\bar{\mathcal{P}}, \mathcal{P})$ and write

$$Z_\Psi = \int \mathcal{D}\varphi \; \exp(iS_{eff}) \,, \tag{25}$$

with

$$\mathcal{D}\varphi \equiv \mathcal{D}q^0 \, \mathcal{D}p_0 \, \mathcal{D}q^i \, \mathcal{D}p_i \, \mathcal{D}\mathcal{C} \, \mathcal{D}\bar{\mathcal{P}} \, \mathcal{D}\bar{\mathcal{C}} \, \mathcal{D}\mathcal{P} \,, \tag{26}$$

effective action

$$S_{eff} = \int dt \left(\dot{q}^i p_i + \dot{q}^0 p_0 + \dot{\mathcal{C}}\bar{\mathcal{P}} + \dot{\bar{\mathcal{C}}}\mathcal{P} - H_\Psi \right) \,, \tag{27}$$

and extended Hamiltonian

$$H_\Psi = U(q^k, p_k) + V(q^k) + \{\Omega, \Psi\} \,, \tag{28}$$

with Ω denoting the BRST charge generating the corresponding BRST transformations. The gauge freedom is captured in the generating functional (25) by the *gauge-fixing fermion* Ψ present in (28).

Ordinary BRST Charge

Quantum commutation relations

$$\begin{aligned} \left[q^k, p_l\right]_- &= i\,\delta^k_l\,, & \left[\mathcal{C}, \bar{\mathcal{P}}\right]_+ &= -\,i\,, \\ \left[q^0, p_0\right]_- &= i\,, & \left[\bar{\mathcal{C}}, \mathcal{P}\right]_+ &= -\,i\,. \end{aligned} \tag{29}$$

The quantum BRST charge in the extended Hamiltonian phase space Ω_b can be written as

$$\Omega_b = i\left(\mathcal{C}\,T(q^k) + \mathcal{P}p_0\right)\,, \tag{30}$$

and is responsible for generating the ordinary BRST symmetry. Given a generic function $F(q^k, p_k, q^0, p_0, \mathcal{C}, \bar{\mathcal{P}}, \bar{\mathcal{C}}, \mathcal{P})$ with well-defined Grassmann parity ϵ_F, we define

$$\delta_b F = [F, \Omega_b]_\pm \equiv F\Omega_b - (-1)^{\epsilon_F}\Omega_b F\,. \tag{31}$$

Ordinary BRST Symmetry

For the fundamental variables, plugging (30) into (31) leads explicitly to

$$\begin{array}{llll} \delta_b q^i = 0\,, & \delta_b q^0 = -\mathcal{P}\,, & \delta_b \mathcal{C} = 0\,, & \delta_b \bar{\mathcal{C}} = p_0\,, \\ \delta_b p_i = \mathcal{C}\, T_i\,, & \delta_b p_0 = 0\,, & \delta_b \bar{\mathcal{P}} = T\,, & \delta_b \mathcal{P} = 0\,. \end{array} \tag{32}$$

The BRST charge Ω_b has ghost number $+1$ and is nillpotent, by which we mean

$$\Omega_b^2 = 0\,, \tag{33}$$

and, as a direct consequence, the transformations (32) are off-shell closed. We may choose the gauge fixing fermion as

$$\Psi = \bar{\mathcal{P}} q^0 + \bar{\mathcal{C}} \chi\,, \tag{34}$$

where

$$\chi \equiv \chi(q^k, p_k, q^0, p_0) \tag{35}$$

is an arbitrary bosonic function which does not depend on the ghost variables.

Gauge-fixed Action

Noting that

$$[\Omega_b, \Psi] = q^0 T + \mathcal{P}\bar{\mathcal{P}} + i\mathcal{C}\bar{\mathcal{C}}\,[T,\chi] + p_0\chi + i\mathcal{P}\,[p_0,\chi]\,\bar{\mathcal{C}}\,, \qquad (36)$$

we may write the quantum BRST invariant Hamiltonian

$$\hat{H} = U + V + q^0 T + p_0\chi + i\mathcal{C}\,[T,\chi]\,\bar{\mathcal{C}} + i\mathcal{P}\,[p_0,\chi]\,\bar{\mathcal{C}} + \mathcal{P}\bar{\mathcal{P}}\,. \qquad (37)$$

