Transverse Momentum Dependent PDFs in Chiral Perturbation Theory

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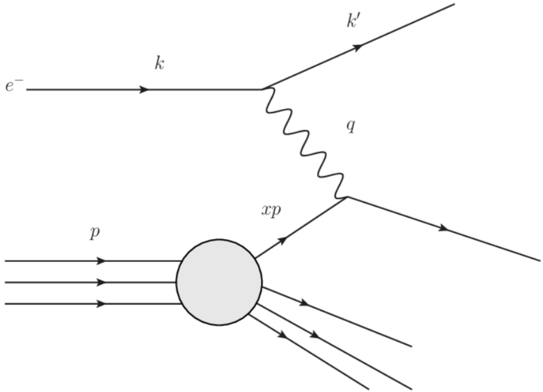
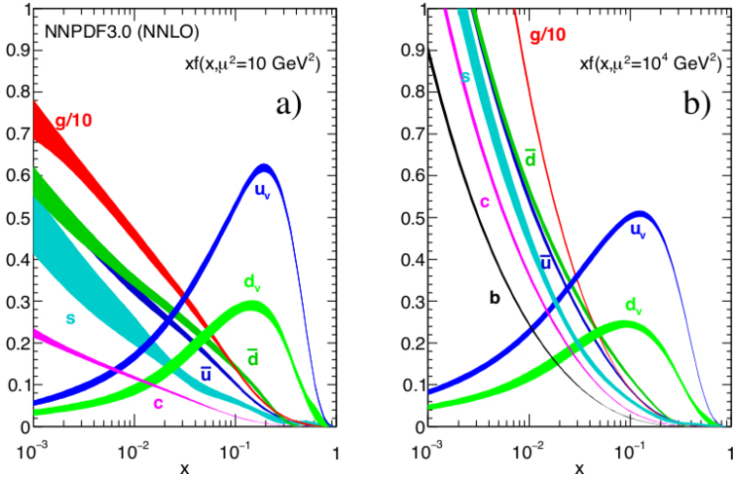
Parton Model

- Assumes that parts of a hadron (partons) carry fraction of a hadron's collinear momentum.



Parton Distribution Functions

- Partons obey distribution functions according to the fraction, x, involved in the scattering.



Parton Distribution Functions

Parton Distribution Functions (PDFs)

Defined in terms of nonlocal QCD operators

$$f_{iH}(x) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle H| \overline{\psi}(b^-) W_n(b^-, 0) \not{n} \psi(0) |H\rangle \, .$$

- Nonperturbative - encode physics happening at hadronic scales ($\Lambda_{QCD} \sim 200$ MeV).

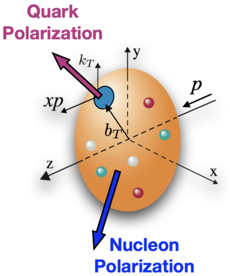
TMD PDFs

Transverse Momementum Dependent PDFs (TMD PDFs)

Generalizations of PDFs that describe the 3D structure of hadrons.

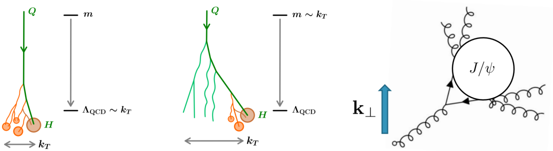
$$f_{iH}(x, b_T) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle H| \overline{\psi_i}(b^-, b_T) W_{\sqsubset}(b, 0) \not{n} \psi_i(0) |H\rangle$$

- Appear in QCD factorization theorems and are extremely interesting to the community.

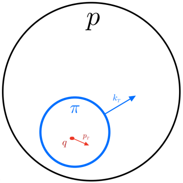


Can you calculate TMD PDFs in an effective theory?

- Systems with heavy quarks use heavy quark effective theory and non-relativistic QCD to study TMDs[2,3].



