Time-ordered perturbation theory calculation of the axial anomaly in QED_{1+1} in LFD

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Motivation [1]

- It is often thought that the simple vacuum structure that comes with the light-front quantization is incompatible with the various nontrivial vacuum phenomena in low-energy hadron physics, for example, the axial anomaly.
- It is well known that regularization procedure in quantum field theory sometimes cannot preserve all of the classical symmetries, so in a quantized theory, the axial vector Ward identity has to be sacrificed in order to preserve the vector one, although on the classical level both are preserved.
- Axial anomaly in the Schwinger model may be understood as stemming from the divergent fermion loop integral, the same which gives the photon its mass.

[1] C.-R. Ji and S. -J. Rey, *Light front view of the axial anomaly*, Phys. Rev. **D53**, 5815 (1996)

Last time

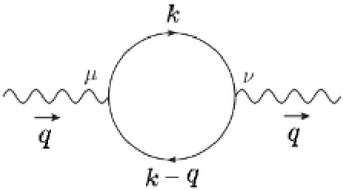
- Last time I talked about the time-ordered perturbation theory calculation of the axial anomaly in the interpolation form of dynamics.
- This time I'm going to talk about the light-front side of the same calculation.

Axial anomaly

- The axial anomaly in QED_2 stems from the two-point function

$$T^5_{\mu\nu}(x-y) = < T\left(J_\mu(x)J^5_\nu(y)\right) > \quad (1)$$

- The Feynman diagram relevant for the axial anomaly in two dimensions is shown below



A naively regularized result will not fulfill the correct Ward Identity (WI) whereas the proper dimensional or Pauli-Villars regularized Feynman integral will do so.

Dimensional regularization

- Before we go into the LFD calculation, let us review how this calculation is traditionally done in the manifestly covariant manner.
- Due to the relationship $\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu$ in two dimensions, it is sufficient to calculate $T^{\mu\nu}$ in order to obtain $T^5_{\mu\nu}$.

$$
\begin{aligned}
&T^{\mu\nu}(q) \\
=&ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{Tr\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{q}+m)\right]}{[k^2-m^2+i\epsilon]\,[(k-q)^2-m^2+i\epsilon]} \\
=&ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{(k_\alpha k_\beta - k_\alpha q_\beta)\,Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2\,Tr\left[\gamma^\mu\gamma^\nu\right]}{[k^2-m^2+i\epsilon]\,[(k-q)^2-m^2+i\epsilon]}. \\
=&ie^2 \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \\
&\times \frac{(k_\alpha k_\beta - k_\alpha q_\beta)\,Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2\,Tr\left[\gamma^\mu\gamma^\nu\right]}{\left[k^2+2(x-1)k\cdot q-(x-1)q^2-m^2\right]^2}.
\end{aligned}
\tag{2}
$$

Divergent integral

There are three terms in Eq. (2), and we shall call them as

$$I^{\mu\nu}_{(1)}(q) \equiv ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{k_\alpha k_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{D_1 D_2}, \tag{3}$$

$$I^{\mu\nu}_{(2)}(q) \equiv -ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{k_\alpha q_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{D_1 D_2}, \tag{4}$$

and

$$I^{\mu\nu}_{(3)}(q) \equiv ie^2 \int \frac{d^2k}{(2\pi)^2} \frac{m^2 Tr\left[\gamma^\mu\gamma^\nu\right]}{D_1 D_2}, \tag{5}$$

where $D_1 = k^2 - m^2 + i\epsilon$ and $D_2 = (k-q)^2 - m^2 + i\epsilon$. We see that $I^{\mu\nu}_{(1)}(q)$ is (naively) logarithmically divergent while $I^{\mu\nu}_{(2)}(q)$ and $I^{\mu\nu}_{(3)}(q)$ are not divergent.