Considering the standard gauge function

$$\chi = \vartheta^{-1}B + \frac{\xi}{2}p_0\,, \qquad \text{with} \qquad (38)$$

$$B = f^{ij}T_i p_j \qquad \text{and} \qquad \vartheta = \omega^{-2} f^{ij} T_i T_j\,, \qquad (39)$$

we obtain the gauge-fixed action

$$\begin{aligned} S_{ext} \;&=\; \int dt\,\Big(\dot{q}^i p_i + \dot{\mathcal{C}}\bar{\mathcal{P}} + \dot{\bar{\mathcal{C}}}\mathcal{P} - U(q^k,p_k) - V(q^k) - q^0 T(q^k) \\ &\quad -\frac{\xi}{2}p_0^2 + p_0\big(\dot{q}^0 - \vartheta^{-1}B\big) + \omega^2\mathcal{C}\bar{\mathcal{C}} - \mathcal{P}\bar{\mathcal{P}}\Big)\,, \end{aligned} \qquad (40)$$

Performing a functional integration over p_0 brings down the term

$$\frac{1}{2\xi}\left(\dot{q}^0 - \vartheta^{-1}B\right)^2 , \tag{41}$$

which is akin to the usual quadratic covariant gauge-fixing term present in the QED and QCD quantum Lagrangians.

$\xi \longrightarrow 0$	Landau gauge
$\xi \longrightarrow 1$	Feynman-'t Hooft
$\xi \longrightarrow 3$	Fried-Yennie
$\xi \longrightarrow \infty$	unitary gauge

4 - BRST-Related Symmetries – Hamiltonian Approach

The following four transformations, generated by their respective charges, leave the extended action (40) invariant:

Ordinary BRST

$$\begin{array}{llll} \delta_b q^i = 0\,, & \delta_b q^0 = -\mathcal{P}\,, & \delta_b \mathcal{C} = 0\,, & \delta_b \bar{\mathcal{C}} = p_0\,, \\ \delta_b p_i = \mathcal{C}\, T_i\,, & \delta_b p_0 = 0\,, & \delta_b \bar{\mathcal{P}} = T\,, & \delta_b \mathcal{P} = 0\,, \end{array}$$

$$\Omega_b = i\left[\mathcal{C}\,T(q^k) + \mathcal{P} p_0\right]\,, \qquad \text{gh}\ \Omega_b = +1\,.$$

Anti-BRST

$$\begin{array}{llll} \bar{\delta}_b q^i = 0\,, & \bar{\delta}_b q^0 = -\bar{\mathcal{P}}\,, & \bar{\delta}_b \mathcal{C} = p_0\,, & \bar{\delta}_b \bar{\mathcal{C}} = 0\,, \\ \bar{\delta}_b p_i = -\bar{\mathcal{C}}\, T_i\,, & \bar{\delta}_b p_0 = 0\,, & \bar{\delta}_b \bar{\mathcal{P}} = 0\,, & \bar{\delta}_b \mathcal{P} = -T\,, \end{array}$$

$$\bar{\Omega}_b = i\left[-\bar{\mathcal{C}}\,T(q^k) + \bar{\mathcal{P}} p_0\right]\,, \qquad \text{gh}\ \bar{\Omega}_b = -1\,.$$

4 - BRST-Related Symmetries – Hamiltonian Approach

Dual-BRST

$$\bar\delta_d q^i = 0\,, \qquad \bar\delta_d q^0 = -\omega\bar{\mathcal{C}}\,, \qquad \bar\delta_d \mathcal{C} = \omega^{-1} T\,, \qquad \bar\delta_d \bar{\mathcal{C}} = 0\,,$$
$$\bar\delta_d p_i = \omega^{-1}\bar{\mathcal{P}} T_i\,, \qquad \bar\delta_d p_0 = 0\,, \qquad \bar\delta_b \bar{\mathcal{P}} = 0\,, \qquad \bar\delta_d \mathcal{P} = \omega p_0\,,$$

$$\bar\Omega_d = i\left[\omega^{-1}\bar{\mathcal{P}} T(q^k) + \omega\bar{\mathcal{C}} p_0\right]\,, \qquad \mathrm{gh}\ \bar\Omega_d = -1\,.$$