- What about systems with only light quarks and gluons? (pions, kaons, protons, etc.)[4]



[2] R. von Kuk, J. Michel, and Z. Sun. JHEP 09 (2023) 205

[3] M. Copeland, S. Fleming, R. Gupta, R. Hodges, and T. Mehen. PRD 109 (2024) 5, 5

[4] F. He and P. Wang. PRD 100 (2019) 7, 074032

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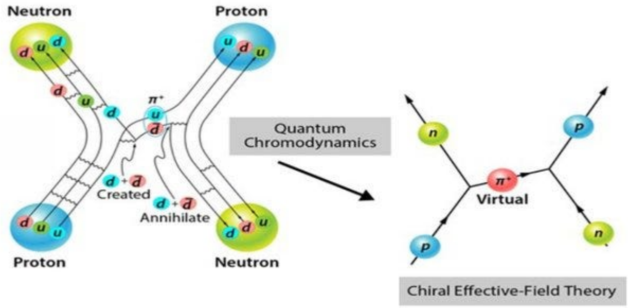
Chiral Perturbation Theory

- Chiral Perturbation Theory (χ-PT) is an effective field theory of QCD where the interactions are written in terms of hadronic degrees of freedom.



SU(2) Chiral Lagrangian

- The Chiral Lagrangian invariant under $SU(2)_L \times SU(2)_R$ transformations is

$$\mathcal{L} = \bar{N}(i\not{D} - M)N + \frac{g_A}{2}\bar{N}\gamma^\mu\gamma_5 u_\mu N + \frac{f_\pi^2}{4}\mathrm{Tr}\left[\partial^\mu\Sigma^\dagger\partial_\mu\Sigma\right].$$

- The sigma fields are in an exponential representation,

$$\Sigma = \exp\left(i\frac{\sqrt{2}M}{f_\pi}\right), \quad M = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}.$$

- Chirally covariant derivative and chiral vielbein

$$D_\mu = \partial_\mu + \Gamma_\mu, \qquad \Gamma_\mu = \frac{1}{2}(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger)$$
$$u_\mu = i(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger),$$

where $\xi^2 = \Sigma$.

Transformation properties

- The fields transform like

$$N(x) \to U(x)N(x), \quad \xi(x) \to L\xi(x)U^\dagger(x) = U(x)\xi(x)R,$$
$$\Sigma(x) \to L\Sigma R$$

  under $SU(2)_L \times SU(2)_R$ chiral transformations.
- This will be important later!

How do you describe PDFs in χPT?

Problem : PDFs are nonlocal

We usually match onto local operators in an effective theory.

$$\text{Local operator in QCD} \rightarrow \text{Local operator(s) in effective theory}$$

How do you describe PDFs in χPT?

- Luckily, PDFs can be written in terms of *local* twist-two operators[5],

$$f^a_{qX}(\zeta) = \int \frac{db^-}{2\pi} e^{-ib^-(\zeta P^+)} \langle X| \overline{\psi}(b^-) W_n(b^-) \tau^a \not{n} W^\dagger(0) \psi(0) |X\rangle$$

$$= \int \frac{db^-}{2\pi} e^{ib^-(\zeta P^+)} \sum_k \frac{(-ib^-)^k}{k!} \langle X| \overline{\psi} \not{n} \tau^a (in \cdot D)^k \psi |X\rangle .$$

[5] J. Chen and X. Ji. PRL 87 (2001) 152002

Matching Collinear PDFs onto χPT

- Matching procedure for local operators is well understood.

$$\text{Local operator} \rightarrow \text{Local operator(s)}$$

$$\mathcal{O}_{QCD} \rightarrow \mathcal{O}_{\chi\text{PT}}$$

Matching Collinear PDFs onto χPT

- Match onto chiral operators with same symmetries as PDF.

$$\overline{\psi}\not{n}\tau^a(in \cdot D)^k\psi = \sum_H c^k_{qH}\mathcal{O}^{k,a}_H$$

- The c^k_{qH} are high energy matching coefficients.
  - They describe physics at scales $p \geq \Lambda_\chi \sim 1.6$ GeV.
- The $\mathcal{O}^{k,a}_H$ are the chiral effective theory operators,
  - They describe low-energy physics at scales $p \ll \Lambda_\chi \sim 1.6$ GeV.

