Interpolating De Sitter and Anti-De Sitter Spaces between IFD and LFD

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Outline

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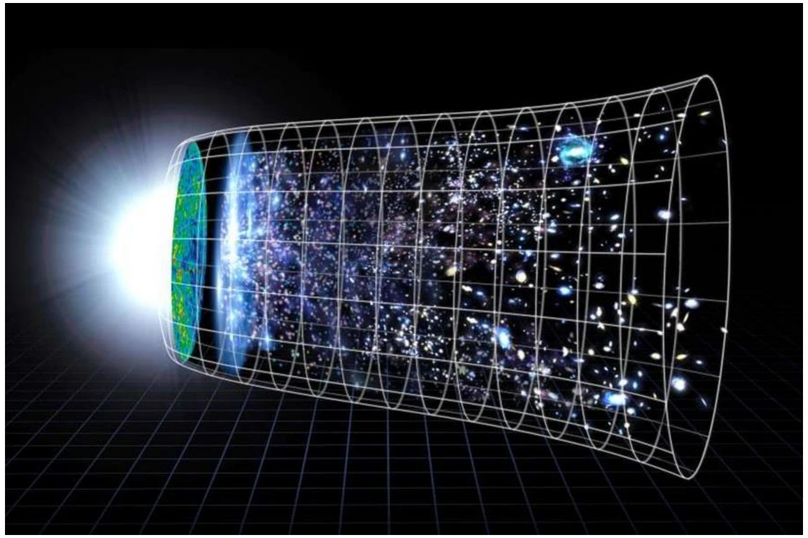
- ❑ Interpolating De sitter Space-Time
  - Geometrical properties
  - Interpolating De Sitter group S0(4,1)
- ❑ Interpolating Anti-De Sitter Space-Time
  - Geometrical properties
  - Interpolating Anti-De Sitter group S0(3,2)
- ❑ Contraction of Interpolating De Sitter and Anti-De Sitter Space-Times
  - Understanding lower dimensions as projections of higher dimensions
- ❑ Conclusion and Outlook

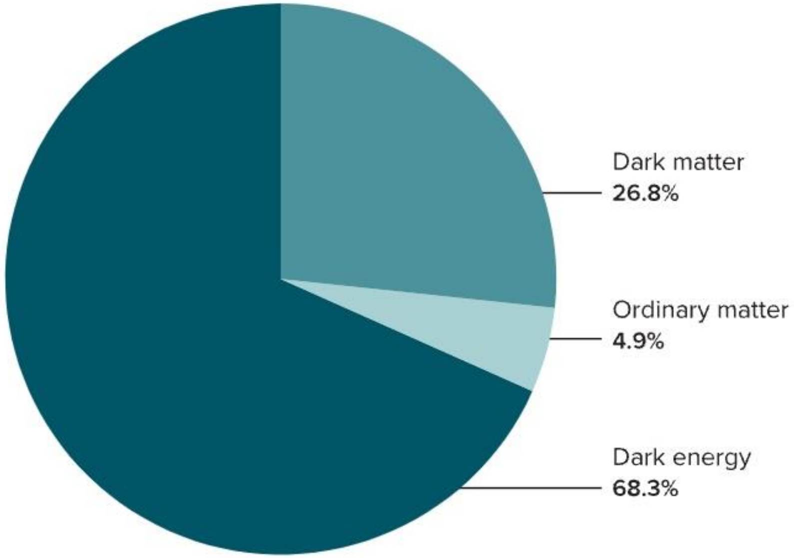
Introduction

Expanding Universe



Source: MIT technology review

Composition of the universe



Source: Adapted from ESA and the Planck collaboration (Knowable magazine)

- In 2011, Adam Riess, Brain P Schmid from High-Z supernova team, along with Saul Perlmutter from the supernova cosmological project were awarded the Nobel Prize in Physics for finding evidence that the expansion of the universe is accelerating.

Positive Vacuum energy density (Λ>**0**)

A. G. Riess et al., "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant", Astronomical Journal 116, 1009 (1998)

Introduction

Einstein Field Equation

$$\underbrace{R_{ab} - \frac{1}{2} R g_{ab}} - \Lambda\, g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

Curvature of the space time (G_{ab})

$T_{ab} = 0 \Rightarrow$ Vaccum

$\Lambda \neq 0 \Rightarrow$ Curvature of the space time is nonzero

$\Lambda =$ Vacuum energy density (Cosmological constant)

T_{ab} = Stress- energy tensor (Matter tensor)

G_{ab} = Einstein Tensor

Other famous solutions

- Friedmann-Lemaître-Robertson-Walker (FLRW) solution
- Schwarzschild Space Time..

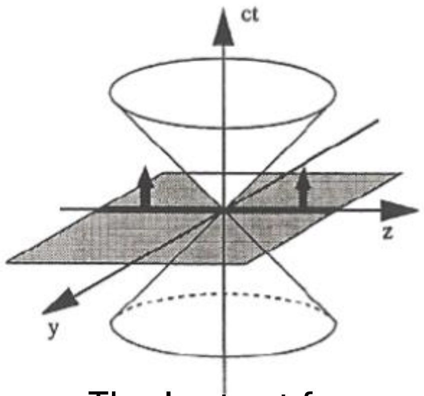
- De Sitter and anti-de Sitter spaces are the maximally symmetric vacuum solutions of Einstein's field equation with positive and negative vacuum energy densities ,respectively

de Sitter, W. "On the relativity of inertia: Remarks concerning Einstein's latest hypothesis" , *Proc. Kon. Ned. Acad. Wet.*, ***19****: 1217–1225 (1917)*

de Sitter, W. " On the curvature of space*" Proc. Kon. Ned. Acad. Wet.*, ***20****: 229–243 (1917)*

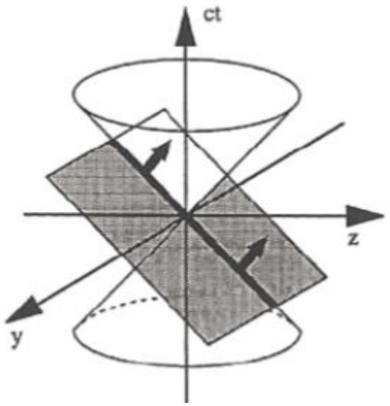
Interpolating between Instant form dynamic (IFD) and light-front dynamic (LFD)

Application in
- LQCD,
T-dept QFT,
IMF ..



The Instant form

1949

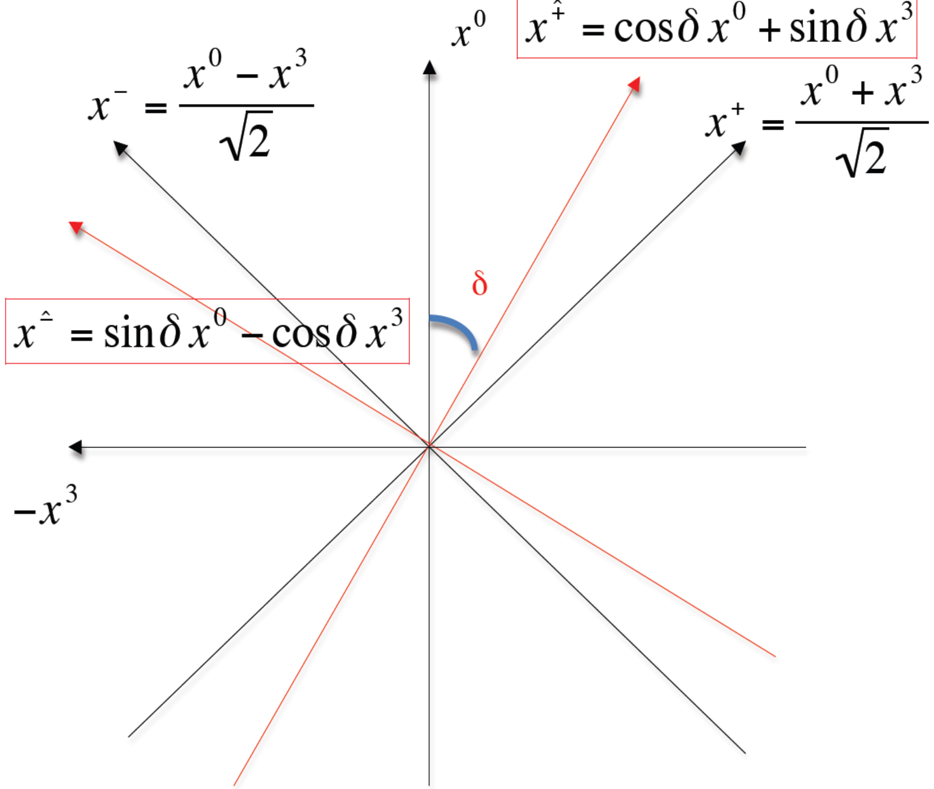


The light-front form

Application in -
PDFs,
DIS,
GPDs ...

