Heavy Mesons within Light Front Quark Model

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Motivation

Table: Current experimental status of meson sector [PDG2024]

		$b\bar{b}$	$c\bar{c}$	B_c	B	B_s	D	D_s
1S	1^3S_1	$\Upsilon(1S)$	$J/\psi(1S)$		$B^*(5325)$	$B_s^*(5415)$	$D^*(2010)$	$D_s^*(2112)$
	1^1S_0	$\eta_b(1S)$	$\eta_c(1S)$	B_c	$B(5279)$	$B_s(5366)$	$D(1867)$	$D_s(1968)$
2S	2^3S_1	$\Upsilon(2S)$	$\psi(1S)$					
	2^1S_0	$\eta_b(2S)$	$\eta_c(2S)$	$B_c(2S)$				
3S	3^3S_1	$\Upsilon(3S)$						
	3^1S_0							
4S	4^3S_1	$\Upsilon(4S)$						
	4^1S_0							
1P	1^3P_2	$\chi_{b2}(1P)$	$\chi_{c2}(1P)$		$B_2^*(5747)$	$B_{s2}(5848)$	$D_2(2460)$	$D_{s2}^*(2573)$
	1^3P_1	$\chi_{b1}(1P)$	$\chi_{c1}(1P)$		$B_1(5721)$	$B_{s1}(5830)$	$D_1(2420)$	$D_{s1}(2536)$
	1^3P_0	$\chi_{b0}(1P)$	$\chi_{c0}(1P)$				$D_0(2300)$	$D_{s0}^*(2317)$
	1^1P_1	$h_b(1P)$	$h_c(1P)$				$D_1(2430)$	$D_{s1}(2460)$
2P	2^3P_2	$\chi_{b2}(2P)$						
	2^3P_1	$\chi_{b1}(2P)$						
	2^3P_0	$\chi_{b0}(2P)$						
	2^1P_1	$h_b(2P)$						
1D	1^3D_3							
	1^3D_2	$\Upsilon_2(1D)$						

S. Navas et al., Particle Data Group, Phys. Rev. D 110, 030001 (2024).

Motivation

Table: Newly observed resonances in meson sector [LHCb,BABAR,CDF]

$b\bar{b}$	$c\bar{c}$	B	D	D_s
$\Upsilon(10860)$	$\psi(4040)$	$B_J(5840)$	$D_J^*(2560)$	$D_{s3}(2860)$
$\Upsilon(11020)$	$\psi_(4415)$	$B_J(5960)$	$D_J^*(2680)$	$D_{s1}(2860)$
	$\psi(3843)$		$D_J(2740)$	$D_{sJ}(3040)$
	$\psi(3823)$		$D_J^*(2760)$	$D_{sJ}(2710)$
	$\psi(3770)$		$D_J(3000)$	$D_{sJ}(2590)$
	$\psi(4160)$		$D_J^*(3000)$	
			$D_2^*(3000)$	

- Problem : Identification, Spin-Parity assignment of newly observed meson states

[LHCb] R. Aaij et al., Phys. Rev. D 94(7), 072001 (2016)

[CDF] T. Aaltonen et al., Phys. Rev. D 90(1), 012013, (2014).

[BABAR] B. Aubert et al., Phys. Rev. D, vol. 80(9), 092003, (2009).

Theoretical Approaches for hadron spectroscopy

- Lattice QCD theories
- QCD Sum Rule methods
- Bag models
- Effective Field theories
- Bethe-Salpter Equation
- **Potential Models**
  - $V(r) = -\frac{4\,\alpha_s}{3r} + Confinement?$
  - Cornell PM
  - Litchenberg PM
  - Quigg and Rosner PM
  - Schroberl PM
  - list is infinite ..!

- Potential Models
  $+$
  Light Front Dynamics
- Light Front Wave Functions $(x, \mathbf{k}_\perp, \lambda_i)$
  $\tau = t + \frac{z}{c}$
  Boost invariant
  Independent of Frame
- The effective bound state mass square
  $M^2_{q\bar{q}} = (P^0_{cm})^2$
  $M^2_{q\bar{q}} = (P^+, P^-, \mathbf{P}_\perp) = (P^0 + P^3, P^0 - P^3, \mathbf{P}_\perp)$

Recent Attempts

FLOW

- Hamiltonian for meson at rest
- Easy to handle potential with simple confinement
- transformations of variables in momentum space
- Mass from variational method using harmonic basis

Mass of the ground pseudoscalar state $M(1^1S_0)$ of heavy-light meson. All are in unit of MeV.