Chiral operators

- The twist two QCD operators are matched onto leading order operators in p_π/Λ_χ, m_π/Λ_χ expansion,

$$\overline{\psi}\not{n}\tau^a(in\cdot D)^k\psi \to$$

$$\mathcal{O}_\pi^{k,a} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\Sigma^\dagger\tau^a(n\cdot\partial)^k\Sigma + \Sigma\tau^a(n\cdot\partial)^k\Sigma^\dagger\right]$$
$$\mathcal{O}_N^{k,a} = \frac{1}{2}\bar{N}\not{n}\left[\xi^\dagger\tau^a(in\cdot\partial)^{k-1}\xi + \xi\tau^a(in\cdot\partial)^{k-1}\xi^\dagger\right]N,$$
$$\mathcal{O}_N^{A;k,a} = \frac{g_A}{2}\bar{N}\not{n}\gamma_5\left[\xi^\dagger\tau^a(in\cdot\partial)^{k-1}\xi - \xi\tau^a(in\cdot\partial)^{k-1}\xi^\dagger\right]N,$$

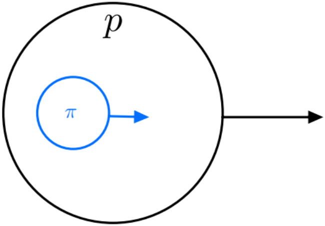
Hadronic Splitting functions

- Chiral operators define moments of hadronic splitting functions,

$$\langle X|\, \mathcal{O}_H^k \,|X\rangle = (P^+)^k \int d\beta\, \beta^{k-1} f_{HX}(\beta).$$

- Inverting the Mellin transform gives an explicit definition,

$$f_{HX}(\beta) = \int \frac{dx^-}{2\pi} e^{-ix^-\beta P^+} \langle X|\, \mathcal{O}_H(x^-,0) \,|X\rangle \,.$$

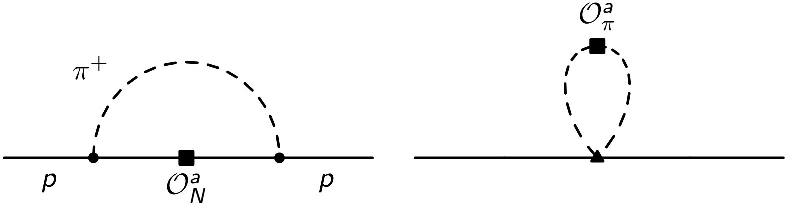


- Chiral operators in splitting functions are *nonlocal*,

$$\mathcal{O}_\pi^a(b^-,0) = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\Sigma^\dagger(b^-)\tau^a(in\cdot\partial)\Sigma(0) + \Sigma(b^-)\tau^a(in\cdot\partial)\Sigma^\dagger(0)\right],$$

$$\mathcal{O}_N^a(b^-,0) = \frac{1}{2}\bar{N}(b^-)\not{n}\left[\xi^\dagger(b^-)\tau^a\xi(0) + \xi(b^-)\tau^a\xi^\dagger(0)\right]N(0),$$

$$\mathcal{O}_N^{A;a}(b^-,0) = \frac{g_A}{2}\bar{N}(b^-)\not{n}\gamma_5\left[\xi^\dagger(b^-)\tau^a\xi(0) - \xi(b^-)\tau^a\xi^\dagger(0)\right]N(0).$$



Side note: Chiral invariance of nonlocal operators

Remark

The nonlocal operators are still chirally invariant.

- The $\xi(x)$ fields cancel out the $U(x)$ dependence from the nucleon's transformation.

$$N(x) \to U(x)N(x), \quad \xi(x) \to L\xi(x)U^\dagger(x) = U(x)\xi(x)R^\dagger$$

so

$$[\xi^\dagger(x)N(x)] \to R[\xi^\dagger(x)N(x)], \quad [\xi(x)N(x)] \to L[\xi(x)N(x)].$$

Side Note: Matching nonlocal operators

- Note, the PDF operator is like a nonlocal vector current in QCD

Example

$$J_V^{\mu,a} = \overline{\psi}(x)\gamma^\mu \tau^a \psi(x)$$
$$\mathcal{O}^a_{\mathrm{PDF}}(b^-, 0) = \overline{\psi}(b^-)W(b,0)\not{n}\tau^a\psi(0)$$

- Our splitting function operators are like nonlocal vector currents in χPT

Example

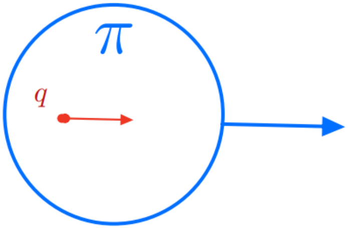
$$J_\pi^{\mu,a} = \frac{if_\pi^2}{4}\mathrm{Tr}\left[\Sigma^\dagger(x)\tau^a\partial^\mu\Sigma(x) + \Sigma(x)\tau^a\partial^\mu\Sigma^\dagger(x)\right]$$
$$\mathcal{O}^a_\pi(b^-, 0) = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\Sigma^\dagger(b^-)\tau^a(in\cdot\partial)\Sigma(0) + \Sigma(b^-)\tau^a(in\cdot\partial)\Sigma^\dagger(0)\right]$$