Naive calculation

If one is being careless and shifts momentum normally in Eq. (2) as one does with Feynman parametrization, one easily gets

$$T^{\mu\nu}(q) = ie^2 \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \Big\{ [k-(x-1)q]_\alpha \, [k-(x-1)q-q]_\beta \times Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2 \, Tr\left[\gamma^\mu\gamma^\nu\right] \Big\} / (k^2-\Delta^2)^2, \tag{6}$$

where

$$\Delta^2 = x(x-1)q^2 + m^2. \tag{7}$$

Going to the Euclidean momentum,

$$T^{\mu\nu}(q) = \frac{e^2}{4\pi^2} \int_0^1 dx \int d^2k_E \left\{ \left[\frac{1}{2} g_{\alpha\beta} k_E^2 + x(x-1) q_\alpha q_\beta \right] \times Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2 \, Tr\left[\gamma^\mu\gamma^\nu\right] \right\} / (k_E^2+\Delta^2)^2. \tag{8}$$

Naive calculation, continued

This is

$$
\begin{aligned}
T^{\mu\nu}(q) = & -\frac{e^2}{4\pi}\int_0^1 dx \frac{x(x-1)q_\alpha q_\beta \, Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] + m^2 \, Tr\left[\gamma^\mu\gamma^\nu\right]}{x(x-1)q^2+m^2} \\
= & -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2+m^2}.
\end{aligned}
\tag{9}
$$

We see that the naive Feynman parametrization calculation misses the anomaly term which should be present. In a divergent integral, one has to be careful and shifting the momentum variable by a constant is not always allowed.

Correct dimensional regularization: reference

RIVISTA DEL NUOVO CIMENTO VOL. 16, N. 8 1993

Overview on the Anomaly and Schwinger Term in Two-Dimensional QED (*).

C. ADAM, R. A. BERTLMANN and P. HOFER
Institut für Theoretische Physik - Universität Wien

(ricevuto il 4 Marzo 1993)

APPENDIX

Feynman integrals in n dimensions:

(A.1) $$I_0 = \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2 + 2kp + M^2)^\alpha} = \frac{i(-\pi)^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)(M^2 - k^2)^{\alpha - n/2}},$$

(A.2) $$I_\mu = \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu}{(p^2 + 2kp + M^2)^\alpha} = -k_\mu I_0,$$

(A.3) $$I_{\mu\nu} = \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{(p^2 + 2kp + M^2)^\alpha} = I_0 \left(k_\mu k_\nu + \frac{1}{2} g_{\mu\nu} \frac{M^2 - k^2}{\alpha - n/2 - 1} \right).$$

We get

$$
\begin{aligned}
& I^{\mu\nu}_{(1)}(q) \\
=& ie^2 \int_0^1 dx \left(\frac{i(-\pi)}{(2\pi)^2(-(x-1)q^2 - m^2 - (x-1)^2 q^2)} \right) \\
& \times \left((x-1)^2 q_\alpha q_\beta + g_{\alpha\beta} \frac{-(x-1)q^2 - m^2 - (x-1)^2 q^2}{2-n} \right) \\
& \times \mathrm{Tr}\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta\right] \\
=& -\frac{e^2}{4\pi} \int_0^1 dx \frac{(x-1)^2 q_\alpha q_\beta \, Tr\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta\right]}{x(x-1)q^2 + m^2} \\
& + \frac{e^2}{4\pi} \int_0^1 dx \frac{1}{2-n} g_{\alpha\beta} \, Tr\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta\right].
\end{aligned}
\tag{10}
$$

According to the n-dimensional formula

$$
g_{\alpha\beta} \, Tr\left[\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta\right] = 2(2-n) g^{\mu\nu}. \tag{11}
$$

Thus,

$$
\begin{aligned}
& I^{\mu\nu}_{(1)}(q) \\
=& -\frac{e^2}{4\pi}\int_0^1 dx \frac{(x-1)^2 q_\alpha q_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{x(x-1)q^2+m^2} + \frac{e^2}{2\pi}g^{\mu\nu} \\
=& -\frac{e^2}{2\pi}\int_0^1 dx \frac{(x-1)^2(2q^\mu q^\nu - g^{\mu\nu}q^2)}{x(x-1)q^2+m^2} + \frac{e^2}{2\pi}g^{\mu\nu}.
\end{aligned} \tag{12}
$$