	Dhiman2019	PDG2024	Difference
B	5212	5279	67
B_s	5313	5366	52
D	1803	1867	64
D_s	1929	1968	39
$a + b\,e^{\alpha r}$	$\chi^2 = 0.012$		
	Arifi2022	**PDG2024**	**Difference**
B	5174	5279	105
B_s	5325	5366	41
D	1731	1867	136
D_s	1938	1968	30
$a + b\,r$	$\chi^2 = 0.025$		

N Dhiman, H Dahiya, C R Ji, and H M Choi. Phys. Rev. D, 100(1), 014026, (2019).

A J Arifi, H M Choi, C R Ji, and Y Oh. Phys. Rev. D, 106(1), 014009, (2022).

Agenda

1. Closer Predictions to PDG Listed Values
2. Must Satisfy Empirical Hierarchy:

- $\Delta M_P > \Delta M_V$, where $\Delta M_{P(V)} = M^{2S}_{P(V)} - M^{1S}_{P(V)}$

	ΔM_V (MeV)	ΔM_P (MeV)
$b\bar{b}$	563	601
$c\bar{c}$	590	654
D	617	680

- $f_{1S} > f_{2S}$

	f_{1S} (MeV)	f_{2S} (MeV)
Υ	689 ± 5	497 ± 5
J/ψ	407 ± 5	294 ± 5

R. L. Workman et al., Particle Data Group, Prog. Theor. Exp. Phys. 2022, 083C01 (2022).

Observations

1. Flavour independent mass gap
   - $\Delta M = M_{2S} - M_{1S} \approx 600$ MeV (Pion to Bottomonium)
2. Power-law potentials and logarithmic potential
   - $V(r) = -Cr^{-\alpha} + Dr^{\epsilon} + V_0$ with $V_0 = a$, $\alpha = 1$ and $\epsilon = \nu$
     $V \sim r^{\epsilon}$ where $\epsilon > 0$, $|\psi(0)|^2 \sim n^{2(\epsilon-1)/(2+\epsilon)}$
   - $V(r) = c\ln(r/r_0)$
     $n\,|\psi(0)|^2 \approx$ Constant

R. L. Workman et al., Particle Data Group, Prog. Theor. Exp. Phys. 2022, 083C01 (2022).

W. Lucha, F. F. Schoberl, and D. Gromes, Phys. Rept. 200, 127 (1991).

C. Quigg, J.L. Rosner, Physics Reports 56(4), 167 (1979)

Light Front Quark Model

The effective Hamiltonian at the center of mass frame is given as

$$H_{q\bar{q}} = \sqrt{m_q^2 + \mathbf{k}^2} + \sqrt{m_{\bar{q}}^2 + \mathbf{k}^2} + V_{q\bar{q}}$$

$$\mathbf{k} = (k_z, k_\perp)$$

$$V_{q\bar{q}} = V_{Coul} + V_{Hyp} + V_{conf}$$
$$= -\frac{4\,\alpha_s}{3r} + \frac{2}{3}\frac{\mathbf{S}_q.\mathbf{S}_{\bar{q}}}{m_q m_{\bar{q}}}\,\nabla^2 V_{Coul} + V_{conf}$$

$$V_{conf} = a + c\ln(r/r_0)$$

$c = 0.733$ GeV
$r_0 = 0.89\ \mathrm{GeV}^{-1}$

C. Quigg, J.L. Rosner, Physics Reports 56(4), 167 (1979)

- The three momentum $\mathbf{k} = (k_z, \mathbf{k}_\perp)$ can be presented as $\mathbf{k} = (x, \mathbf{k}_\perp)$ through the relation

$$k_z = \left(x - \frac{1}{2}\right) M_0 + \frac{m_{\bar{q}}^2 - m_q^2}{2M_0}$$

- M_0^2 is the invariant mass square of $q\bar{q}$ system can be obtained

$$M_0^2 = \frac{\mathbf{k}_\perp^2 + m_q^2}{x} + \frac{\mathbf{k}_\perp^2 + m_{\bar{q}}^2}{1-x}$$