Back to the derivation: Matching coefficients

- High energy matching coefficients define moments of QCD PDFs of parton in intermediate hadron,

$$c_{qH}^{k} = \int d\alpha \alpha^{k-1} q_{H}^{(0)}(\alpha).$$

- Still describes the high energy physics of the system ($p \geq \Lambda_\chi \sim 1.6$ GeV).



Convolution formalism

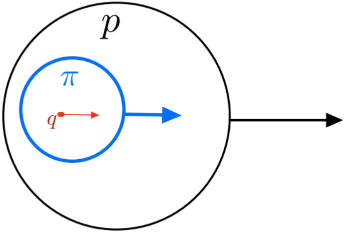
- Use

$$\langle X|\,\mathcal{O}_H^k\,|X\rangle = (P^+)^k \int d\beta\beta^{k-1} f_{HX}(\beta)$$

$$c_{qH}^k = \int d\alpha\alpha^{k-1} q_H^{(0)}(\alpha)$$

to derive,

$$f_{qX}(\zeta) = \sum_H \int_\xi^1 \frac{d\alpha}{\alpha} q_H^{(0)}(\alpha) f_{HX}\left(\frac{\zeta}{\alpha}\right).$$



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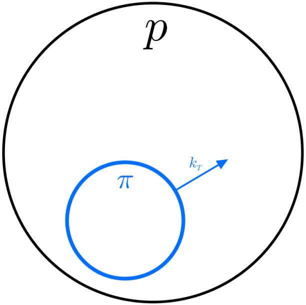
TMD Splitting Functions

- To extend this approach to the TMD PDF, define *TMD hadronic splitting functions*

$$\tilde{f}^a_{HX}(\beta, \mathbf{b}_T) = \int \frac{db^-}{2\pi} e^{-ib^-\beta P^+} \langle X| \mathcal{O}^a_H(b,0) |X\rangle \,.$$

where the chiral operators now have transverse and lightlike seperation, $b = (b^-, 0, \mathbf{b}_T)$.



TMD Splitting Functions

- Like the nonlocal operators we saw before, but now with $\mathbf{b}_T$,

$$\mathcal{O}_\pi^a(b,0) = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\Sigma^\dagger(b)\tau^a(n\cdot\partial)\Sigma(0) + \Sigma(b)\tau^a(n\cdot\partial)\Sigma^\dagger(0)\right],$$

$$\mathcal{O}_N^a(b,0) = \frac{1}{2}\bar{N}(b)\not{n}\left[\xi^\dagger(b)\tau^a\xi(0) + \xi(b)\tau^a\xi^\dagger(0)\right]N(0),$$

$$\mathcal{O}_N^{A;a}(b,0) = \frac{g_A}{2}\bar{N}(b)\not{n}\gamma_5\left[\xi^\dagger(b)\tau^a\xi(0) - \xi(b)\tau^a\xi^\dagger(0)\right]N(0).$$

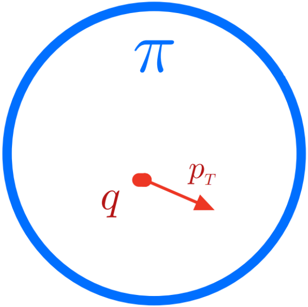
- These are still chirally invariant!

Matching coefficients

- The matching coefficients are now TMD distributions in the intermediate hadrons (in the chiral limit)

$$\tilde{q}_H^{(0)}(\alpha, \mathbf{b}_T) = \int \frac{db^-}{2\pi} e^{-ib^-(\alpha P^+)} \langle H | \overline{\psi}(b) \tau^a W_{\sqsubset}(b,0) \not{n} \psi(0) | H \rangle |_{m_\pi = 0}.$$

- These still describe high energy physics!