The other two terms can be calculated without difficulty as

$$
\begin{aligned}
I^{\mu\nu}_{(2)}(q) =& -\frac{e^2}{4\pi}\int_0^1 dx \frac{(x-1) q_\alpha q_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{x(x-1)q^2+m^2} \\
=& -\frac{e^2}{2\pi}\int_0^1 dx \frac{(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2)}{x(x-1)q^2+m^2}
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
I^{\mu\nu}_{(3)}(q) =& -\frac{e^2}{4\pi}\int_0^1 dx \frac{m^2 Tr\left[\gamma^\mu\gamma^\nu\right]}{x(x-1)q^2+m^2} \\
=& -\frac{e^2}{2\pi}\int_0^1 dx \frac{g^{\mu\nu} m^2}{x(x-1)q^2+m^2}.
\end{aligned} \tag{14}
$$

Combining the three pieces, we get

$$T^{\mu\nu}(q) = -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2 + m^2} + \frac{e^2}{2\pi}g^{\mu\nu}, \tag{15}$$

which satisfies gauge invariance

$$T^{\mu\nu}(q) = T(q^2)\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) \tag{16}$$

Then the vector—axial vector two point function can be obtained by $T_5^{\mu\nu} = \varepsilon^{\nu\lambda}T^\mu_\lambda$:

$$T_5^{\mu\nu}(q) = T(q^2)\varepsilon^{\nu\lambda}\left(\frac{q^\mu q_\lambda}{q^2} - g^\mu_\lambda\right). \tag{17}$$

It fulfils the vector current conservation,

$$q_\mu T_5^{\mu\nu}(q) = T(q^2)\varepsilon^{\nu\lambda}q_\mu\left(\frac{q^\mu q_\lambda}{q^2} - g^\mu_\lambda\right) = 0, \tag{18}$$

and the anomalous axial vector current,

$$q_\nu T_5^{\mu\nu}(q) = T(q^2)\varepsilon^{\nu\lambda}q_\nu\left(\frac{q^\mu q_\lambda}{q^2} - g^\mu_\lambda\right) = -T(q^2)q_\nu\varepsilon^{\nu\mu}. \tag{19}$$

Pauli-Villars regularization

Another, equally valid way of dealing with the divergent integral is Pauli-Villars regularization. It is defined as

$$T^{\mu\nu}_{PV}(q) = T^{\mu\nu}(q,m) - T^{\mu\nu}(q,\Lambda_{PV})|_{\Lambda_{PV}\to\infty}. \tag{20}$$

In the previous, naive calculation, we obtained

$$T^{\mu\nu}(q,m) = -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2 + m^2}. \tag{21}$$

Thus,

$$T^{\mu\nu}(q,\Lambda_{PV})|_{\Lambda_{PV}\to\infty} = -\frac{e^2}{2\pi}g^{\mu\nu}. \tag{22}$$

Therefore,

$$T^{\mu\nu}_{PV}(q) = -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2 + m^2} + \frac{e^2}{2\pi}g^{\mu\nu}, \tag{23}$$

same as the dimensionally regularized result.

LFD calculation of the axial anomaly Feynman diagram

For the LFD calculation, we use Pauli-Villars regularization

$$T^{\mu\nu}_{PV}(q) = T^{\mu\nu}(q,m) - T^{\mu\nu}(q,\Lambda_{PV})|_{\Lambda_{PV}\to\infty}. \tag{24}$$

Starting from Eqs. (3), (4) and (5), we write them in the light-front coordinates.