- variable transformation $(k_z, \mathbf{k}_\perp) \to (x, \mathbf{k}_\perp)$

$$\frac{\partial k_z}{\partial x} = \frac{M_0}{4x(1-x)}\left[1 - \frac{(m_q^2 - m_{\bar{q}}^2)^2}{M_0^4}\right]$$

Light Front Wave Function

- The Light Front Wave Function (LFWF) $\Psi_{q\bar{q}} = \Psi_{nS}^{J,J_z}$ for vector and pseudoscalar states in momentum space for S-wave is given by [Choi2015]

$$\psi_{nS}^{JJ_z}(x, \mathbf{k}_\perp, \lambda_i) = \Phi_{nS}(x, \mathbf{k}_\perp)\, \mathcal{R}_{\lambda_q \lambda_{\bar{q}}}^{JJ_z}(x, \mathbf{k}_\perp)$$

- For the $1S$ and $2S$ states

$$\phi_{1S}(x, \mathbf{k}_\perp) = \frac{4\pi^{3/4}}{\beta^{3/2}} \sqrt{\frac{\partial k_z}{\partial x}}\, \exp^{-\mathbf{k}^2/2\beta^2}$$

$$\phi_{2S}(x, \mathbf{k}_\perp) = \frac{4\pi^{3/4}}{\sqrt{6}\beta^{7/2}} \left(2\mathbf{k}^2 - 3\beta^2\right) \sqrt{\frac{\partial k_z}{\partial x}}\, \exp^{-\mathbf{k}^2/2\beta^2}$$

- Relationship between Φ_{nS} and ϕ_{nS}

$$\begin{pmatrix} \Phi_{1S} \\ \Phi_{2S} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_{1S} \\ \phi_{2S} \end{pmatrix}$$

Pure $(\Phi_{1S}, \Phi_{2S}) = (\phi_{1S}, \phi_{2S})\ ;\ \theta = 0$
Mixed $(\Phi_{1S}, \Phi_{2S})\ ;\ \theta \neq 0$

H.M. Choi, C.R. Ji, Z. Li, H.Y. Ryu, Phys. Rev. C 92(5), 055203 (2015)

Mass Formula

$$M_{q\bar{q}}^{1S} = \frac{\beta}{\sqrt{\pi}} \sum_{i=q,\bar{q}} \Bigg\{ z_i e^{z_i/2} \left[\frac{1}{3} c_2^2 (3 - z_i) z_i K_2\left(\frac{z_i}{2}\right) + \frac{1}{6}\left(9 - 3c_1^2 + 2c_2^2 z_i^2 - 6\sqrt{6} c_1 c_2\right) K_1\left(\frac{z_i}{2}\right)\right]$$

$$+ \sqrt{\pi}\left(\sqrt{6} c_1 c_2 - 3c_2^2\right) U\left(-1/2, -2, z_i\right) \Bigg\} \xrightarrow{\text{Kinetic term}}$$

$$+a + b\left(1 - \frac{\gamma_E}{2} - \log(2\beta r_0) - \frac{2c_1 c_2}{\sqrt{6}} + \frac{c_2^2}{3}\right) \xrightarrow{\text{Confinement term}}$$

$$-\frac{4\alpha_s \beta}{9\sqrt{\pi}}\left(5 + c_1^2 + 6\sqrt{\frac{2}{3}} c_1 c_2\right) \xrightarrow{\text{Coulomb interaction term}}$$

$$+\frac{16\alpha_s \beta^3 \mathbf{S}_q \cdot \mathbf{S}_{\bar{q}}}{9 m_q m_{\bar{q}} \sqrt{\pi}} (3 - c_1^2 + 2\sqrt{6} c_1 c_2) \xrightarrow{\text{Hyperfine interaction term}}$$

$$M_{q\bar{q}}^{2S} = M_{q\bar{q}}^{1S}(c_1 \to -c_2,\ c_2 \to c_1)$$

$$z_i = m_i^2/\beta^2, \quad c_1 = \cos\theta, \quad c_2 = \sin\theta$$

Mixing Angle and Variational Parameters

- $\Delta M_P > \Delta M_V$ (Recall)
  $\Delta M_{P(V)} = M^{2S}_{P(V)} - M^{1S}_{P(V)}$
- Decomposition:

$$\Delta M_{P(V)} = \Delta M^{kin}_{P(V)} + \Delta M^{Conf}_{P(V)} + \Delta M^{Coul}_{P(V)} + \Delta M^{Hyp}_{P(V)}$$