IFD ($\delta = 0$)

LFD ($\delta = \frac{\pi}{4}$)



$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos(\delta) & 0 & 0 & \sin(\delta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\delta) & 0 & 0 & -\cos(\delta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolation space time matrix

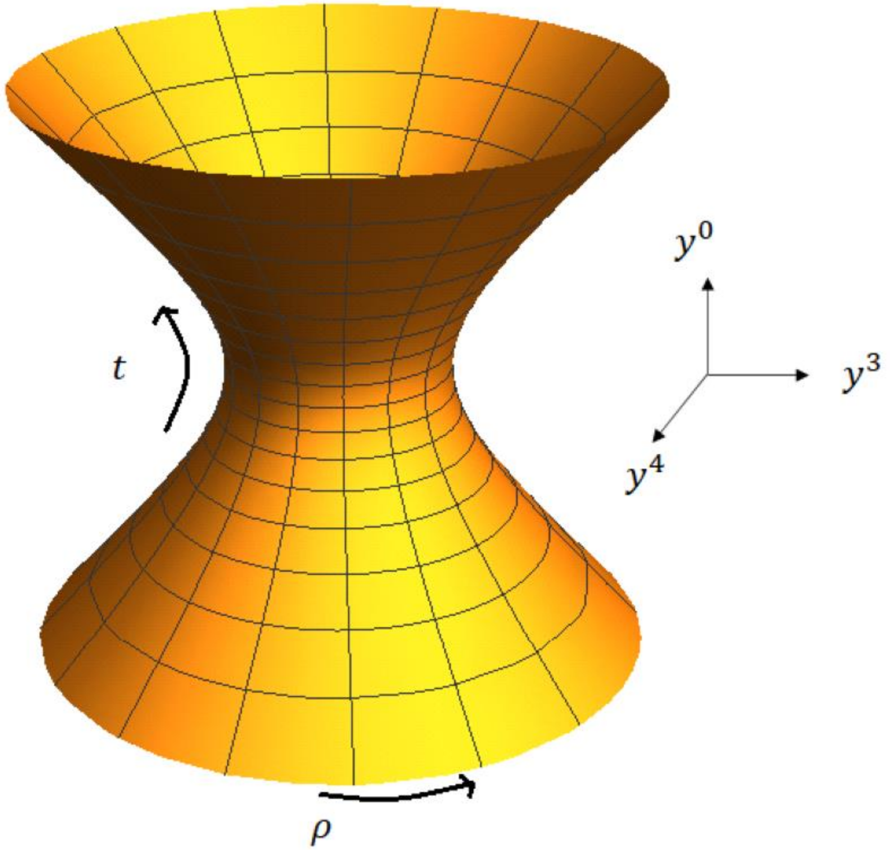
$$g_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{bmatrix}$$

$\mathbb{S} = Sin(2\delta)$
$\mathbb{C} = Cos(2\delta)$

- Relate IFD and LFD and show the whole landscape in between.
- Magnify the zero-mode by varying the δ parameter.
- Clarify any conceivable confusion between the Infinity momentum frame and LFD

De Sitter Space (Λ>0)

- can be visualized as the hyperboloid in flat five-dimensional space



- $y^0 = l_1\, Sinh\left(\frac{t}{l_1}\right)$
- $y^1 = l_1 Cosh\left(\frac{t}{l_1}\right)\, Sin(\rho)Sin(\theta)Cos(\varphi)$
- $y^2 = l_1 Cosh\left(\frac{t}{l_1}\right)\, Sin(\rho)Sin(\theta)Sin(\varphi)$
- $y^3 = l_1 Cosh\left(\frac{t}{l_1}\right)\, Sin(\rho)Cos(\theta)$
- $y^4 = l_1 Cosh\left(\frac{t}{l_1}\right)\, Cos(\rho)$

- De sitter space-time matrix tensor

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

- Space-time invariant $\quad y_0^2 - y_1^2 - y_2^2 - \underline{y_3^2 -} y_4^2 = -{l_1}^2 \qquad l_1 = \sqrt{\frac{3}{\Lambda_1}}$

l_1="De Sitter radius"

- Line element of the de sitter space

$$d{s_1}^2 = dt^2 - {l_1}^2 Cosh^2\left(\frac{t}{l_1}\right)[d\rho^2 + Sin^2\rho(d\theta^2 + Sin^2\theta d\varphi^2)]$$

- Two dimensions are suppressed in the figures

- Five- dimension interpolating transformation matrix

$$G^{\hat{\alpha}}_{\beta} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

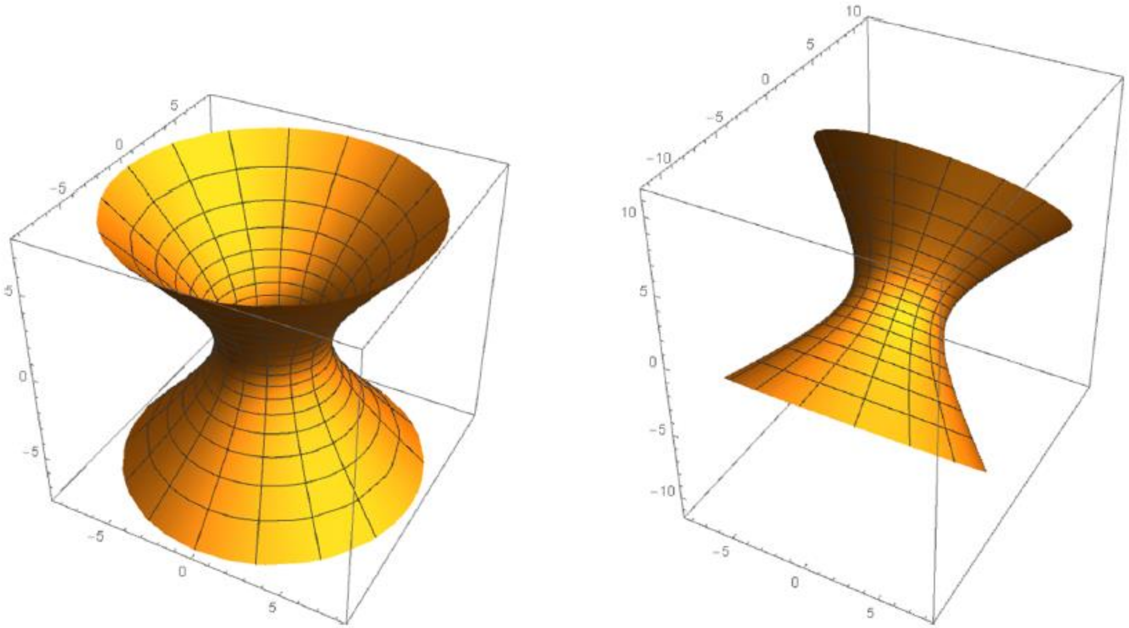
- Interpolating de Sitter space-time matrix tensor

$$\eta^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\hat{\alpha}\hat{\beta}}$$

- Interpolating de Sitter coordinates $y^{\hat{\alpha}} = G^{\hat{\alpha}}_{\beta} y^{\beta}$

$$\begin{aligned}
y^{\hat{+}} &= l_1 \sinh(t/l_1)\cos(\delta) + l_1 \cosh(t/l_1)\sin(\rho)\cos(\theta)\sin(\delta) \\
y^{\hat{1}} &= l_1 \cosh(t/l_1)\sin(\rho)\sin(\theta)\cos(\phi) \\
y^{\hat{2}} &= l_1 \cosh(t/l_1)\sin(\rho)\sin(\theta)\sin(\phi) \\
y^{\hat{-}} &= l_1 \sinh(t/l_1)\sin(\delta) - l_1 \cosh(t/l_1)\sin(\rho)\cos(\theta)\cos(\delta) \\
y^{\hat{4}} &= l_1 \cosh(t/l_1)\cos(\rho)
\end{aligned}$$

$$ds_1^2 = \eta_{\hat{\alpha}\hat{\beta}} dy^{\hat{\alpha}} dy^{\hat{\beta}} = (2dy^+ dy^- - dy_\perp^2 - dy_4^2)$$



(a) IFD De Sitter Space ($\theta = \phi = 0$) (b) LFD De Sitter Space ($\theta = \phi = 0$)

De Sitter Horizon

- De Sitter line element

$$ds_1{}^2 = dt^2 - l_1{}^2 Cosh^2\left(\frac{t}{l_1}\right) d\rho^2$$

$$ds_1{}^2 = 0 \Rightarrow \qquad \rho - \rho_0 = \pm ArcSin\left(Tanh\left(\frac{t}{l_1}\right)\right)$$

Solution-1

$$\rho - \rho_0 = ArcSin\left(Tanh\left(\frac{t}{l_1}\right)\right) \text{ and } \rho_0 = \frac{\pi}{2}$$

Solution-2

$$\rho - \rho_0 = -ArcSin\left(Tanh\left(\frac{t}{l_1}\right)\right) \text{ and } \rho_0 = -\frac{\pi}{2}$$

De Sitter interpolating time

$$y^{\hat{+}} = l_1 Sinh\left(\frac{t}{l_1}\right) cos\delta + l_1 Cosh\left(\frac{t}{l_1}\right) \sin(\rho)\sin(\delta)$$