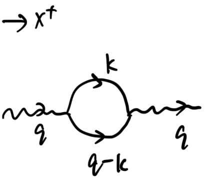
$$I^{\mu\nu}_{(1)}(q) = \frac{ie^2}{4\pi^2}\int dk^+ \int dk^- \\ \times \frac{k_\alpha k_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}; \tag{25}$$

$$I^{\mu\nu}_{(2)}(q) = -\frac{ie^2}{4\pi^2}\int dk^+ \int dk^- \\ \times \frac{k_\alpha q_\beta Tr\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}; \tag{26}$$

and

$$I^{\mu\nu}_{(3)}(q) = \frac{ie^2}{4\pi^2} \int dk^+ \int dk^- \times \frac{m^2\, Tr\left[\gamma^\mu \gamma^\nu\right]}{\left[2k^+k^- - m^2 + i\epsilon\right]\left[2(k-q)^+(k-q)^- - m^2 + i\epsilon\right]}. \tag{27}$$

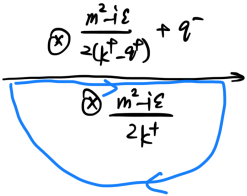
There is only one time-ordered diagram:

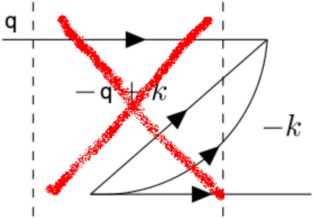


$q^+ > 0$

$k^+ > 0$

$\& \; q^+ - k^+ > 0$





The simplest term, $I^{\mu\nu}_{(3)}(q)$, can be calculated easily as the following.

$$\begin{aligned}
I^{\mu\nu}_{(3)}(q) =& \frac{ie^2}{4\pi^2} 2g^{\mu\nu} m^2 \int dk^+ \int dk^- \\
& \times \frac{1}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} \\
=& \frac{ie^2}{4\pi^2} 2g^{\mu\nu} m^2 (-2\pi i) q^+ \\
& \times \int_0^1 dx \frac{1}{2k^+ 2(k-q)^+ \left(\frac{m^2}{2k^+} - \frac{m^2}{2(k-q)^+} - \frac{q^2}{2q^+}\right)} \\
=& g^{\mu\nu} \frac{e^2 m^2}{2\pi} \int_0^1 dx \frac{1}{x(x-1)\left(\frac{m^2}{x} - \frac{m^2}{x-1} - q^2\right)} \\
=& g^{\mu\nu} \frac{e^2 m^2}{2\pi} \int_0^1 dx \frac{-1}{x(x-1)q^2 + m^2}, \qquad (28)
\end{aligned}$$

in agreement with the covariant calculation, Eq. (14).

Now, for the second term, $I^{\mu\nu}_{(2)}(q)$, we can calculate as follows, applying the trace of four gamma matrices formula.

$$I^{\mu\nu}_{(2)}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \times \frac{k^\mu q^\nu - g^{\mu\nu} k\cdot q + q^\mu k^\nu}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}, \tag{29}$$

and now we can write out and calculate each and every component of $I^{\mu\nu}_{(2)}(q)$ as the following

$$\begin{aligned} I^{++}_{(2)}(q) =& -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \\ & \times \frac{2k^+q^+}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} \\ =& \frac{e^2}{2\pi}(2q^+q^+)\int_0^1 dx \frac{x}{x(x-1)q^2 + m^2} \\ =& \frac{e^2}{2\pi}(2q^+q^+)\int_0^1 dx \frac{1-x}{x(x-1)q^2 + m^2}. \end{aligned} \tag{30}$$

Other components are

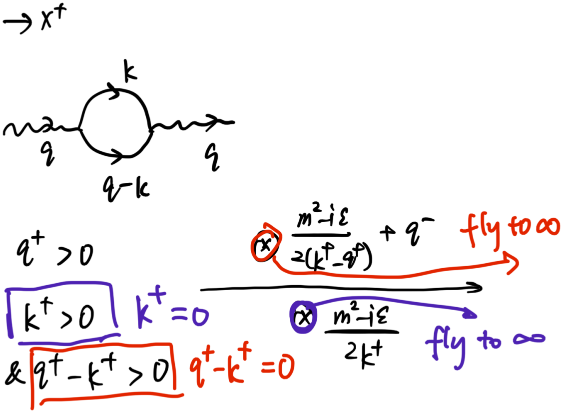
$$I_{(2)}^{+-}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \times \frac{k^+q^- - k\cdot q + q^+k^-}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} = 0; \tag{31}$$