- Mass Gap Difference:

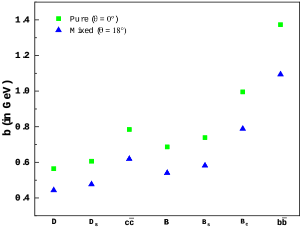
$$\Delta M_P - \Delta M_V = \Delta M^{hyp}_P - \Delta M^{hyp}_V$$

- Mixing Contribution:

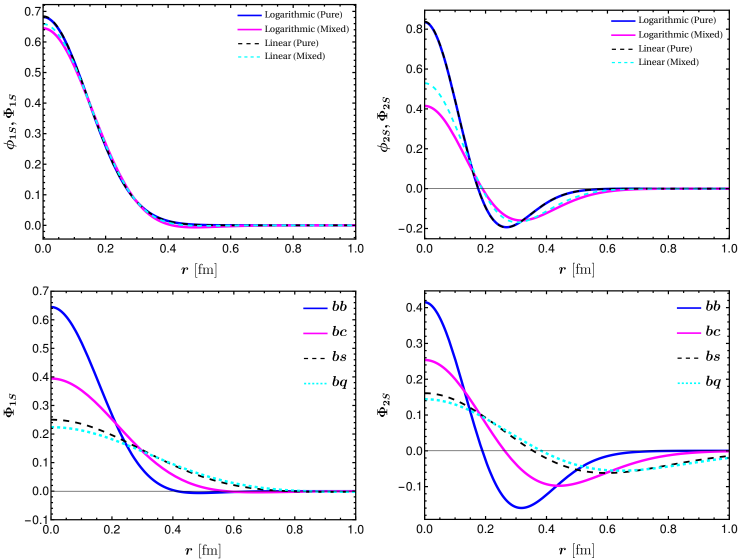
$$\Delta M_P - \Delta M_V = C\,(2\sqrt{6}\sin 2\theta - \cos 2\theta)$$

$$C = \frac{16\,\alpha_s\,\beta^3}{9\,m_q\,m_{\bar{q}}\,\sqrt{\pi}}$$

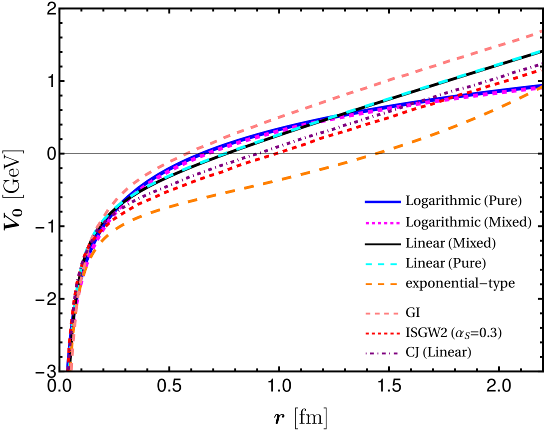
- Mixing Angle:
  $\frac{1}{2}\cot^{-1}(2\sqrt{6}) < \theta < \frac{\pi}{4}$
  $6^\circ < \theta < 45^\circ$



Radial wave functions for $b\bar{b}$ and other b flavoured mesons



Potential between quark and anti-quark



A.J. Arifi, H.M. Choi, C.R. Ji, Y. Oh, Phys. Rev. D 106(1), 014009 (2022).

N. Dhiman, H. Dahiya, C.R. Ji, H.M. Choi, Phys. Rev.D 100(1), 014026 (2019).

S.Godfrey, N. Isgur, Phys. Rev. D 32, 189 (1985).

D.Scora and N. Isgur,Phys.Rev.D 52,2783 (1995).