TMD convolution

- Remember the collinear convolution is

$$f_{qX}(\zeta) = \sum_H \int_\xi^1 \frac{d\alpha}{\alpha} q_H^{(0)}(\alpha) f_{HX}\left(\frac{\zeta}{\alpha}\right).$$

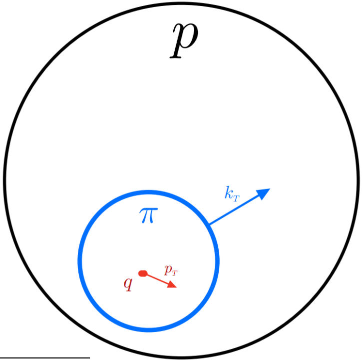
- The TMD analog is,

$$\tilde{f}_{qX}(\zeta, \mathbf{b}_T) = \sum_H \int_\xi^1 \frac{d\alpha}{\alpha} \tilde{q}_H^{(0)}(\alpha, \mathbf{b}_T) \tilde{f}_{HX}\left(\frac{\zeta}{\alpha}, \mathbf{b}_T\right).$$

TMD convolution

- Importantly, this conserves momentum,

$$f_{qX}(\xi, \mathbf{q}_T) = \sum_H \int d^2\mathbf{p}_T d^2\mathbf{q}_T \int_\xi^1 \frac{d\alpha}{\alpha} q_H^{(0)}(\alpha, \mathbf{p}_T) f_{HX}(\frac{\zeta}{\alpha}, \mathbf{k}_T) \times \delta^{(2)}(\mathbf{p}_T + \mathbf{k}_T - \mathbf{q}_T).$$



[4] F. He and P. Wang. PRD 100 (2019) 7, 074032

Low energy limit of the formalism

- The coefficients describe the high energy physics of the system ($p \geq \Lambda_\chi$).
  - $\Lambda_\chi \gg \Lambda_{QCD} \sim 200$ MeV.
- If the total transverse momentum is small, $\mathbf{q}_T \sim m_\pi \sim \Lambda_{QCD}$,

$$f^a_X(\zeta, \mathbf{q}_T) = \sum_H \int_\zeta^1 \frac{d\alpha}{\alpha} q_H^{(0)}(\alpha) f^a_{HX}\left(\frac{\zeta}{\alpha}, \mathbf{q}_T\right).$$

- i.e., matching coefficients have nothing to say about the transverse momentum dependence.

High energy limit of the formalism

- Likewise, if the total transverse momentum is large, $\mathbf{q}_T \gg m_\pi \sim \Lambda_{QCD}$, then we can expand in $\Lambda_{QCD}^2/\mathbf{q}_T^2$,

$$f_X^a(\zeta, \mathbf{q}_T) = \sum_{H,p} \int_\zeta^1 \frac{d\alpha}{\alpha} \int_\alpha^1 \frac{d\sigma}{\sigma} C_{pq}\left(\frac{\alpha}{\sigma}, \mathbf{q}_T\right) q_H^{(0)}(\sigma) f_{HX}^a\left(\frac{\zeta}{\alpha}\right) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{\mathbf{q}_T^2}\right).$$

- Like the usual operator product expansion (OPE) applied to TMDs.
  - C_{pq} is a perturbatively calculable coefficient.

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Calculating splitting functions

- Need to calculate the TMD hadronic splitting functions in χPT.
- The chiral operators are in terms of exponentials of pion fields.
  - Not practical!

$$\xi = \exp\left(i\frac{M}{f_\pi}\right), \quad M = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}.$$

- Need to expand in m_π/f_π or p_π/f_π.

Expanding operators

- Expand the nucleon operator up to $\mathcal{O}(m_\pi^2/f_\pi^2)$,

$$\mathcal{O}_N^a(b,0) = \frac{1}{2}\bar{N}(b)\not{n}\left[\xi^\dagger(b)\tau^a\xi(0) + \xi(b)\tau^a\xi^\dagger(0)\right]N(0),$$

$$\to \bar{N}(b)\not{n}\Bigg[\tau^a + \frac{1}{4f_\pi^2}\Big(\tau\cdot\pi(b)\tau^a\tau\cdot\pi(0)\Big) - \frac{1}{8f_\pi^2}\Big((\tau\cdot\pi(b))^2\tau^a + \tau^a(\tau\cdot\pi(0))^2\Big) + \mathcal{O}\left(\frac{p_\pi^3}{\Lambda_\chi^3}, \frac{m_\pi^3}{\Lambda_\chi^3}\right)\Bigg]N(0).$$

- Three types of terms!