$$I_{(2)}^{-+}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \times \frac{k^-q^+ - k\cdot q + q^-k^+}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} = 0; \tag{32}$$

and

$$I_{(2)}^{--}(q) = -\frac{ie^2}{2\pi^2}\int dk^+ \int dk^- \times \frac{2k^-q^-}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}. \tag{33}$$

Eq. (33) is exactly the calculation that was done in [2], where it was shown that a naive calculation missing the LF zero mode does not give the correct answer.



[2] B. Bakker, M. DeWitt, C. -R. Ji, and Y. Mishchenko, *Restoring the equivalence between the light-front and manifestly covariant formalisms*, Phys. Rev. **D72**, 076005 (2005)

In our case here, because of equal mass in the two propagators, accounting for the arc contribution is enough to obtain the correct answer. We get

$$
\begin{aligned}
I_{(2)}^{--}(q) =& -\frac{ie^2}{\pi^2} q^- \int dk^+ \int dk^- \\
& \times \frac{k^-}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} \\
=& -\frac{ie^2}{\pi^2} q^- (-2\pi i) q^+ \\
& \times \int_0^1 dx \frac{\frac{m^2}{2k^+}}{2k^+ 2(k-q)^+ \left(\frac{m^2}{2k^+} - \frac{m^2}{2(k-q)^+} - \frac{q^2}{2q^+}\right)} \\
& + \frac{ie^2}{\pi^2} q^- \int dk^+ \lim_{R\to\infty} \int_0^{-\pi} iRe^{i\theta} d\theta \frac{Re^{i\theta}}{2k^+ 2(k-q)^+ \left(Re^{i\theta}\right)^2} \\
=& \frac{e^2}{\pi} q^- q^- \frac{1}{q^2} \int_0^1 dx \left\{ \frac{m^2}{x\,[x(x-1)q^2 + m^2]} + \frac{1}{2x(x-1)} \right\}.
\end{aligned}
\tag{34}
$$

In which

$$
\frac{m^2}{x\,[x(x-1)q^2 + m^2]} = \frac{(1-x)q^2}{x(x-1)q^2 + m^2} + \frac{1}{x} \tag{35}
$$

and

$$
\begin{aligned}
&\int_0^1 dx \frac{1}{2x(x-1)} \\
=& -\frac{1}{2}\left(\int_0^1 dx \frac{1}{x} + \int_0^1 dx \frac{1}{1-x}\right) \\
=& -\frac{1}{2}\left(\int_0^1 dx \frac{1}{x} + \int_0^1 dx \frac{1}{x}\right) \\
=& -\int_0^1 dx \frac{1}{x}. \qquad (36)
\end{aligned}
$$

Thus, the answer is

$$
\begin{aligned}
I_{(2)}^{--}(q) &= \frac{e^2}{\pi} q^- q^- \frac{1}{q^2} \int_0^1 dx \left\{ \frac{(1-x)q^2}{x(x-1)q^2+m^2} + \frac{1}{x} - \frac{1}{x} \right\} \\
&= \frac{e^2}{\pi} q^- q^- \int_0^1 dx \frac{(1-x)}{x(x-1)q^2+m^2}. \qquad (37)
\end{aligned}
$$

Looking at all four components of the calculation of $I_{(2)}^{\mu\nu}(q)$, we see that the answer agrees with the result from the covariant way, Eq. (13).