Mass Spectrum of mesons

Ground and first radial excited state mass spectra of B mesons (All are in units of MeV)

State	Pure	Mixed (18°)	[PDG2022]	Linear [Arifi2022]	Expo. [Dhiman2019]
1^3S_1	5325	5327	5324.71 ± 0.21	5325	5242
1^1S_0	5242	5247	5279.34 ± 0.12	5174	5212
2^3S_1	6075	5913	...	5968	...
2^1S_0	5951	5881	...	5740	...

LHCb: $B(5840)^0$ M $= 5862.9 \pm 5.0 \pm 6.7 \pm 0.2$ MeV

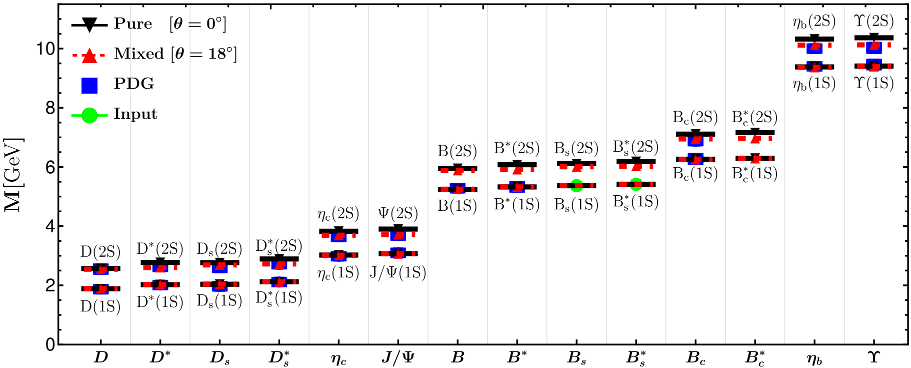
[LHCb] R.Aaij et al., J. High Energy. Phys. 04, 024 (2015).

R L Workman et al., Particle Data Group, Prog. Theor. Exp. Phys. 2022, 083C01 (2022).

N Dhiman, H Dahiya, C R Ji, and H M Choi. Phys. Rev. D, 100(1), 014026, (2019).

A J Arifi, H M Choi, C R Ji, and Y Oh. Phys. Rev. D, 106(1), 014009, (2022).

Mass Spectrum of Mesons



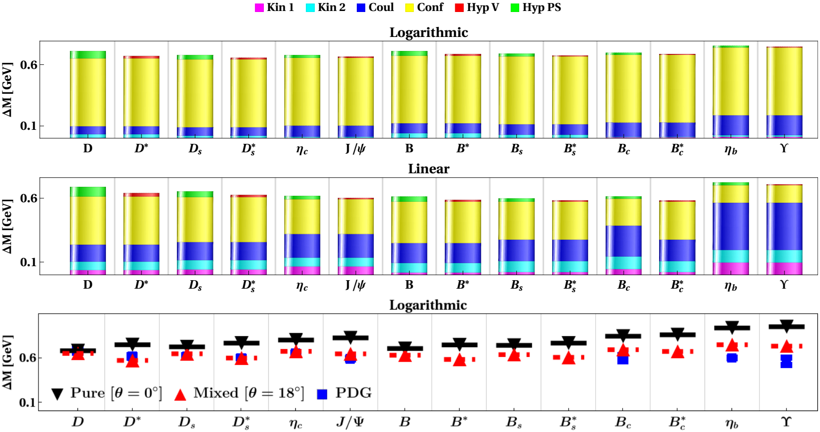
χ^2 Results:

Linear PURE: 0.024 Linear Mixed: 0.009

Log PURE: 0.019 Log Mixed: 0.003

[Linear] A.J. Arifi, H.M. Choi, C.R. Ji, Y. Oh, Phys. Rev. D 106(1), 014009 (2022)

Mass Gap and Its Components



Log:

Conf $\propto \log(2\beta r_0)$

Coul $\propto \beta$

Linear :

Conf $\propto \frac{1}{\beta}$

Coul $\propto \beta$

Decay Constants

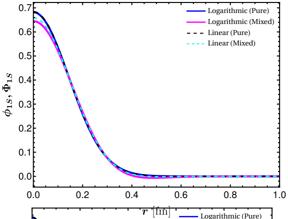
$$\langle 0|\bar{q}\gamma^{\mu}\gamma^{5}q|P_{\mu}\rangle = if_{P}\, P^{\mu}$$