Anti-dual BRST

$$\delta_d q^i = 0\,, \qquad \delta_d q^0 = -\omega\mathcal{C}\,, \qquad \delta_d \mathcal{C} = 0\,, \qquad \delta_d \bar{\mathcal{C}} = -\omega^{-1} T\,,$$
$$\delta_d p_i = -\omega^{-1}\mathcal{P} T_i\,, \qquad \delta_d p_0 = 0\,, \qquad \delta_d \bar{\mathcal{P}} = \omega p_0\,, \qquad \delta_d \mathcal{P} = 0\,,$$

$$\Omega_d = i\left[-\omega^{-1}\mathcal{P} T(q^k) + \omega\mathcal{C} p_0\right]\,, \qquad \mathrm{gh}\ \Omega_d = +1\,.$$

The dual symmetries, $\bar{\delta}_d$ and δ_d, can be obtained from δ_b and $\bar{\delta}_b$ by

$$\text{(anti-)BRST} \longrightarrow \text{(anti-)dual-BRST}$$

$$a: \quad \mathcal{C} \longrightarrow \omega^{-1}\bar{\mathcal{P}}\,, \quad \bar{\mathcal{C}} \longrightarrow \omega^{-1}\mathcal{P}\,, \quad \mathcal{P} \longrightarrow \omega\bar{\mathcal{C}}\,, \quad \bar{\mathcal{P}} \longrightarrow \omega\mathcal{C}\,, \tag{42}$$

and the anti- symmetries, $\bar{\delta}_b$ and δ_d, from δ_b and $\bar{\delta}_d$ by

$$\text{(dual-)BRST} \longrightarrow \text{(dual-)anti-BRST}$$

$$b: \quad \mathcal{C} \longrightarrow -\bar{\mathcal{C}}\,, \quad \bar{\mathcal{C}} \longrightarrow \mathcal{C}\,, \quad \mathcal{P} \longrightarrow \bar{\mathcal{P}}\,, \quad \bar{\mathcal{P}} \longrightarrow -\mathcal{P}\,. \tag{43}$$

Both equations (42) and (43) above represent canonical transformations:

$$a: \quad \begin{array}{rcl} \left[\mathcal{C}, \bar{\mathcal{P}}\right]_+ &=& -\,i\,, \quad \rightarrow \quad \left[\omega^{-1}\bar{\mathcal{P}}, \omega\mathcal{C}\right]_+ \;=\; -\,i\,, \\ \left[\bar{\mathcal{C}}, \mathcal{P}\right]_+ &=& -\,i\,, \quad \rightarrow \quad \left[\omega^{-1}\mathcal{P}, \omega\bar{\mathcal{C}}\right]_+ \;=\; -\,i\,. \end{array} \tag{44}$$

$$b: \quad \begin{array}{rcl} \left[\mathcal{C}, \bar{\mathcal{P}}\right]_+ &=& -\,i\,, \quad \rightarrow \quad \left[-\bar{\mathcal{C}}, -\mathcal{P}\right]_+ \;=\; -\,i\,, \\ \left[\bar{\mathcal{C}}, \mathcal{P}\right]_+ &=& -\,i\,, \quad \rightarrow \quad \left[\mathcal{C}, \bar{\mathcal{P}}\right]_+ \;=\; -\,i\,. \end{array} \tag{45}$$

What is the origin of all those similar, seemingly related, symmetries?

It turns out that action (40) is invariant wrt the group $\mathbb{Z}_4 \times \mathbb{Z}_2$:

$$\begin{aligned} S_{ext} &= \int dt \Big(\dot{q}^i p_i + \dot{\mathcal{C}}\bar{\mathcal{P}} + \dot{\bar{\mathcal{C}}}\mathcal{P} - U(q^k, p_k) - V(q^k) - q^0 T(q^k) \\ &\quad - \frac{\xi}{2} p_0^2 + p_0\big(\dot{q}^0 - \vartheta^{-1} B\big) + \omega^2 \mathcal{C}\bar{\mathcal{C}} - \mathcal{P}\bar{\mathcal{P}} \Big) \, . \end{aligned} \tag{46}$$

Transformations (42) and (43) may be mapped into the two $\mathbb{Z}_4 \times \mathbb{Z}_2$ generators a and b, characterizing the group as

$$\mathbb{Z}_4 \times \mathbb{Z}_2 = < a, b \,|\, a^2 = b^4 = e, ab = ba > , \tag{47}$$

which can be fully realized by the $\pm$BRST, $\pm$anti-BRST, $\pm$dual-BRST and $\pm$anti-dual-BRST transformations.