Now, let us turn to the divergent term, $I^{\mu\nu}_{(1)}(q)$.

$$I^{\mu\nu}_{(1)}(q) = \frac{ie^2}{2\pi^2} \int dk^+ \int dk^- \times \frac{2k^\mu k^\nu - g^{\mu\nu} k^2}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}. \tag{38}$$

In terms of LF components, again the $++$ component can be calculated easily

$$\begin{aligned} I^{++}_{(1)}(q) &= \frac{ie^2}{2\pi^2} \int dk^+ \int dk^- \times \frac{2k^+k^+}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} \\ &= -\frac{e^2}{2\pi}(2q^+q^+) \int_0^1 dx \frac{x^2}{x(x-1)q^2 + m^2} \\ &= -\frac{e^2}{2\pi}(2q^+q^+) \int_0^1 dx \frac{(1-x)^2}{x(x-1)q^2 + m^2}; \end{aligned} \tag{39}$$

The $+-$ and $-+$ components are again zeros

$$I_{(1)}^{+-}(q) = \frac{ie^2}{2\pi^2} \int dk^+ \int dk^- \times \frac{2k^+k^- - k^2}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} = 0; \quad (40)$$

$$I_{(1)}^{-+}(q) = \frac{ie^2}{2\pi^2} \int dk^+ \int dk^- \times \frac{2k^-k^+ - k^2}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]} = 0; \quad (41)$$

and

$$I_{(1)}^{--}(q) = \frac{ie^2}{2\pi^2} \int dk^+ \int dk^- \times \frac{2k^-k^-}{[2k^+k^- - m^2 + i\epsilon]\,[2(k-q)^+(k-q)^- - m^2 + i\epsilon]}. \quad (42)$$

Eq. (42) again if calculated naively, will lead to incorrect result. Following the flying pole paper, we calculate Eq. (42) in a similar way.

$$I_{(1)}^{--}(q) = \frac{ie^2}{\pi^2}\int dk^+ \int dk^- \frac{(k^-)^2}{D_1 D_2}, \tag{43}$$

where

$$D_1 = 2k^+k^- - m^2 + i\epsilon, \tag{44}$$

$$D_2 = 2(k-q)^+(k-q)^- - m^2 + i\epsilon. \tag{45}$$

We will utilize the "asymptotic method" discussed in the flying pole paper. When $k^- \to \infty$ and $k^+ \to 0$,

$$V_{asy1} = \frac{ie^2}{\pi^2}\int dk^+ \int dk^- \frac{(k^-)^2}{D_1 2(-q^+)k^-} = -\frac{ie^2}{2\pi^2 q^+}\int dk^+ \int dk^- \frac{k^-}{D_1}. \tag{46}$$

Taking derivative with respect to m^2 gives

$$\frac{\partial}{\partial m^2} V_{asy1} = -\frac{ie^2}{2\pi^2 q^+}\int dk^+ \int_{-R}^{R} dk^- \frac{k^-}{D_1^2}$$
$$= -\frac{ie^2}{2\pi^2 q^+}\int dk^+ \left[\frac{-\frac{m^2}{2k^+k^- - m^2} + \ln\left(m^2 - 2k^+k^-\right)}{4(k^+)^2}\right]_{k^-=-R}^{R}$$
$$= -\frac{ie^2}{2\pi^2 q^+}\int dk^+ \frac{i\pi}{4(k^+)^2}, \tag{47}$$

where $k^+ \to 0$. When $k^- \to \infty$ and $k^+ \to q^+$,

$$V_{asy2} = \frac{ie^2}{\pi^2}\int dk^+ \int dk^- \frac{(k^-)^2}{D_2 2q^+ k^-} = \frac{ie^2}{2\pi^2 q^+}\int dk^+ \int dk^- \frac{k^-}{D_2}. \tag{48}$$

Taking derivative with respect to m^2 gives

$$\frac{\partial}{\partial m^2} V_{asy2} = \frac{ie^2}{2\pi^2 q^+}\int dk^+ \int_{-R}^{R} dk^- \frac{k^-}{D_2^2}$$
$$= \frac{ie^2}{2\pi^2 q^+}\int dk^+ \left[\frac{-\frac{2(k^+-q^+)q^- + m^2}{2(k^+-q^+)(k^- - q^-) - m^2} + \ln\left(m^2 - 2(k^+ - q^+)(k^- - q^-)\right)}{4(k^+ - q^+)^2}\right]_{k^-=-R}^{R}$$
$$= \frac{ie^2}{2\pi^2 q^+}\int dk^+ \frac{i\pi}{4(k^+ - q^+)^2}, \tag{49}$$

where $k^+ - q^+ \to 0$.