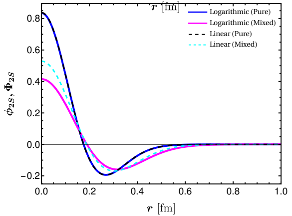
$$\langle 0|\bar{q}_2\gamma^{\mu}q_1|V(P,\lambda)\rangle = f_V\, M_V \epsilon^{\mu}_{\lambda}$$

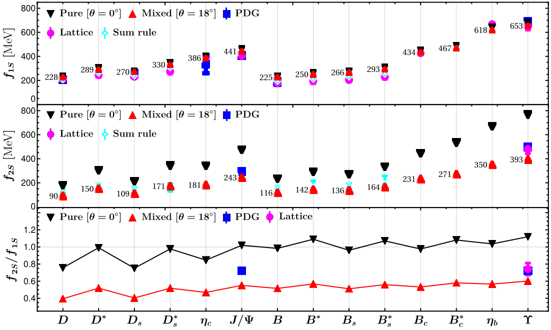
$$f_{P(V)} = \sqrt{6}\int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{(2\pi^3)} \frac{\phi_{nS}(x,\mathbf{k}_\perp)}{\sqrt{\mathcal{A}^2+\mathbf{k}_\perp^2}} \mathcal{O}_{P(V)}$$

$$\mathcal{O}_P = \mathcal{A},\ \mathcal{O}_V = \mathcal{A} + \frac{2\mathbf{k}_\perp^2}{\mathbf{D}_{LF}}$$

$$\mathcal{A} = (1-x)m_Q + x\, m_{\bar{q}},\quad \mathbf{D}_{LF} = M_0 + m_Q + m_{\bar{q}}$$



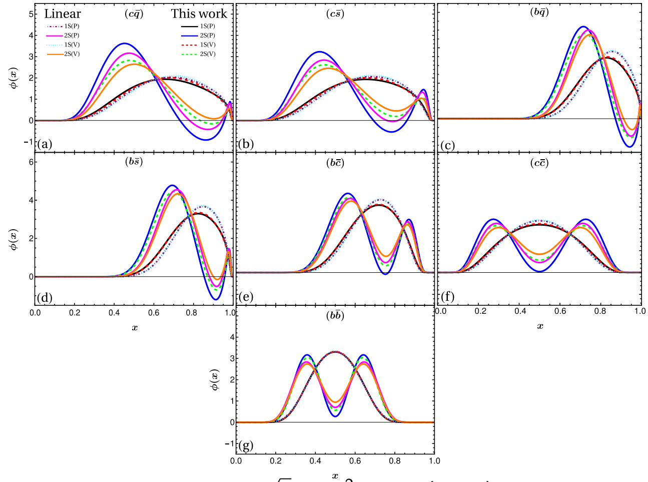




$f_{1S} > f_{2S}$

PRD 102(5), 054511, (2020); PRD 103(5), 054512, (2021); EPJC 82(10), 869 (2022); EPJC 75(1), (2015)

Twist-2 Distribution Amplitudes



$$\phi^{\text{tw-2}}_{P(V)}(x,\mu) = \frac{2\sqrt{6}}{f^{(+)}_{P(V)}} \int \frac{d^2\mathbf{k}_\perp}{(2\pi^3)} \frac{\phi_{1S}(x,\mathbf{k}_\perp)}{\sqrt{\mathcal{A}^2+\mathbf{k}_\perp^2}} \mathcal{O}_{P(V)}$$

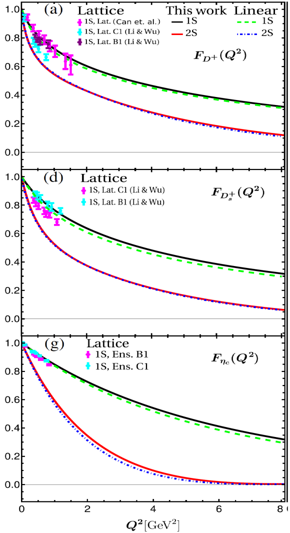
Electromagnetic Form Factors

$$F(Q^2) = e_q\, I^+(Q^2, m_q, m_{\bar{q}}) + e_{\bar{q}}\, I^+(Q^2, m_q, m_{\bar{q}})$$