De Sitter light-front time: $y^+ = 0 \Rightarrow$

$$\rho - \rho_0 = -ArcSin\left(Tanh\left(\frac{t}{l_1}\right)\right)$$

$t = -\infty \qquad \rho(-\infty) = 0$

$t = \infty \qquad \rho(\infty) = \pi$

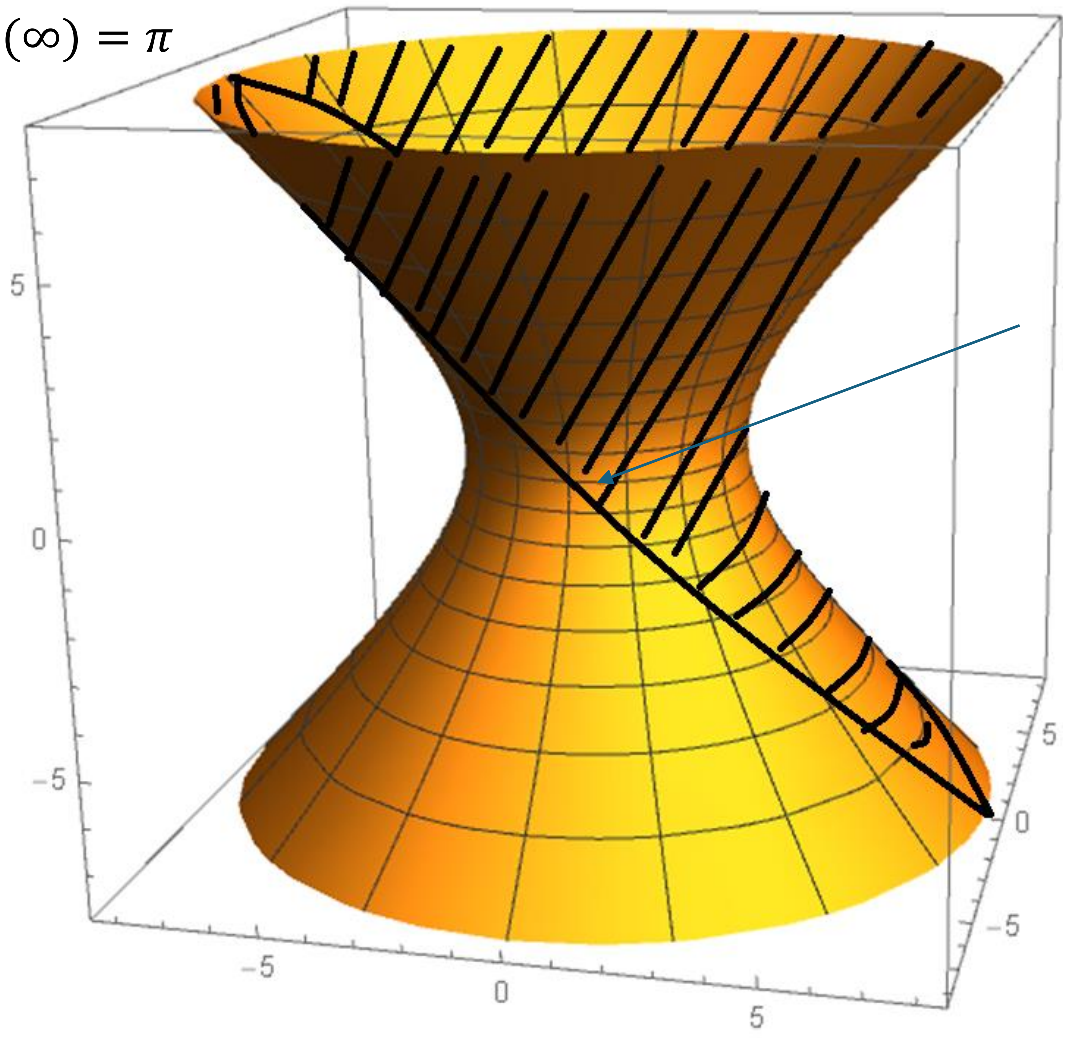
De Sitter Horizon

$t = -\infty$ $\rho(-\infty) = 0$

$t = \infty$ $\rho(\infty) = \pi$

It takes infinity amount of time for light to get one end of the circle to the other end of the circle