So actually,

$$V_{asy1} + V_{asy2} = 0. \tag{50}$$

$$\begin{aligned}
I^{--}_{(1)}(q) &= \left[I^{--}_{(1)}(q) - V_{asy1} - V_{asy2} \right] + V_{asy1} + V_{asy2} \\
&= \frac{ie^2}{\pi^2} \frac{1}{2q^+} \int dk^+ \int dk^- k^- \frac{2k^- q^+ + D_2 - D_1}{D_1 D_2} + V_{asy1} + V_{asy2} \\
&= \frac{ie^2}{\pi^2} \frac{1}{2q^+} \int dk^+ \int dk^- k^- \frac{2q^-(q^+ - k^+)}{D_1 D_2} + V_{asy1} + V_{asy2} \\
&= \frac{ie^2}{\pi^2} \int dk^+ \int dk^- \frac{k^- q^- (1 - k^+/q^+)}{D_1 D_2} + V_{asy1} + V_{asy2}.
\end{aligned} \tag{51}$$

Now the power of k^- has been reduced, and the k^- integration is exactly as in Eq. (34). Thus, we obtain

$$I^{--}_{(1)}(q) = -\frac{e^2}{\pi} q^- q^- \int_0^1 dx \frac{(1-x)^2}{x(x-1)q^2 + m^2}.$$

Combining the results from all four components calculation of $I^{\mu\nu}_{(1)}(q)$, and adding it to the results of $I^{\mu\nu}_{(2)}(q)$ and $I^{\mu\nu}_{(3)}(q)$, we get

$$T^{\mu\nu}(q,m) = -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2+m^2}. \tag{52}$$

Repeating the calculations for $m = \Lambda_{PV}$, we get

$$T^{\mu\nu}(q,\Lambda_{PV}) = -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}\Lambda_{PV}^2}{x(x-1)q^2+\Lambda_{PV}^2}. \tag{53}$$

Taking $\Lambda_{PV} \to \infty$,

$$T^{\mu\nu}(q,\Lambda_{PV})|_{\Lambda_{PV}\to\infty} = -\frac{e^2}{2\pi}g^{\mu\nu}. \tag{54}$$

Thus,

$$\begin{aligned} T^{\mu\nu}_{PV}(q) &= T^{\mu\nu}(q,m) - T^{\mu\nu}(q,\Lambda_{PV})|_{\Lambda_{PV}\to\infty} \\ &= -\frac{e^2}{2\pi}\int_0^1 dx \frac{x(x-1)(2q^\mu q^\nu - g^{\mu\nu}q^2) + g^{\mu\nu}m^2}{x(x-1)q^2+m^2} + \frac{e^2}{2\pi}g^{\mu\nu}. \end{aligned} \tag{55}$$

- From this we see that LFD calculation gives the same result agreeing with the manifestly covariant calculations.
- The anomaly can be understood as coming from the divergent integral in the $I^{--}_{(1)}(q)$ computation contributed from the LF zero mode $\sim \delta(k^+)$ and $\sim \delta(q^+ - k^+)$.
- Because of gauge invariance $q_\mu T^{\mu\nu} = 0$, we can relate e.g., $T^{+-} = -\frac{q_-}{q_+} T^{--}$. Thus, by properly calculating the T^{--} component, we obtain the correct axial anomaly term $\frac{e^2}{2\pi} g^{\mu\nu}$ which manifests only in the T^{+-} and T^{-+} components in LFD due to the metric, $g^{++} = g^{--} = 0$.

Thank you for your attention!