$$I^+(Q^2,\ m_q,\ m_{\bar{q}}) = \int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\,\Phi\left(x, \mathbf{k}_\perp\right)\Phi^*\left(x, \mathbf{k}'_\perp\right) \times \frac{\mathcal{A}^2 + \mathbf{k}_\perp . \mathbf{k}'_\perp}{\sqrt{\mathcal{A}^2 + \mathbf{k}_\perp^2}\sqrt{\mathcal{A}^2 + \mathbf{k}'^2_\perp}}$$

$$\mathbf{k}'_\perp = \mathbf{k}_\perp + (1-x)\,\mathbf{q}_\perp$$

$$F_{\rm em}(0) = e_q + e_{\bar{q}}$$



H. M.Choi and C.R.Ji, Phys. Rev.D 59, 074015 (1999).

K.U. Can et. al, Phys. Lett. B 719, 103(2013)

N.Li and Y. J.Wu, Eur.Phys.J.A 53, 56(2017)

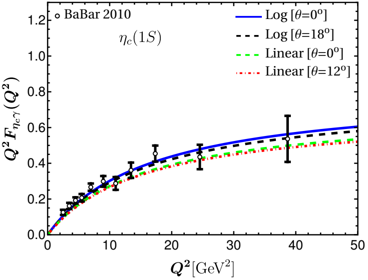
N.Li, C. C.Liu and Y.J. Wu, Eur. Phys. J. A

56,242(2020)

Transition Form Factors of Charmonium

$$F^{nS}_{\eta_{c(b)}\gamma}(q^2) = e^2_{c(b)} \frac{\sqrt{2N_c}}{4\pi^3} \int_0^1 \frac{\mathrm{d}x}{(1-x)} \int \mathrm{d}^2\mathbf{k}_\perp \frac{1}{M_0^2 - q^2} \times \Psi^{nS}_{\frac{\uparrow\downarrow-\downarrow\uparrow}{\sqrt{2}}}(x, \mathbf{k}_\perp),$$

$$\Psi^{nS}_{\frac{\uparrow\downarrow-\downarrow\uparrow}{\sqrt{2}}}(x, \mathbf{k}_\perp) = \frac{m_Q}{\sqrt{\mathbf{k}_\perp^2 + m_Q^2}} \Phi_{nS}(x, \mathbf{k}_\perp).$$



[BABAR] J.P.Lees et al, Phys. Rev. D 81, 052010 (2010), [DSE/BSE] J.Chen et al, Phys.Rev.D 95, 016010 (2017),[BLFQ, BLFQ/DA] Y. Li, Phys. Rev. D 105, L071901 (2022)

Di-leptonic Decays

The BR of leptonic decays of pseudoscalar state of charged meson is computed using the relation given by

$$\mathcal{BR}(P^+ \to \ell^+ \nu_\ell) = \frac{G_F^2}{8\pi} f_P^2 \, |V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2 M_P \times \tau_{P^+}$$

The BR of di-leptonic rare transitions of neutral charge mesons is expressed as

$$\mathcal{BR}(B_q^0 \to \ell^+ \ell^-) = \frac{G_F^2}{\pi} \frac{\alpha^2 f_{B_q}^2 m_\ell^2}{(4\pi sin^2\Theta_W)^2} \, m_{B_q} \sqrt{1 - 4\frac{m_\ell^2}{m_{B_q}^2}} |V_{tb}^* V_{tq}|^2 |C_{10}|^2 \times \tau_{B_q^0}$$

$$C_{10} = \eta_Y \frac{x_t}{8} \left[\frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t\right]$$

C. Bobeth et. al Physical review letters 112,101801 (2014).

C. Bobeth, M. Gorbahn, and E. Stamou, Physical Review D 89, 034023 (2014).

G. Buchalla and A. J. Buras, Nuclear Physics B 400, 225(1993)

A. J. Buras, arXiv preprint hep-ph/9806471 (1998).

H.S. Lee, arXiv preprint arXiv:1511.03783 (2015).

Di-leptonic Decays of charged open flavour Meson

Table: The branching ratios *BR* (in %) for leptonic decays of heavy-light mesons.