5 - BRST-Related Symmetries in Configuration Space

We maintain p_k in configuration space, keeping the gauge symmetry simple and interpreting the system as defined by a first-order Lagrangian.

Elimination of p_0 spoils the BRST off-shell nilpotency which become closed only on shell. We see that p_0 is a Nakanishi-Lautrup variable.

$$Z_\Psi = \int \mathcal{D}\varphi \; \exp(iS_{eff}) \tag{48}$$

We perform the functional integration in the ghost momenta variables $\mathcal{P}$ and $\bar{\mathcal{P}}$ and obtain a neat first-order action given by

$$\begin{aligned} S &= \int dt \Big(\dot{q}^i p_i - \dot{\mathcal{C}}\dot{\bar{\mathcal{C}}} - U(q^k, p_k) - V(q^k) - q^0 T(q^k) \\ &\quad - \frac{\xi}{2} p_0^2 + p_0 \big(\dot{q}^0 - \vartheta^{-1} B \big) + \omega^2 \mathcal{C}\bar{\mathcal{C}} \Big) \, . \end{aligned} \tag{49}$$

5 - BRST-Related Symmetries in Configuration Space

Hence, we come to a Lagrangian framework with corresponding BRST-Related symmetries and conserved charges given by

Ordinary BRST

$$\begin{array}{lll} s_b\, q^i = 0\,, & s_b\, q^0 = -\dot{\mathcal{C}}\,, & s_b\, \mathcal{C} = 0\,, \\ s_b\, p_i = \mathcal{C}\, T_i\,, & s_b\, p_0 = 0\,, & s_b\, \bar{\mathcal{C}} = p_0\,, \end{array} \tag{50}$$

$$\mathcal{Q}_b = i\left[\mathcal{C}\, T(q^k) + \dot{\mathcal{C}} p_0\right]\,, \qquad \text{gh}\ \mathcal{Q}_b = +1\,; \tag{51}$$

Anti-BRST

$$\begin{array}{lll} \bar{s}_b\, q^i = 0\,, & \bar{s}_b\, q^0 = \dot{\bar{\mathcal{C}}}\,, & \bar{s}_b\, \mathcal{C} = p_0\,, \\ \bar{s}_b\, p_i = -\bar{\mathcal{C}}\, T_i\,, & \bar{s}_b\, p_0 = 0\,, & \bar{s}_b\, \bar{\mathcal{C}} = 0\,, \end{array} \tag{52}$$

$$\bar{\mathcal{Q}}_b = i\left[-\bar{\mathcal{C}}\, T(q^k) - \dot{\bar{\mathcal{C}}} p_0\right]\,, \qquad \text{gh}\ \bar{\mathcal{Q}}_b = -1\,; \tag{53}$$

Dual-BRST

$$\begin{aligned} &\bar{s}_d\, q^i = 0\,, && \bar{s}_d\, q^0 = -\omega\bar{\mathcal{C}}\,, && \bar{s}_d\,\mathcal{C} = \omega^{-1}\,T\,, \\ &\bar{s}_d\, p_i = -\omega^{-1}\dot{\bar{\mathcal{C}}}\,T_i\,, && \bar{s}_d\, p_0 = 0\,, && \bar{s}_d\,\bar{\mathcal{C}} = 0\,, \end{aligned} \tag{54}$$

$$\bar{\mathcal{Q}}_d = i\left[-\omega^{-1}\dot{\bar{\mathcal{C}}}\,T(q^k) + \omega\bar{\mathcal{C}}p_0\right]\,, \qquad \text{gh}\ \bar{\mathcal{Q}}_d = -1\,; \tag{55}$$