$\rho(\infty) = \pi$



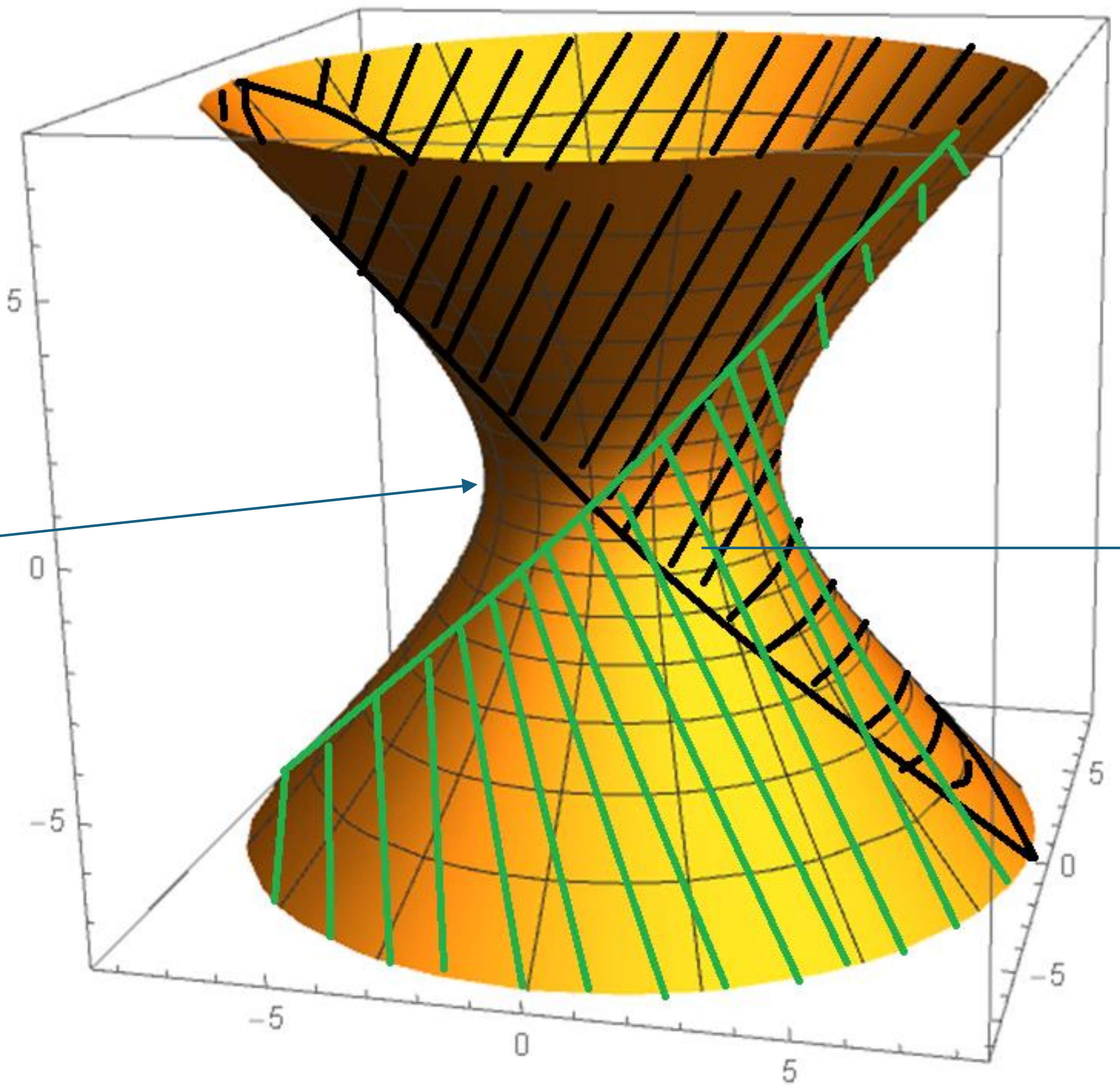
De sitter horizon

$\rho(-\infty) = 0$

9

De Sitter Horizon

Region one can send and receive signals



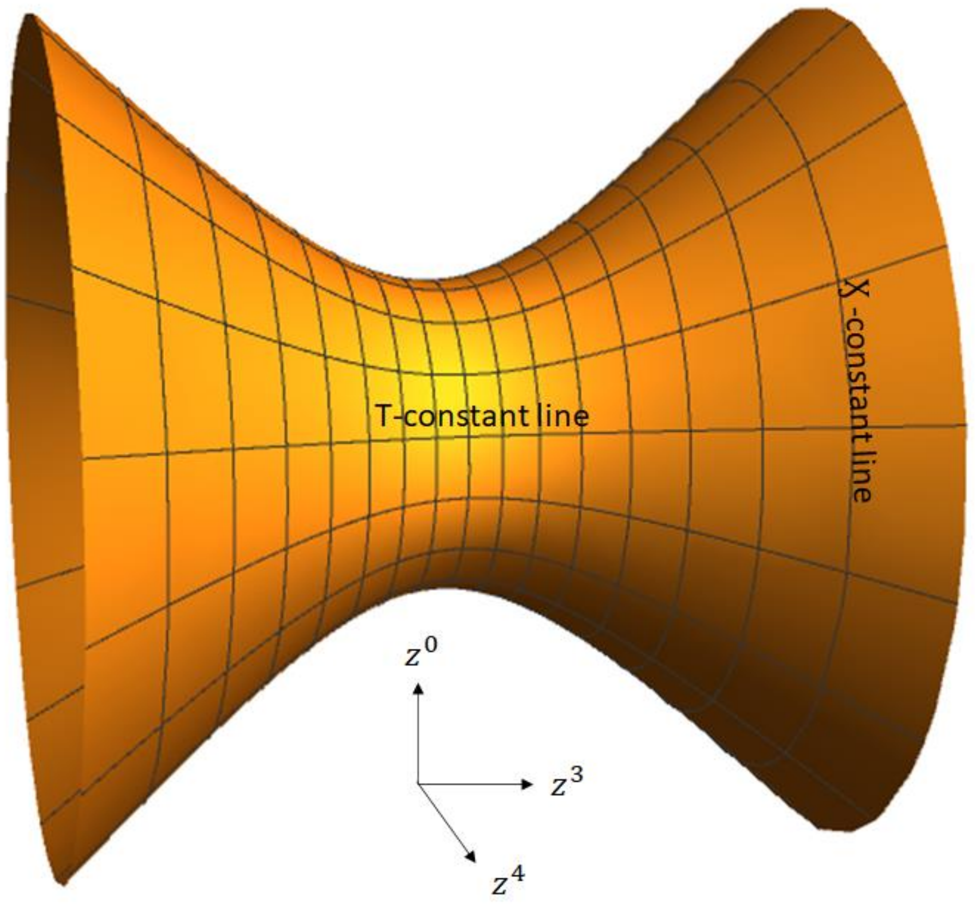
The region neither a single can be sent nor can be received

Static patch

Causal diamond

Anti-de Sitter Space ($\Lambda < 0$)

- can be visualized as the hyperboloid in flat five-dimensional space



- $z^0 = l_2 Cosh(\text{Ꭓ}) Sin\left(\frac{T}{l_2}\right)$
- $z^1 = l_2 Sinh(\text{Ꭓ})\ Sin(\theta) Cos(\varphi)$
- $z^2 = l_2 Sinh(\text{Ꭓ}) Sin(\theta) Sin(\varphi)$
- $z^3 = l_2 Sinh(\text{Ꭓ})\ Cos(\theta)$
- $z^4 = l_2 Cosh(\text{Ꭓ}) Cos\left(\frac{T}{l_2}\right)$

l_2="Anti de Sitter radius"

Space-time invariant

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 + z_4^2 = {l_2}^2$$

$$l_2 = \sqrt{\frac{-3}{\Lambda_2}}$$

$$d{s_2}^2 = Cosh^2\text{Ꭓ} dT^2 - {l_2}^2[d\text{Ꭓ}^2 + Sinh^2\text{Ꭓ}(d\theta^2 + Sin^2\theta d\varphi^2)]$$

- T is periodic
- Closed timelike worldlines.

- ❑ Ads conformal structure unwrapped the closed timelike worldline.

Interpolating Anti-de Sitter space-time between IFD and LFD

- Interpolating de Sitter coordinates

$$z^{\widehat{\alpha}} = G^{\widehat{\alpha}}_{\beta} z^{\beta}$$

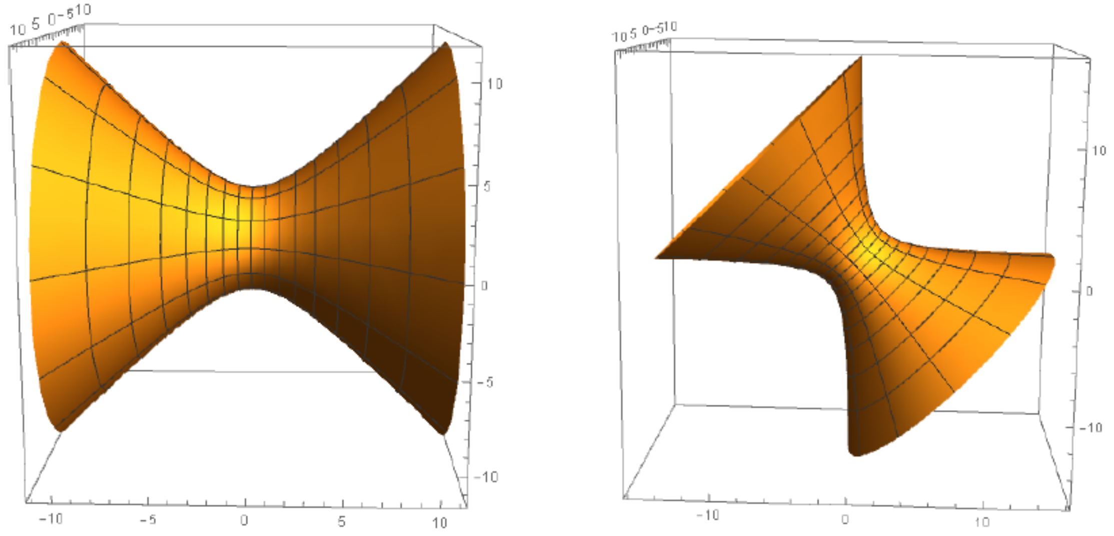
$$\begin{aligned}
z^{\hat{+}} &= l_2 \cosh(\chi) \sin(T/l_2) \cos(\delta) + l_2 \sinh(\chi) \cos(\theta) \sin(\delta) \\
z^{\hat{1}} &= l_2 \sinh(\chi) \sin(\theta) \cos(\phi) \\
z^{\hat{2}} &= l_2 \sinh(\chi) \sin(\theta) \sin(\phi) \\
z^{\hat{-}} &= l_2 \cosh(\chi) \sin(T/l_2) \sin(\delta) - l_2 \sinh(\chi) \cos(\theta) \cos(\delta) \\
z^{\hat{4}} &= l_2 \cosh(\chi) \cos(T/l_2)
\end{aligned}$$

$$l_2 = \sqrt{\frac{-3}{\Lambda_2}}$$

$$ds_2^2 = g_{\hat{\alpha}\hat{\beta}} dz^{\hat{\alpha}} dz^{\hat{\beta}} = (2dz^{+}dz^{-} - dz_{\perp}^2 + dz_4^2)$$

- Interpolating de Sitter space-time matrix tensor

$$g^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = g_{\hat{\alpha}\hat{\beta}}$$



(a) IFD ant-De Sitter Space ($\theta = \phi = 0$)

(b) LFD anti-De Sitter Space ($\theta = \phi = 0$)

De Sitter Group : $SO(4, 1)$

Ant-De Sitter Group : $SO(3, 2)$

- Ten homogenous transformations

Matrix of operators

$$R^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix}$$

De sitter space-time matrix tensor

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Infinitesimal operators of the group $= m_{ab}$

$m_{ab} \Rightarrow$ 6 cyclic operators
4 hyperbolic operators

Lie-Algebra

$$[R^{\alpha\beta}, R^{\gamma\delta}] = i(\eta^{\beta\gamma} R^{\alpha\delta} - \eta^{\beta\delta} R^{\alpha\gamma} - \eta^{\alpha\gamma} R^{\beta\delta} + \eta^{\alpha\delta} R^{\beta\gamma})$$

De Sitter translation operators

$$\alpha = \beta = \gamma = \delta = 0, 1, 2, 3, 4$$

$$\Gamma^\mu = R^{4\mu},$$

$$\begin{aligned} [R^{\mu\nu}, R^{\rho\lambda}] &= i(\eta^{\nu\rho} R^{\mu\lambda} - \eta^{\nu\lambda} R^{\mu\rho} - \eta^{\mu\rho} R^{\nu\lambda} + \eta^{\mu\lambda} R^{\nu\rho}) \\ [R^{\mu\nu}, \Gamma^\rho] &= i(\eta^{\nu\rho}\Gamma^\mu - \eta^{\mu\rho}\Gamma^\nu) \\ [\Gamma^\mu, \Gamma^\nu] &= i R^{\mu\nu} \end{aligned}$$

$$\mu = \nu = \lambda = \rho = 0, 1, 2, 3$$

Matrix of operators

$$J^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Pi^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Pi^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Pi^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Pi^3 \\ \Pi^0 & \Pi^1 & \Pi^2 & \Pi^3 & 0 \end{pmatrix}$$

Anti-de sitter space-time matrix tensor

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$m_{ab} \Rightarrow$ 4 cyclic operators
6 hyperbolic operators

Lie-Algebra

$$[J^{\alpha\beta}, J^{\gamma\delta}] = i(g^{\beta\gamma} J^{\alpha\delta} - g^{\beta\delta} J^{\alpha\gamma} - g^{\alpha\gamma} J^{\beta\delta} + g^{\alpha\delta} J^{\beta\gamma})$$

Anti de Sitter translation operators

$$\Pi^\mu = J^{4\mu}$$

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\lambda}] &= i(g^{\nu\rho} J^{\mu\lambda} - g^{\nu\lambda} J^{\mu\rho} - g^{\mu\rho} J^{\nu\lambda} + g^{\mu\lambda} J^{\nu\rho}) \\ [J^{\mu\nu}, \Pi^\rho] &= i(g^{\nu\rho}\Pi^\mu - g^{\mu\rho}\Pi^\nu) \\ [\Pi^\mu, \Pi^\nu] &= -\, i J^{\mu\nu} \end{aligned}$$

- Translation operators in five-dimensional spaces do not commute with each other.