Expanded Nucleon Operator

- One nucleon at each point.

$$\mathcal{O}_N^a(b,0) = \bar{N}(b)\not{n}\Big[\tau^a + \frac{1}{4f_\pi^2}\Big(\tau\cdot\pi(b)\tau^a\tau\cdot\pi(0)\Big) - \frac{1}{8f_\pi^2}\Big((\tau\cdot\pi(b))^2\tau^a + \tau^a(\tau\cdot\pi(0))^2\Big) + \mathcal{O}\Big(\frac{p_\pi^3}{\Lambda_\chi^3}, \frac{m_\pi^3}{\Lambda_\chi^3}\Big)\Big]N(0).$$

Expanded Nucleon Operator

- One nucleon and one pion at each point.

$$\mathcal{O}_N^a(b,0) = \bar{N}(b)\not{n}\Bigg[\tau^a + \frac{1}{4f_\pi^2}\Big(\tau\cdot\pi(b)\tau^a\tau\cdot\pi(0)\Big) - \frac{1}{8f_\pi^2}\Big((\tau\cdot\pi(b))^2\tau^a + \tau^a(\tau\cdot\pi(0))^2\Big) + \mathcal{O}\left(\frac{p_\pi^3}{\Lambda_\chi^3}, \frac{m_\pi^3}{\Lambda_\chi^3}\right)\Bigg]N(0).$$

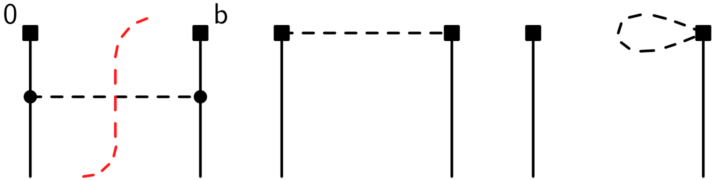
Expanded Nucleon Operator

- One nucleon at each point and two pions at b or 0.

$$\mathcal{O}_N^a(b,0) = \bar{N}(b)\not{n}\Bigg[\tau^a + \frac{1}{4f_\pi^2}\Big(\tau\cdot\pi(b)\tau^a\tau\cdot\pi(0)\Big) - \frac{1}{8f_\pi^2}\Big((\tau\cdot\pi(b))^2\tau^a + \tau^a(\tau\cdot\pi(0))^2\Big) + \mathcal{O}\left(\frac{p_\pi^3}{\Lambda_\chi^3}, \frac{m_\pi^3}{\Lambda_\chi^3}\right)\Bigg]N(0).$$

Nucelon Feynman Diagrams

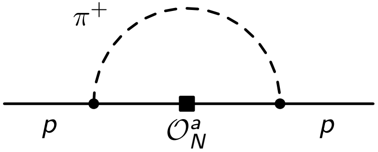
- These three terms generate different Feynman diagrams.



- Use Feynman rules to calculate the splitting functions.

Why draw the diagrams like this?

- Operator $\mathcal{O}_N^a$ has fields at b and 0.



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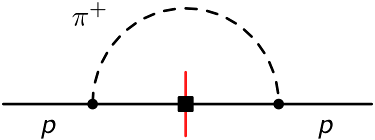

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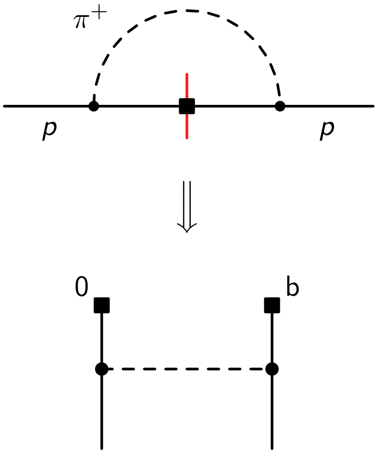


Nonlocal Feynman Rule

- Generically we have,

$$\int \frac{db^- d^2\mathbf{b}_T}{(2\pi)^3} e^{-i\beta P^+ b^-} e^{-i\mathbf{b}_T\cdot\mathbf{q}_T} \langle N| \mathcal{O}(b)\mathcal{O}(0) |N\rangle \,.$$

This is equivalent to

$$\langle N| \mathcal{O}(0)\delta^{(2)}(\hat{\mathcal{P}}_T - \mathbf{q}_T)\delta(n\cdot\hat{\mathcal{P}} - (1-\beta)P^+)\mathcal{O}(0) |N\rangle \,.$$

Where $\mathcal{P}$ projects out intermediate state momentum.