Transition	This work	IQM [Ciftci2000]	[PDG2022]
$B^+ \to e^+\nu_e$	1.30×10^{-11}	$\cdots$	$< 9.8 \times 10^{-7}$
$B^+ \to \mu^+\nu_\mu$	5.75×10^{-7}	4.82×10^{-7}	$< 8.6 \times 10^{-7}$
$B^+ \to \tau^+\nu_\tau$	1.29×10^{-4}	9.25×10^{-5}	$(1.09 \pm 0.24) \times 10^{-4}$
$D^+ \to e^+\nu_e$	1.02×10^{-8}	...	$< 8.8 \times 10^{-6}$
$D^+ \to \mu^+\nu_\mu$	4.47×10^{-4}	2.87×10^{-4}	$(3.74 \pm 0.17) \times 10^{-4}$
$D^+ \to \tau^+\nu_\tau$	1.76×10^{-3}	0.75×10^{-3}	$(1.20 \pm 0.27) \times 10^{-3}$
$D_s^+ \to e^+\nu_e$	1.45×10^{-7}	...	$< 8.3 \times 10^{-5}$
$D_s^+ \to \mu^+\nu_\mu$	6.39×10^{-3}	4.41×10^{-3}	$(5.49 \pm 0.16) \times 10^{-3}$
$D_s^+ \to \tau^+\nu_\tau$	10.71×10^{-2}	4.30×10^{-2}	$(11.67 \pm 0.33) \times 10^{-2}$

H. Ciftci and H. Koru, International Journal of Modern Physics E9, 407 (2000).

R L Workman et al., Particle Data Group, Prog. Theor. Exp. Phys. 2022, 083C01 (2022).

Di-leptonic Decays of neutral open flavour Meson

Table: The branching ratios for rare leptonic decays of charge neutral mesons

Transition	Branching Ratio		
	This work	EFT [Bobeth2014]	PDG2022
$B^0 \to e^+e^-$	3.93×10^{-15}	$(2.48 \pm 0.21) \times 10^{-15}$	$< 2.5 \times 10^{-9}$
			$< 3.0 \times 10^{-9}$ [*LHCb* 2020]
$B^0 \to \mu^+\mu^-$	1.73×10^{-10}	$(1.06 \pm 0.09) \times 10^{-10}$	$< 5^{+17}_{-15} \times 10^{-11}$
			$< 2.6 \times 10^{-10}$ [*LHCb* 2022, *LHCb* 2021]
			$< 3.6 \times 10^{-10}$ [*CMS* 2020]
$B^0 \to \tau^+\tau^-$	3.65×10^{-8}	$(2.22 \pm 0.19) \times 10^{-8}$	$< 2.1 \times 10^{-3}$
$B_s^0 \to e^+e^-$	13.19×10^{-14}	$(8.54 \pm 0.55) \times 10^{-14}$	$< 9.4 \times 10^{-9}$
			$< 11.2 \times 10^{-9}$ [*LHCb* 2020]
$B_s^0 \to \mu^+\mu^-$	5.81×10^{-9}	$(3.65 \pm 0.23) \times 10^{-9}$	$(2.9 \pm 0.4) \times 10^{-9}$
			$(3.09^{+0.46}_{-0.43} \pm 0.14) \times 10^{-9}$[LHCb 2022]
			$(2.9 \pm 0.7 \pm 0.2) \times 10^{-10}$[*CMS* 2020]
$B_s^0 \to \tau^+\tau^-$	12.47×10^{-7}	$(7.73 \pm 0.49) \times 10^{-7}$	$< 6.8 \times 10^{-3}$

[LHCb] R Aaij et al. PRL, 124(21):211802, (2020), PRD, 105:012010, (2022), PRL, 128(4):041801, (2022)
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Summary

Take-Home Messages

- Small mixing is necessary.
- $B_J(5840)$ could be identified as 2^1S_0.
- Several discoveries of radial and orbital excited states require more efforts to understand their internal structure.
- Similar mass gaps between baryons and mesons highlight the role of different potential component contributions.
- Wave functions can be expanded using a harmonic oscillator (HO) basis.

Future Scope

- Implementation of the Gaussian expansion method.
- Extension of studies to baryons and exotic states.
- Exploration of various decay processes.

Thank You for your attention!

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