Interpolating De Sitter Group ($SO(4,1)$)

- Matrix of interpolating de Sitter operators

$$R^{\hat{\alpha}\hat{\beta}} = G^{\hat{\alpha}}_{\gamma} R^{\gamma\delta} G^{\hat{\beta}}_{\delta} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^{1} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^{2} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{+}} & \Gamma^{1} & \Gamma^{2} & \Gamma^{\hat{-}} & 0 \end{pmatrix}$$

$$\Gamma^{\hat{+}} = \Gamma^0 \cos\delta + \Gamma^3 \sin\delta \ , \ \Gamma^{\hat{-}} = \Gamma^0 \sin\delta - \Gamma^3 \cos\delta$$

$$E^{\hat{1}} = J^2 \sin\delta + K^1 \cos\delta$$
$$E^{\hat{2}} = -J^1 \sin\delta + K^2 \cos\delta$$
$$F^{\hat{1}} = K^1 \sin\delta - J^2 \cos\delta$$
$$F^{\hat{2}} = K^3 \sin\delta + J^1 \cos\delta$$

Lie-Algebra

$$[R^{\hat{\alpha}\hat{\beta}}, R^{\hat{\gamma}\hat{\delta}}] = i(\eta^{\hat{\beta}\hat{\gamma}} R^{\hat{\alpha}\hat{\delta}} - \eta^{\hat{\beta}\hat{\delta}} R^{\hat{\alpha}\hat{\gamma}} - \eta^{\hat{\alpha}\hat{\gamma}} R^{\hat{\beta}\hat{\delta}} + \eta^{\hat{\alpha}\hat{\delta}} R^{\hat{\beta}\hat{\gamma}})$$

$$\hat{\alpha} = \hat{\beta} = \hat{\gamma} = \hat{\delta} = \hat{+}, \hat{1}, \hat{2}, \hat{-}, \hat{4}$$

Interpolating de Sitter translation operators $\Gamma^{\hat{\mu}} = R^{4\hat{\mu}}$

$$\begin{aligned} [R^{\hat{\mu}\hat{\nu}}, R^{\hat{\rho}\hat{\lambda}}] &= i(\eta^{\hat{\nu}\hat{\rho}} R^{\hat{\mu}\hat{\lambda}} - \eta^{\hat{\nu}\hat{\lambda}} R^{\hat{\mu}\hat{\rho}} - \eta^{\hat{\mu}\hat{\rho}} R^{\hat{\nu}\hat{\lambda}} + \eta^{\hat{\mu}\hat{\lambda}} R^{\hat{\nu}\hat{\rho}}) \\ [R^{\hat{\mu}\hat{\nu}}, \Gamma^{\hat{\rho}}] &= i(\eta^{\hat{\nu}\hat{\rho}} \Gamma^{\hat{\mu}} - \eta^{\hat{\mu}\hat{\rho}} \Gamma^{\hat{\nu}}) \\ [\Gamma^{\hat{\mu}}, \Gamma^{\hat{\nu}}] &= i R^{\hat{\mu}\hat{\nu}} \end{aligned}$$

$$\hat{\mu} = \hat{\nu} = \hat{\rho} = \hat{\lambda} = \hat{+}, \hat{1}, \hat{2}, \hat{-}$$

Interpolating Anti-de Sitter Group ($SO(3,2)$)

- Matrix of interpolating anti-de Sitter operators

$$J^{\hat{\alpha}\hat{\beta}} = G^{\hat{\alpha}}_{\gamma} J^{\gamma\delta} G^{\hat{\beta}}_{\delta} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^{1} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^{2} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{+}} & \Pi^{1} & \Pi^{2} & \Pi^{\hat{-}} & 0 \end{pmatrix}$$

$$\Pi^{\hat{+}} = \Pi^0 \cos\delta + \Pi^3 \sin\delta \ , \ \Pi^{\hat{-}} = \Pi^0 \sin\delta - \Pi^3 \cos\delta$$

Lie-Algebra

$$[J^{\hat{\alpha}\hat{\beta}}, J^{\hat{\gamma}\hat{\delta}}] = i(g^{\hat{\beta}\hat{\gamma}} R^{\hat{\alpha}\hat{\delta}} - g^{\hat{\beta}\hat{\delta}} J^{\hat{\alpha}\hat{\gamma}} - g^{\hat{\alpha}\hat{\gamma}} J^{\hat{\beta}\hat{\delta}} + g^{\hat{\alpha}\hat{\delta}} J^{\hat{\beta}\hat{\gamma}})$$

Interpolating anti-de Sitter translation operators $\Pi^{\hat{\mu}} = J^{4\hat{\mu}}$

$$\begin{aligned} [J^{\hat{\mu}\hat{\nu}}, J^{\hat{\rho}\hat{\lambda}}] &= i(g^{\hat{\nu}\hat{\rho}} J^{\hat{\mu}\hat{\lambda}} - g^{\hat{\nu}\hat{\lambda}} J^{\hat{\mu}\hat{\rho}} - g^{\hat{\mu}\hat{\rho}} J^{\hat{\nu}\hat{\lambda}} + g^{\hat{\mu}\hat{\lambda}} J^{\hat{\nu}\hat{\rho}}) \\ [J^{\hat{\mu}\hat{\nu}}, \Pi^{\hat{\rho}}] &= i(g^{\hat{\nu}\hat{\rho}} \Pi^{\hat{\mu}} - g^{\hat{\mu}\hat{\rho}} \Pi^{\hat{\nu}}) \\ [\Pi^{\hat{\mu}}, \Pi^{\hat{\nu}}] &= -i J^{\hat{\mu}\hat{\nu}} \end{aligned}$$

Interpolating De Sitter Group ($SO(4,1)$)

Interpolating Anti-de Sitter Group ($SO(3,2)$)