Nonlocal Feynman Rule

Additional Feynman Rule

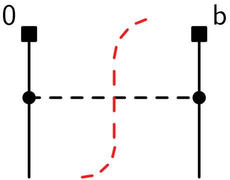
Add delta functions conserving momentum at the point b

$$\delta^{(2)}(\hat{\mathcal{P}}_T - \mathbf{q}_T)\delta(n \cdot \hat{\mathcal{P}} - (1-\beta)P^+)$$

Nucleon Splitting Function

- Using χPT Feynman rules, the first diagram gives

$$f^a_{NN}(\beta, \mathbf{q}_T) = \frac{g_A^2 I^a_{NN}}{4f_\pi^2} \int \frac{d^4k}{(2\pi)^4} \overline{u}(P) \not{k} \gamma_5 \frac{i(\not{P} - \not{k} + M)}{(P-k)^2 - M^2 + i\epsilon} \frac{i}{k^2 - m_\pi^2 + i\epsilon} \not{n}$$
$$\times \frac{i(\not{P} - \not{k} + M)}{(P-k)^2 - M^2 + i\epsilon} \gamma_5 \not{k} u(P) \delta^{(2)}(\mathbf{k}_T - \mathbf{q}_T) \delta(k^+ - (1-\beta)P^+).$$



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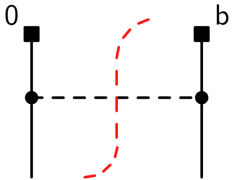
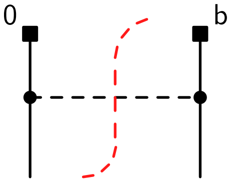
$$\times \frac{i(\not{P} - \not{k} + M)}{(P-k)^2 - M^2 + i\epsilon} \gamma_5 \not{k} u(P) \delta^{(2)}(\mathbf{k}_T - \mathbf{q}_T) \delta(k^+ - (1-\beta)P^+).$$



Nucleon Splitting Function Answer

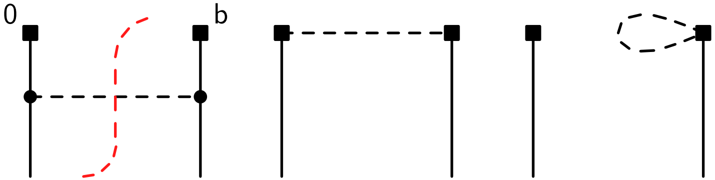
- Average over spins and evaluate the k^- integral:

$$f^a_{NN}(\beta, \mathbf{q}_T) = \frac{g_A^2 I^a_{NN}}{16\pi^3 M f_\pi^2} \left(\frac{M^2 m_\pi^2 \beta(\beta - 1)}{\mathbf{q}_T^2 + M^2(1-\beta)^2 + m_\pi^2 \beta} - \frac{1}{4} \log\left(\frac{\mathbf{q}_T^2 + m_\pi^2}{\mu^2} \right) \delta(1-\beta) \right)$$

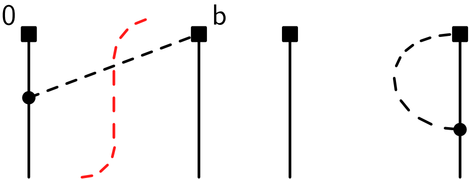


All Diagrams

$\mathcal{O}_N^a$:

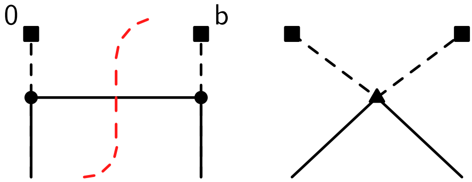


$\mathcal{O}_N^{A,a}$:



All Diagrams

$\mathcal{O}^a_\pi$:



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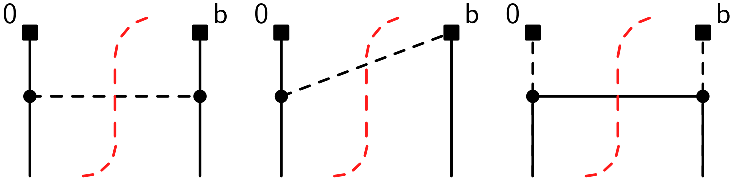
TMD diagrams