Anti-Dual-BRST

$$\begin{aligned} &s_d\, q^i = 0\,, && s_d\, q^0 = -\omega\mathcal{C}\,, && s_d\,\mathcal{C} = 0\,, \\ &s_d\, p_i = -\omega^{-1}\dot{\mathcal{C}}\,T_i\,, && s_d\, p_0 = 0\,, && s_d\,\bar{\mathcal{C}} = -\omega^{-1}\,T\,, \end{aligned} \tag{56}$$

$$\mathcal{Q}_d = i\left[-\omega^{-1}\dot{\mathcal{C}}\,T(q^k) + \omega\mathcal{C}p_0\right]\,, \qquad \text{gh}\ \mathcal{Q}_d = +1\,. \tag{57}$$

In BRST-cohomology terms, the BRST-Related invariances allow the action to be decomposed as a sum between a BRST-exact and a BRST-closed parts.

$$\begin{aligned} S &= \int dt \Big\{ \dot{q}^i p_i - U(q^k, p_k) - V(q^k) - q^0 T(q^k) \\ &\quad + \frac{1}{2} s_b\, \bar{s}_b \, \left[\xi \mathcal{C} \bar{\mathcal{C}} - (q^0)^2 - \omega^{-2} \vartheta^{-2} B^2 \right] \Big\} \end{aligned} \tag{58}$$

or

$$\begin{aligned} S &= \int dt \Big\{ - U(q^k, p_k) - V(q^k) - \frac{\xi}{2} p_0^2 + p_0 \left(\dot{q}^0 - \vartheta^{-1} B \right) \\ &\quad + \frac{1}{2} s_d\, \bar{s}_d \left[\left(\frac{\dot{q}^i p_i}{\dot{T}} \right)^2 - (q^0)^2 \right] \Big\} . \end{aligned} \tag{59}$$

We have seen an important feature distinguishing the (anti-)BRST from the (anti-)dual-BRST symmetries.
Additionally, they allow for a physical realization of a Hodge theory with a rich algebraic structure.

6 - BRST Algebra

The four BRST-Related charges (51), (53), (55) and (57) are fully off-shell nillpotent fermionic operators,

$$\mathcal{Q}_b^2 = \bar{\mathcal{Q}}_b^2 = \bar{\mathcal{Q}}_d^2 = \mathcal{Q}_d^2 = 0 \tag{60}$$

conserved under time evolution modulo equations of motion.
We introduce the ghost number operator

$$\mathcal{G} = i\left(\dot{\mathcal{C}}\bar{\mathcal{C}} - \mathcal{C}\dot{\bar{\mathcal{C}}}\right) , \tag{61}$$

satisfying

$$\begin{aligned} [\mathcal{G}, \mathcal{Q}_b] = \mathcal{Q}_b \,, \qquad & \left[\mathcal{G}, \bar{\mathcal{Q}}_b\right] = -\bar{\mathcal{Q}}_b \,, \\ \left[\mathcal{G}, \bar{\mathcal{Q}}_d\right] = -\bar{\mathcal{Q}}_d \,, \qquad & [\mathcal{G}, \mathcal{Q}_d] = \mathcal{Q}_d \,. \end{aligned} \tag{62}$$

Ghost number conservation is then warranted by the global scale symmetry

$$\mathcal{C} \longrightarrow e^{\lambda}\mathcal{C} \,, \qquad \bar{\mathcal{C}} \longrightarrow e^{-\lambda}\bar{\mathcal{C}} \,, \tag{63}$$

with λ denoting a continuous constant parameter.

6 - BRST Algebra

The two (anti-)BRST operators commute among themselves

$$\left[\mathcal{Q}_b, \bar{\mathcal{Q}}_b\right] = 0\,, \tag{64}$$

as well as the (anti-)dual-BRST ones

$$\left[\bar{\mathcal{Q}}_d, \mathcal{Q}_d\right] = 0\,, \tag{65}$$

and we have

$$\left[\mathcal{Q}_b, \bar{\mathcal{Q}}_d\right] = \left[\bar{\mathcal{Q}}_b, \mathcal{Q}_d\right] = i(\omega^{-1}T^2 + \omega p_0^2) \equiv 2\mathcal{W}\,. \tag{66}$$

$\mathcal{W}$ represents a Casimir operator for the superalgebra generated by the BRST charges and generates the symmetry

$$s_W F \equiv [F, \mathcal{W}]\,. \tag{67}$$

For the fundamental variables, the non-null s_W transformations read

$$s_W p_i = \omega^{-1} T T_i\,, \qquad s_W q^0 = -\omega p_0\,, \tag{68}$$

leaving (49) invariant.