$$
\begin{aligned}
\left[\Gamma^{\hat{+}},\Gamma^{\hat{1}}\right] &= iE^{\hat{1}} \\
\left[\Gamma^{\hat{+}},\Gamma^{\hat{2}}\right] &= iE^{\hat{2}} \\
\left[\Gamma^{\hat{+}},\Gamma^{\hat{-}}\right] &= -iK^{3} \\
\left[\Gamma^{\hat{1}},\Gamma^{\hat{2}}\right] &= iJ^{3} \\
\left[\Gamma^{\hat{1}},\Gamma^{\hat{-}}\right] &= -iF^{\hat{1}} \\
\left[\Gamma^{\hat{2}},\Gamma^{\hat{-}}\right] &= -iF^{\hat{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Gamma^{\hat{+}},K^{3}\right] &= -i(\mathbb{C}\Gamma^{\hat{-}} - \mathbb{S}\Gamma^{\hat{+}}) \\
\left[\Gamma^{\hat{+}},E^{\hat{1}}\right] &= i\mathbb{C}\Gamma^{\hat{1}} \\
\left[\Gamma^{\hat{+}},E^{\hat{2}}\right] &= i\mathbb{C}\Gamma^{\hat{2}} \\
\left[\Gamma^{\hat{+}},J^{3}\right] &= 0 \\
\left[\Gamma^{\hat{+}},F^{\hat{1}}\right] &= i\mathbb{S}\Gamma^{\hat{1}} \\
\left[\Gamma^{\hat{+}},F^{\hat{2}}\right] &= i\mathbb{S}\Gamma^{\hat{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Gamma^{\hat{1}},K^{3}\right] &= 0 \\
\left[\Gamma^{\hat{1}},E^{\hat{1}}\right] &= i\Gamma^{\hat{+}} \\
\left[\Gamma^{\hat{1}},E^{\hat{2}}\right] &= 0 \\
\left[\Gamma^{\hat{1}},J^{3}\right] &= -i\Gamma^{\hat{2}} \\
\left[\Gamma^{\hat{1}},F^{\hat{1}}\right] &= i\Gamma^{\hat{-}} \\
\left[\Gamma^{\hat{1}},F^{\hat{2}}\right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left[\Gamma^{\hat{2}},K^{3}\right] &= 0 \\
\left[\Gamma^{\hat{2}},E^{\hat{1}}\right] &= 0 \\
\left[\Gamma^{\hat{2}},E^{\hat{2}}\right] &= i\Gamma^{\hat{+}} \\
\left[\Gamma^{\hat{2}},J^{3}\right] &= i\Gamma^{\hat{1}} \\
\left[\Gamma^{\hat{2}},F^{\hat{1}}\right] &= 0 \\
\left[\Gamma^{\hat{2}},F^{\hat{2}}\right] &= i\Gamma^{\hat{-}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Gamma^{\hat{-}},K^{3}\right] &= -i(\mathbb{S}\Gamma^{\hat{-}} + \mathbb{C}\Gamma^{\hat{+}}) \\
\left[\Gamma^{\hat{-}}E^{\hat{1}}\right] &= i\mathbb{S}\Gamma^{\hat{1}} \\
\left[\Gamma^{\hat{-}},E^{\hat{2}}\right] &= i\mathbb{S}\Gamma^{\hat{2}} \\
\left[\Gamma^{\hat{-}},J^{3}\right] &= 0 \\
\left[\Gamma^{\hat{-}},F^{\hat{1}}\right] &= -i\mathbb{C}\Gamma^{\hat{1}} \\
\left[\Gamma^{\hat{-}},F^{\hat{2}}\right] &= -i\mathbb{C}\Gamma^{\hat{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Pi^{\hat{+}}\Pi^{\hat{1}}\right] &= -iE^{\hat{1}} \\
\left[\Pi^{\hat{+}},\Pi^{\hat{2}}\right] &= -iE^{\hat{2}} \\
\left[\Pi^{\hat{+}},\Pi^{\hat{-}}\right] &= iK^{3} \\
\left[\Pi^{\hat{1}},\Pi^{\hat{2}}\right] &= -iJ^{3} \\
\left[\Pi^{\hat{1}},\Pi^{\hat{-}}\right] &= iF^{\hat{1}} \\
\left[\Pi^{\hat{2}},\Pi^{\hat{-}}\right] &= iF^{\hat{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Pi^{\hat{+}},K^{3}\right] &= -i(\mathbb{C}\Pi^{\hat{-}} - \mathbb{S}\Pi^{\hat{+}}) \\
\left[\Pi^{\hat{+}},E^{\hat{1}}\right] &= i\mathbb{C}\Pi^{\hat{1}} \\
\left[\Pi^{\hat{+}},E^{\hat{2}}\right] &= i\mathbb{C}\Pi^{\hat{2}} \\
\left[\Pi^{\hat{+}},J^{3}\right] &= 0 \\
\left[\Pi^{\hat{+}},F^{\hat{1}}\right] &= i\mathbb{S}\Pi^{\hat{1}} \\
\left[\Pi^{\hat{+}},F^{\hat{2}}\right] &= i\mathbb{S}\Pi^{\hat{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Pi^{\hat{1}},K^{3}\right] &= 0 \\
\left[\Pi^{\hat{1}},E^{\hat{1}}\right] &= i\Pi^{\hat{+}} \\
\left[\Pi^{\hat{1}},E^{\hat{2}}\right] &= 0 \\
\left[\Pi^{\hat{1}},J^{3}\right] &= -i\Pi^{\hat{2}} \\
\left[\Pi^{\hat{1}},F^{\hat{1}}\right] &= i\Pi^{\hat{-}} \\
\left[\Pi^{\hat{1}},F^{\hat{2}}\right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left[\Pi^{\hat{2}},K^{3}\right] &= 0 \\
\left[\Pi^{\hat{2}},E^{\hat{1}}\right] &= 0 \\
\left[\Pi^{\hat{2}},E^{\hat{2}}\right] &= i\Pi^{\hat{+}} \\
\left[\Pi^{\hat{2}},J^{3}\right] &= i\Pi^{\hat{1}} \\
\left[\Pi^{\hat{2}},F^{\hat{1}}\right] &= 0 \\
\left[\Pi^{\hat{2}},F^{\hat{2}}\right] &= i\Pi^{\hat{-}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\Pi^{\hat{-}},K^{3}\right] &= -i(\mathbb{S}\Pi^{\hat{-}} + \mathbb{C}\Pi^{\hat{+}}) \\
\left[\Pi^{\hat{-}}E^{\hat{1}}\right] &= i\mathbb{S}\Pi^{\hat{1}} \\
\left[\Pi^{\hat{-}},E^{\hat{2}}\right] &= i\mathbb{S}\Pi^{\hat{2}} \\
\left[\Pi^{\hat{-}},J^{3}\right] &= 0 \\
\left[\Pi^{\hat{-}},F^{\hat{1}}\right] &= -i\mathbb{C}\Pi^{\hat{1}} \\
\left[\Pi^{\hat{-}},F^{\hat{2}}\right] &= -i\mathbb{C}\Pi^{\hat{2}}
\end{aligned}
$$

- Gamma matrices in chiral basis

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\gamma^0 \Rightarrow$ Time- like Hermitian matrix

$\gamma^i \Rightarrow$ Space- like anti-Hermitian matrices

W.A. Hepner. IL NUOVO CIMENTO VOL XXVI,N.2 (1962)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

- ❑ De Sitter Group , S0(4,1)

$\gamma^4 = -i\gamma^5 \Rightarrow$ anti-Hermitian matrix

Matrix of operators

$$R^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^0\gamma^4 \\ \gamma^1\gamma^0 & 0 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^1\gamma^4 \\ \gamma^2\gamma^0 & \gamma^2\gamma^1 & 0 & \gamma^2\gamma^3 & \gamma^2\gamma^4 \\ \gamma^3\gamma^0 & \gamma^3\gamma^1 & \gamma^3\gamma^2 & 0 & \gamma^3\gamma^4 \\ \gamma^4\gamma^0 & \gamma^4\gamma^1 & \gamma^4\gamma^2 & \gamma^4\gamma^3 & 0 \end{pmatrix}$$

- ❑ Anti-de Sitter Group, S0(3,2)

$\gamma^5 \Rightarrow$ Hermitian matrix

Matrix of operators

$$J^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Pi^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Pi^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Pi^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Pi^3 \\ \Pi^0 & \Pi^1 & \Pi^2 & \Pi^3 & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^0\gamma^5 \\ \gamma^1\gamma^0 & 0 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^1\gamma^5 \\ \gamma^2\gamma^0 & \gamma^2\gamma^1 & 0 & \gamma^2\gamma^3 & \gamma^2\gamma^5 \\ \gamma^3\gamma^0 & \gamma^3\gamma^1 & \gamma^3\gamma^2 & 0 & \gamma^3\gamma^5 \\ \gamma^5\gamma^0 & \gamma^5\gamma^1 & \gamma^5\gamma^2 & \gamma^5\gamma^3 & 0 \end{pmatrix}$$

- Satisfy corresponding Lie algebra

Clifford algebra

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} I$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\alpha, \beta = 0,1,2,3,4$$

Clifford algebra

$$\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta} I$$

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\alpha, \beta = 0,1,2,3,5$$

- Interpolating Gamma matrices in chiral basis

$$\gamma^{\hat{+}} = \gamma^0 \cos\delta + \gamma^3 \sin\delta,\ \gamma^{\hat{1}} = \gamma^1\ ,\gamma^{\hat{2}} = \gamma^2,\ \gamma^{\hat{-}} = \gamma^0 \sin\delta - \gamma^3 \cos\delta\ ,\gamma^{\hat{4}} = \gamma^4,\ \gamma^{\hat{5}} = \gamma^5$$

De Sitter operators

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{+}} & \Gamma^1 & \Gamma^2 & \Gamma^{\hat{-}} & 0 \end{pmatrix} = \frac{i}{4}\begin{pmatrix} 0 & [\gamma^{\hat{+}},\gamma^1] & [\gamma^{\hat{+}},\gamma^2] & [\gamma^{\hat{+}},\gamma^{\hat{-}}] & [\gamma^{\hat{+}},\gamma^4] \\ [\gamma^1,\gamma^{\hat{+}}] & 0 & [\gamma^1,\gamma^2] & [\gamma^1,\gamma^{\hat{-}}] & [\gamma^1,\gamma^4] \\ [\gamma^2,\gamma^{\hat{+}}] & [\gamma^2,\gamma^1] & 0 & [\gamma^2,\gamma^{\hat{-}}] & [\gamma^2,\gamma^4] \\ [\gamma^{\hat{-}},\gamma^{\hat{+}}] & [\gamma^{\hat{-}},\gamma^1] & [\gamma^{\hat{-}},\gamma^2] & 0 & [\gamma^{\hat{-}},\gamma^4] \\ [\gamma^4,\gamma^{\hat{+}}] & [\gamma^4,\gamma^1] & [\gamma^4,\gamma^2] & [\gamma^4,\gamma^{\hat{-}}] & 0 \end{pmatrix}$$

$$\eta^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\hat{\alpha}\hat{\beta}}$$