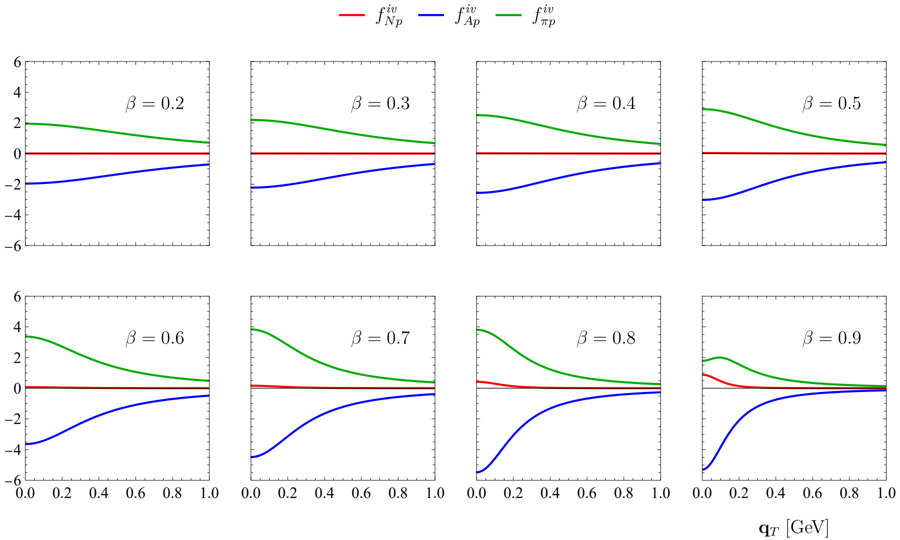
- Only the diagrams with "cuts" have nontrivial transverse momentum dependence.



Plotting

- Domain of plots:
  - Stay in "small" $\mathbf{q}_T$ region : $0 < \mathbf{q}_T < 1$ GeV.
  - Stay away from "lightfront zero-mode", $\delta(1-\beta)$, so plot $0.2 < \beta < 0.8$.
- Plot isovector splitting functions for $u - d$ distribution, i.e., take take $\tau^a = \tau^3 = \sigma_z$.
  - The isospin factors are $I_{Np}^{(iv)} = -1$, $I_{Np}^{A;(iv)} = 8$, and $I_{\pi p}^{(iv)} = 8$.

TMD hadronic Splitting Functions



Nontrivial Check

Ward Identity

Note: $f_p^{A,iv} + f_{\pi p}^{iv} + 4 f_{Np}^{iv} = 0.$

- A constraint required by the gauge invariance of the theory.
  - Our results satisfy this for all values of $\mathbf{k}_T$ and β.

Summary

- Reviewed how χPT can be applied to PDFs.
- Extended this framework to match TMD PDFs onto χPT.
- Calculated the TMD hadronic splitting functions at next-to-leading order in the chiral expansion.
- Studied the TMD hadronic splitting functions numerically.

Theoretical Outlook

- Many future directions.
  - Include Δ resonance.
  - Extend to SU(3).
  - Study polarized TMDs.
  - Study TMDs of other hadrons (like pions).
  - Include higher order corrections.
  - Study TMD fragmentation functions.

Phenomenological Outlook

- Want to compare TMD splitting function calculations with data!
  - Need pion (TMD?) PDFs, polarized proton PDFs, etc.
  - Want to explore validity of different "regimes".

Low energy

$$f_X^a(\zeta, \mathbf{q}_T) = \sum_H \int_\zeta^1 \frac{d\alpha}{\alpha} q_H^{(0)}(\alpha) f_{HX}^a\left(\frac{\zeta}{\alpha}, \mathbf{q}_T\right).$$

vs.

High Energy

$$f_X^a(\zeta, \mathbf{q}_T) = \sum_{H,p} \int_\zeta^1 \frac{d\alpha}{\alpha} \int_\alpha^1 \frac{d\sigma}{\sigma} C_{pq}\left(\frac{\alpha}{\sigma}, \mathbf{q}_T\right) q_H^{(0)}(\sigma) f_{HX}^a\left(\frac{\zeta}{\alpha}\right) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{\mathbf{q}_T^2}\right).$$