7 - New Symmetries

The quantum action (49) enjoys the further nillpotent symmetries

$$\Delta_1 p_i = \mathcal{C} T_i \,, \tag{69}$$

$$\Delta_1 p_0 = -\frac{2}{\xi}(\omega^2 \mathcal{C} + \ddot{\mathcal{C}}) \,, \qquad \Delta_1 q^0 = -\dot{\mathcal{C}} \,, \tag{70}$$

$$\Delta_1 \bar{\mathcal{C}} = -p_0 + \frac{2}{\xi}\left(\dot{q}^0 - \vartheta^{-1} B\right) , \tag{71}$$

and

$$\bar{\Delta}_1 p_i = -\bar{\mathcal{C}} T_i \,, \tag{72}$$

$$\bar{\Delta}_1 p_0 = \frac{2}{\xi}(\omega^2 \bar{\mathcal{C}} + \ddot{\bar{\mathcal{C}}}) \,, \qquad \bar{\Delta}_1 q^0 = \dot{\bar{\mathcal{C}}} \,, \tag{73}$$

$$\bar{\Delta}_1 \mathcal{C} = p_0 - \frac{2}{\xi}\left(\dot{q}^0 - \vartheta^{-1} B\right) . \tag{74}$$

7 - New Symmetries

We report here a brand new set of non-local symmetries

$$\Delta_2 p_0 = \frac{1}{\xi}(\omega^2 \mathcal{C} + \ddot{\mathcal{C}}) \,, \tag{75}$$

$$\Delta_2 q^0 = \frac{1}{2}\dot{\mathcal{C}} + \frac{\omega^2}{2}\int \mathcal{C}\, dt \,, \tag{76}$$

$$\Delta_2 \bar{\mathcal{C}} = \frac{1}{2}p_0 - \frac{1}{\xi}(\dot{q}^0 - \vartheta^{-1}B) - \frac{1}{2}\int T\, dt \,, \tag{77}$$

and

$$\bar{\Delta}_2 p_0 = -\frac{1}{\xi}(\omega^2 \bar{\mathcal{C}} + \ddot{\bar{\mathcal{C}}}) \,, \tag{78}$$

$$\bar{\Delta}_2 q^0 = -\frac{1}{2}\dot{\bar{\mathcal{C}}} - \frac{\omega^2}{2}\int \bar{\mathcal{C}}\, dt \,, \tag{79}$$

$$\bar{\Delta}_2 \mathcal{C} = \frac{1}{2}p_0 - \frac{1}{\xi}(\dot{q}^0 - \vartheta^{-1}B) - \frac{1}{2}\int T\, dt \,, \tag{80}$$

7 - New Symmetries

which also leave the action

$$S = \int dt \left(\dot{q}^i p_i - \dot{\mathcal{C}}\dot{\bar{\mathcal{C}}} - U(q^k, p_k) - V(q^k) - q^0 T(q^k) - \frac{\xi}{2} p_0^2 + p_0 \left(\dot{q}^0 - \vartheta^{-1} B \right) + \omega^2 \mathcal{C}\bar{\mathcal{C}} \right) \qquad (81)$$

invariant.

The Δ_2 symmetries are clearly distinct from the Δ_1 ones, as the former do not affect p_i, i.e.,

$$\Delta_2 p_i = \bar{\Delta}_2 p_i = 0\,, \qquad (82)$$

while the latter contains terms corresponding to integrals of T and $\mathcal{C}$. A comparative analysis of the Δ symmetries as well as its relevance in specific quantum field theory models is currently under investigation.

Conclusion and Final Remarks

- The BFV quantization of our prototypical first-class system allowed us to realize various forms of BRST-Related transformations.
- A group of symmetries of the ghosts sector has been clarified, connecting the BRST-Related transformations.
- BRST-Related charges exhibit Hodge theory properties. A Casimir operator leads to a bosonic symmetry closing a Lie superalgebra.
- Simplicity and generality of the prototypical system permits the extension of our results to similar models in the literature.
- We reported brand new forms of non-local symmetries fully realized in the prototypical first-class system.

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