Clifford algebra $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}} I$

$\hat{\alpha} = \hat{\beta} = \hat{+}, 1, 2, \hat{-}, 4$

Anti-de Sitter operators

$$J^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{+}} & \Pi^1 & \Pi^2 & \Pi^{\hat{-}} & 0 \end{pmatrix} = \frac{i}{4}\begin{pmatrix} 0 & [\gamma^{\hat{+}},\gamma^1] & [\gamma^{\hat{+}},\gamma^2] & [\gamma^{\hat{+}},\gamma^{\hat{-}}] & [\gamma^{\hat{+}},\gamma^5] \\ [\gamma^1,\gamma^{\hat{+}}] & 0 & [\gamma^1,\gamma^2] & [\gamma^1,\gamma^{\hat{-}}] & [\gamma^1,\gamma^5] \\ [\gamma^2,\gamma^{\hat{+}}] & [\gamma^2,\gamma^1] & 0 & [\gamma^2,\gamma^{\hat{-}}] & [\gamma^2,\gamma^5] \\ [\gamma^{\hat{-}},\gamma^{\hat{+}}] & [\gamma^{\hat{-}},\gamma^1] & [\gamma^{\hat{-}},\gamma^2] & 0 & [\gamma^{\hat{-}},\gamma^5] \\ [\gamma^5,\gamma^{\hat{+}}] & [\gamma^5,\gamma^1] & [\gamma^5,\gamma^2] & [\gamma^5,\gamma^{\hat{-}}] & 0 \end{pmatrix}$$

$$g^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = g_{\hat{\alpha}\hat{\beta}}$$

Clifford algebra $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2g^{\hat{\alpha}\hat{\beta}} I$

$\hat{\alpha} = \hat{\beta} = \hat{+}, 1, 2, \hat{-}, 5$

- Interpolating operators in the spinor representation can be summarized as commutation among the interpolating gamma matrices, and they all satisfy the corresponding Lie algebra.

- Classical mechanics as the limiting case of relativistic mechanics

Relativistic Boost

$$\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{bmatrix} Cosh[\eta] & Sinh[\eta] \\ Sinh[\eta] & Cosh[\eta] \end{bmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}$$

$$Cosh[\eta] = \gamma = \frac{1}{\sqrt{1-(\beta)^2}},$$

$$Sinh[\eta] = \gamma\, \beta$$

$$Tanh[\eta] = \beta = v/c$$

$$c \to \infty$$

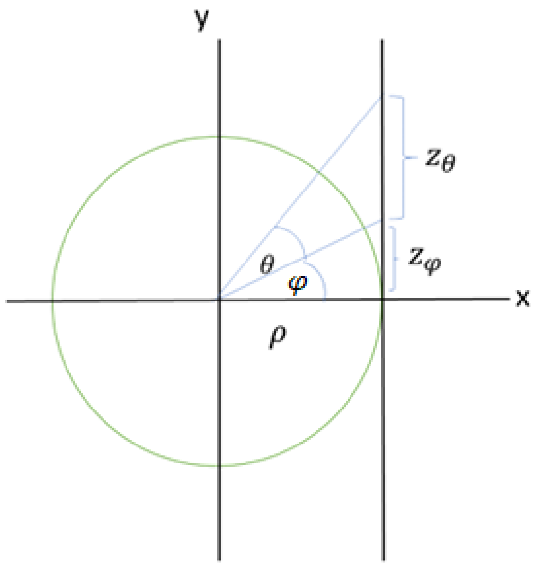
Non-relativistic Boost

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \qquad \begin{aligned} t' &= t \\ z' &= vt + z \end{aligned}$$

- This method violate the relativistic causality

“On the contraction of groups and their representation”
By E. Inonu and E.P Wigner (1958)

- Projection of 2-D rotation in to 1-D translation



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho\, Cos(\varphi) \\ \rho\, Sin(\varphi) \end{pmatrix}$$

Rotation of 2D space

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} Cos[\theta] & -Sin[\theta] \\ Sin[\theta] & Cos[\theta] \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rho \to \infty$$

$$\theta \to 0$$

$$\rho\theta \to z_\theta$$

(Vanishing the curvature)

Translation in 1D space

$$\begin{pmatrix} 1 \\ z_\theta + z_\varphi \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ z_\theta & 1 \end{bmatrix} \begin{pmatrix} 1 \\ z_\varphi \end{pmatrix}$$

Contracting de Sitter and anti-de Sitter spaces

•The curvature of five-dimensional space-time is due to the vacuum energy density.

$\Lambda \to 0$

(Eliminating curvature $\Rightarrow l_1 \to \infty, l_2 \to \infty$)

Four-dimensional flat space time

Line-element

$$ds^2 = dt^2 - {l_1}^2 Cosh^2\left(\frac{t}{l_1}\right) d\rho^2$$

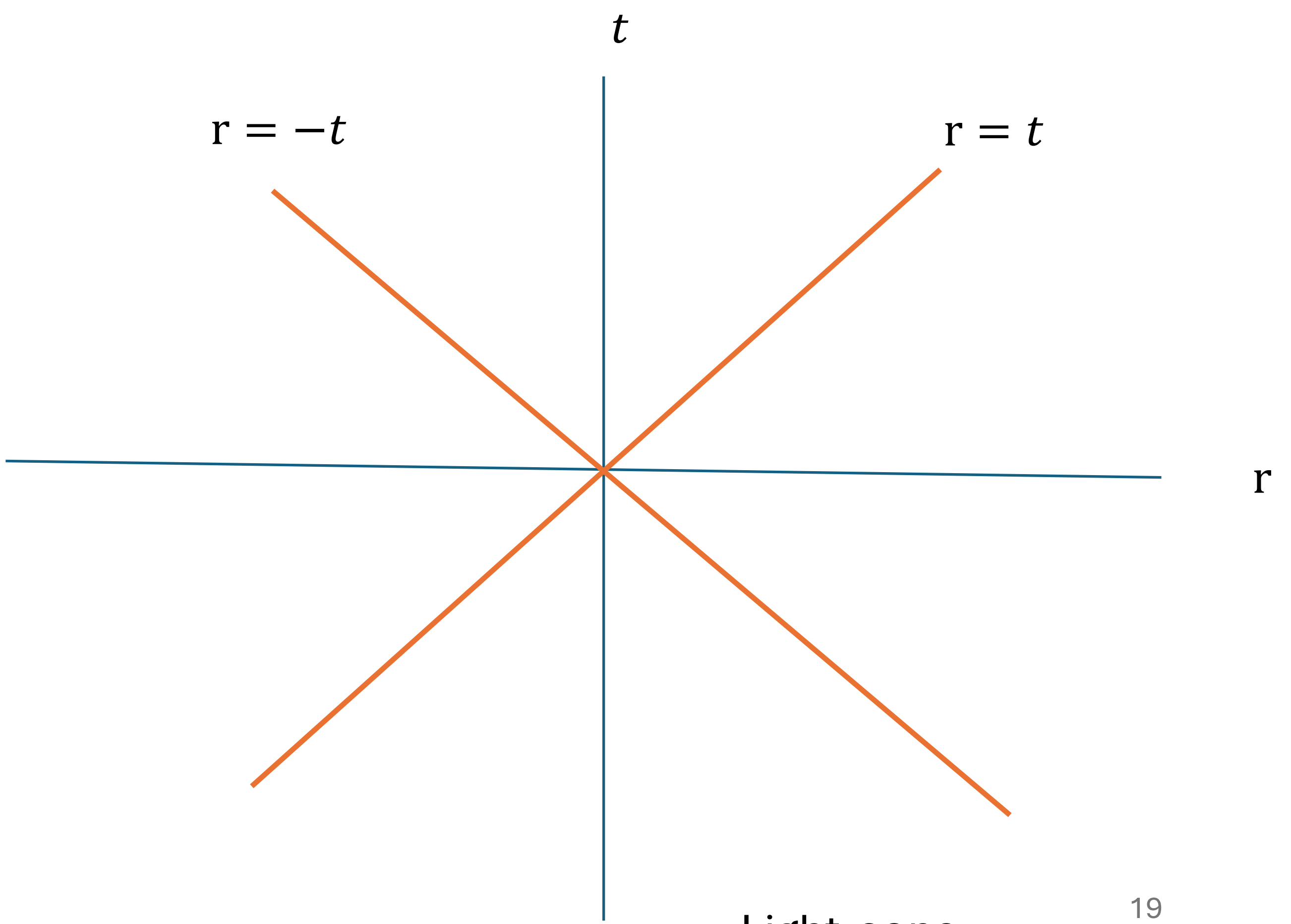
$$ds^2 = dt^2 - Cosh^2\left(\frac{t}{l_1}\right) dr^2$$

$$ds^2 = 0$$

$$\mathrm{r}(t) = \pm l_1 Arctan\left(Sinh\left(\frac{t}{l_1}\right)\right)$$

$$l_1 \to \infty$$

$$\mathrm{r} = t \qquad \mathrm{r} = -t$$



Light-cone

Contracting de Sitter and anti-de Sitter spaces into Poincare space

$$\Lambda \to 0$$

$\boldsymbol{K}$: Boost Operators

$\boldsymbol{J}$: Rotation Operators

Remain unchanged

$\boldsymbol{\Gamma}$: De Sitter translation operators

Π: Anti-De Sitter translation operators

Become Poincare translation operator P

- $\boldsymbol{\Gamma}$ =De sitter translation : De sitter is transitive under a combination of translation and proper conformal transformation

$$\boldsymbol{\Gamma} = e^{id^{\mu}\boldsymbol{\Gamma}_{\mu}} = e^{i\,\eta_{\mu\nu}\frac{a^{\mu}}{l_1}\boldsymbol{\Gamma}^{\nu}} = e^{i\,\eta_{\mu\nu}\frac{a^{\mu}}{l_1}(l_1 P^{\nu} - \frac{1}{4l_1}Ꝃ^{\nu})}$$

$$l_1 \to \infty$$

Ꝃ =Special conformal transformation

$$\boldsymbol{\Gamma} \to P = e^{i\,\eta_{\mu\nu}a^{\mu}P^{\nu}}$$

Contraction of Interpolating de Sitter and anti-de Sitter spaces into Interpolating Poincare space (vector representation)

Contraction in the space-time matrix

$$(ds_1^2)_{l_1\to\infty} = (ds_2^2)_{l_2\to\infty} = ds^2 = \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\nu}} dx^{\hat{\nu}} = (2dx^{+}dx^{-} - dx_{\perp}^2)$$

Interpolating de Sitter translation

$$y'^{\hat{\alpha}} = G_{\beta}^{\hat{\alpha}} \left(e^{i\,\eta_{\hat{\mu}\hat{\nu}} \frac{a^{\hat{\mu}}}{l_1} \Gamma^{\hat{\nu}}}\right)_{\gamma}^{\beta} (G^{-1})_{\hat{\beta}}^{\gamma}\, y^{\hat{\beta}}$$

Interpolating anti-de Sitter translation

$$z'^{\hat{\alpha}} = G_{\beta}^{\hat{\alpha}} \left(e^{i\,\eta_{\hat{\mu}\hat{\nu}} \frac{a^{\hat{\mu}}}{l_2} \Pi^{\hat{\nu}}}\right)_{\gamma}^{\beta} (G^{-1})_{\hat{\beta}}^{\gamma}\, z^{\hat{\beta}}$$

$(y'^{\hat{\alpha}})_{l_1\to\infty}$

$(z'^{\hat{\alpha}})_{l_2\to\infty}$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & a^{\hat{+}} \\ 0 & 1 & 0 & 0 & a^{\hat{1}} \\ 0 & 0 & 1 & 0 & a^{\hat{1}} \\ 0 & 0 & 0 & 1 & a^{\hat{-}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \\ 1 \end{pmatrix}$$

$$[P^{\hat{\mu}}, P^{\hat{\nu}}] = 0$$

Interpolating Poincare translation

Contraction of Interpolating de Sitter and anti-de Sitter spaces into Interpolating Poincare space

Five-dimensional de Sitter coordinates

(Λ>0)

Group algebra S0(4,1)

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{+}} & \Gamma^1 & \Gamma^2 & \Gamma^{\hat{-}} & 0 \end{pmatrix}$$

$$\underline{[\Gamma^{\hat{\mu}}, \Gamma^{\hat{\nu}}] = iR^{\hat{\mu}\hat{\nu}}}$$

Five-dimensional anti-de Sitter coordinates

(Λ <0)

Group algebra S0(3,2) ,

$$J^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{+}} & \Pi^1 & \Pi^2 & \Pi^{\hat{-}} & 0 \end{pmatrix}$$

$$\underline{[\Pi^{\hat{\mu}}, \Pi^{\hat{\nu}}] = -iJ^{\hat{\mu}\hat{\nu}}}$$

Four-dimensional

Poincare space coordinates

Poincare group algebra IS0(3,1)

- Six homogenous operators

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix}$$

- Four inhomogeneous operators

$$\begin{pmatrix} P^{\hat{+}} \\ P^1 \\ P^2 \\ P^{\hat{-}} \end{pmatrix}$$

Lie algebra

$$[M^{\hat{\mu}\hat{\nu}}, M^{\hat{\rho}\hat{\lambda}}] = i(g^{\hat{\nu}\hat{\rho}}M^{\hat{\mu}\hat{\lambda}} - g^{\hat{\nu}\hat{\lambda}}M^{\hat{\mu}\hat{\rho}} - g^{\hat{\mu}\hat{\rho}}M^{\hat{\nu}\hat{\lambda}} + g^{\hat{\mu}\hat{\lambda}}M^{\hat{\nu}\hat{\rho}})$$

$$[P^{\hat{\rho}}, M^{\hat{\mu}\hat{\nu}}] = i(g^{\hat{\rho}\hat{\mu}}P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}}P^{\hat{\mu}})$$

$$\underline{[P^{\hat{\mu}}, P^{\hat{\nu}}] = 0}$$

Kinematic and dynamic operators for different interpolation angle

Since kinematic operators leave the time-invariant, their usage is beneficial in describing the characteristics of the motion with a simpler time-variant expression.

	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, \Gamma^1, \Gamma^2, \Gamma^3, \Pi^1, \Pi^2, \Pi^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, \Gamma^0, \Pi^0$
$0 \le \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, \Gamma^1, \Gamma^2, \Pi^1, \Pi^2, \Gamma_{\hat{-}}, \Pi_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, \Gamma_{\hat{+}}, \Pi_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, \Gamma^1, \Gamma^2, \Pi^1, \Pi^2, \Gamma^+, \Pi^+$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, \Gamma^-, \Pi^-$

- In the de Sitter group ($\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, \Gamma_1, \Gamma_2$,and $\Gamma_{\hat{-}}$) are always kinematic.

  $\rightarrow y^{\hat{+}} = 0$ plane is intact under the transformation.
- In the anti-de Sitter group ($\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, \Pi_1, \Pi_2$,and $\Pi_{\hat{-}}$) are always kinematic.
- K^3 becomes kinematic exactly at the LF

	Kinematic	Dynamic
$\delta = 0$	$\mathcal{K}^{\hat{1}} = -J^2, \mathcal{K}^{\hat{2}} = J^1, J^3, P^1, P^2, P^3$	$\mathcal{D}^{\hat{1}} = -K^1, \mathcal{D}^{\hat{2}} = -K^2, K^3, P^0$
$0 \le \delta < \pi/4$	$\mathcal{K}^{\hat{1}}, \mathcal{K}^{\hat{2}}, J^3, P^1, P^2, P_{\hat{-}}$	$\mathcal{D}^{\hat{1}}, \mathcal{D}^{\hat{2}}, K^3, P_{\hat{+}}$
$\delta = \pi/4$	$\mathcal{K}^{\hat{1}} = -E^1, \mathcal{K}^{\hat{2}} = -E^2, J^3, K^3, P^1, P^2, P^+$	$\mathcal{D}^{\hat{1}} = -F^1, \mathcal{D}^{\hat{2}} = -F^2, P^-$

- In the Poincaré group ($\mathcal{K}^{\ddot{1}}, \mathcal{K}^{\ddot{2}}, J^3, P_1, P_2$,and $P_{\hat{-}}$) are always kinematic.

Conclusion

- The geometry of the spacetime is deeply connected with the corresponding groups and algebras.

- The Interpolating dynamic method can be applied to the five-dimensional spaces despite the curvature of their space time.

- After establishing interpolating group algebra, we confirm interpolating de Sitter and anti-de Sitter groups can be contracted into interpolating Poincare group in the limit of vacuum energy densities of their spaces go to zero, making the curvature of the spaces vanish.

- We can write all the operators in terms of gamma matrices, which emphasizes that they have their own physical meaning that connects with the dimension and probably with the geometry of the space.

- Even though the contraction process does not change the kinematic and dynamic characteristics of interpolating operators, it does reduce the number of homogenous operators

❑ Outlook

- Investigating the utility of the interpolation formalism in higher dimensional spacetimes such as De Sitter and Anti-De sitter spaces.

- The Friedmann-Lemaître-Robertson-Walker (FLRW) solution is a cosmological model that describes a homogeneous and isotropic universe. It is a key solution to Einstein's field equations in general relativity and forms the foundation for modern cosmology, particularly in the context of the Big Bang theory.

Metric: The FLRW metric is expressed in the form:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right],$$

where:

- $a(t)$ is the scale factor, describing how the size of the universe changes over time.
- k is the curvature parameter, which can take values 0 (flat), 1 (closed), or -1 (open).
- $d\Omega^2$ represents the angular part of the metric.

- The scalar factor $a(t)$ can have different forms depending on the contents of the universe. For example, in a matter-dominated universe , $a(t) \propto t^{2/3}$, while in a radiation-dominated universe $a(t) \propto t^{1/2}$.

- If dark energy (cosmological constant) is significant, it leads to accelerated expansion.

Schwarzschild Space Time

Schwarzschild spacetime describes the gravitational field outside a spherically symmetric, non-rotating mass, such as a planet, star, or black hole. It is a solution to Einstein's field equations of general relativity and is one of the first exact solutions discovered

Metric: The Schwarzschild metric is given by the line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where:

- G is the gravitational constant,
- M is the mass of the object,
- c is the speed of light,
- r is the radial coordinate (the circumferential radius),
- $d\Omega^2$ represents the angular part of the metric.