# Developing a Foundation

for the Implementation of  the Relativistic QFT in Quantum Computing.

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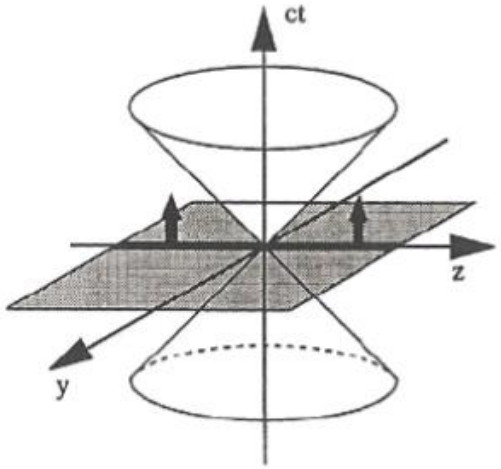
# Out Line

- Motivation
- Instant Form Dynamics and  Light Front Dynamics
- Continuum field theory calculations
- Lattice field theory calculations
- Rescaling the lattice calculations
- Quantum circuit

❑ Main goal: The implementation of the relativistic QFT on a quantum computer.



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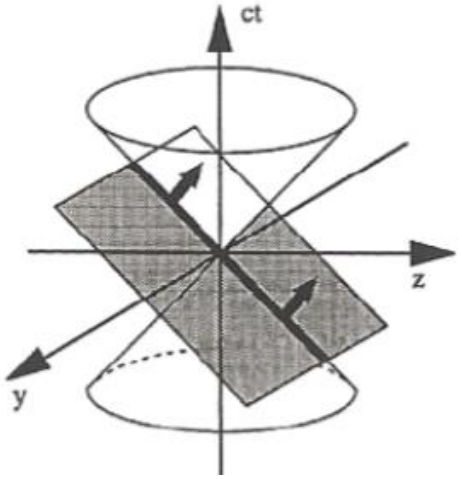



The Instant form ⟷ The light-front form

P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949)

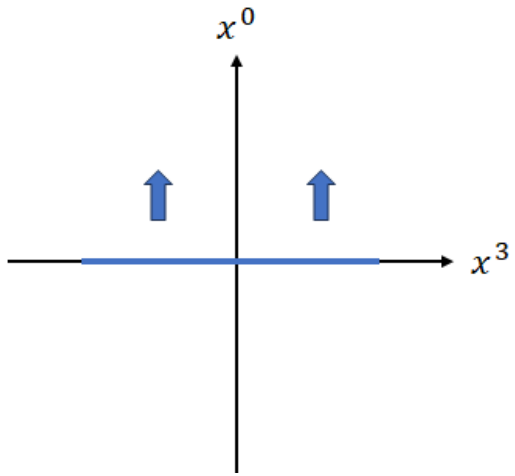
## ❑ Instant Form Dynamics

- Traditional approach evolve with relativistic dynamics
- Closely related to the Euclidian space
- Application in - LQCD, T-dept QFT, IMF ..

## ❑ Light Front Dynamics

- Innovative approach for relativistic dynamics
- Strictly in Minkowski space
- Application in -PDFs, DIS, GPDs ...

## ❑ Energy-momentum dispersion relations




- Instant-form dynamics

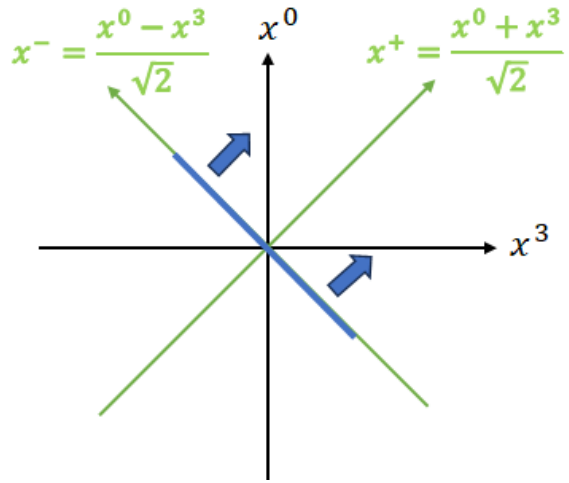
$$P^0 = \sqrt{\vec{\mathbf{P}}^2 + m^2}$$

Irrational relationship

- Light-front form dynamics

$$P^- = \frac{\mathbf{P}_\perp^2 + m^2}{2P^+}$$

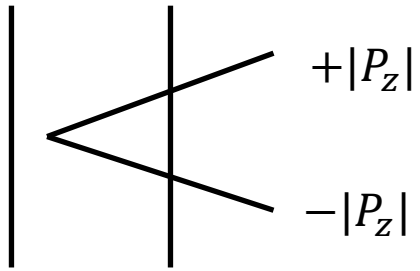
Rational relationship



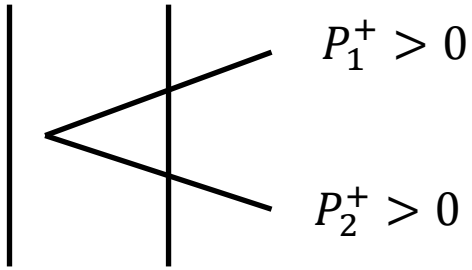

## ❑ Time-ordered diagrams : Vacuum fluctuation

Instant time $x^0 \rightarrow$

Allowed




Light-front time $x^+ \rightarrow$




Not Allowed
Unless $P_1^+ = P_2^+ = 0$
(LF zero-modes)

# Poincare matrix

- Instant-form dynamics

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

- Light-front form dynamics

$$M^{\mu\nu}_{LF} = \begin{pmatrix} 0 & E^1 & E^2 & -K^3 \\ -E^1 & 0 & J^3 & -F^1 \\ -E^2 & -J^3 & 0 & -F^2 \\ K^3 & F^1 & F^2 & 0 \end{pmatrix}$$

**IFD**

- Rotation ($J^1$, $J^2$, $J^3$ ) – Kinematic operators
- Boost ($K^1$, $K^2$, $K^3$ ) – Dynamic operators

**LFD**

- Transverse Rotation ($F^1$, $F^2$) – Dynamic operators
- Longitudinal Rotation ($J^3$ ) - Kinematic operator
- Boost ($E^1$, $E^2$, $K^3$ ) – Kinematic operators

$$\begin{aligned} E^1 &= \frac{1}{\sqrt{2}}(K^1 + J^2), \\ E^2 &= \frac{1}{\sqrt{2}}(K^2 - J^1), \\ F^1 &= \frac{1}{\sqrt{2}}(K^1 - J^2), \\ F^2 &= \frac{1}{\sqrt{2}}(K^2 + J^1). \end{aligned}$$

Kirby, W. M., Hadi, S., Kreshchuk, M., & Love, P. J. *Quantum simulation of second-quantized Hamiltonians in compact encoding*. *Physical Review A, 104*(4), 042607 (2021). .

Kreshchuk, M., Jia, S., Kirby, W. M., Goldstein, G., Vary, J. P., & Love, P. J. *Simulating hadronic physics on noisy intermediate-scale quantum devices using basis light-front quantization*. *Physical Review A, 103*(6), 062601 (2021).

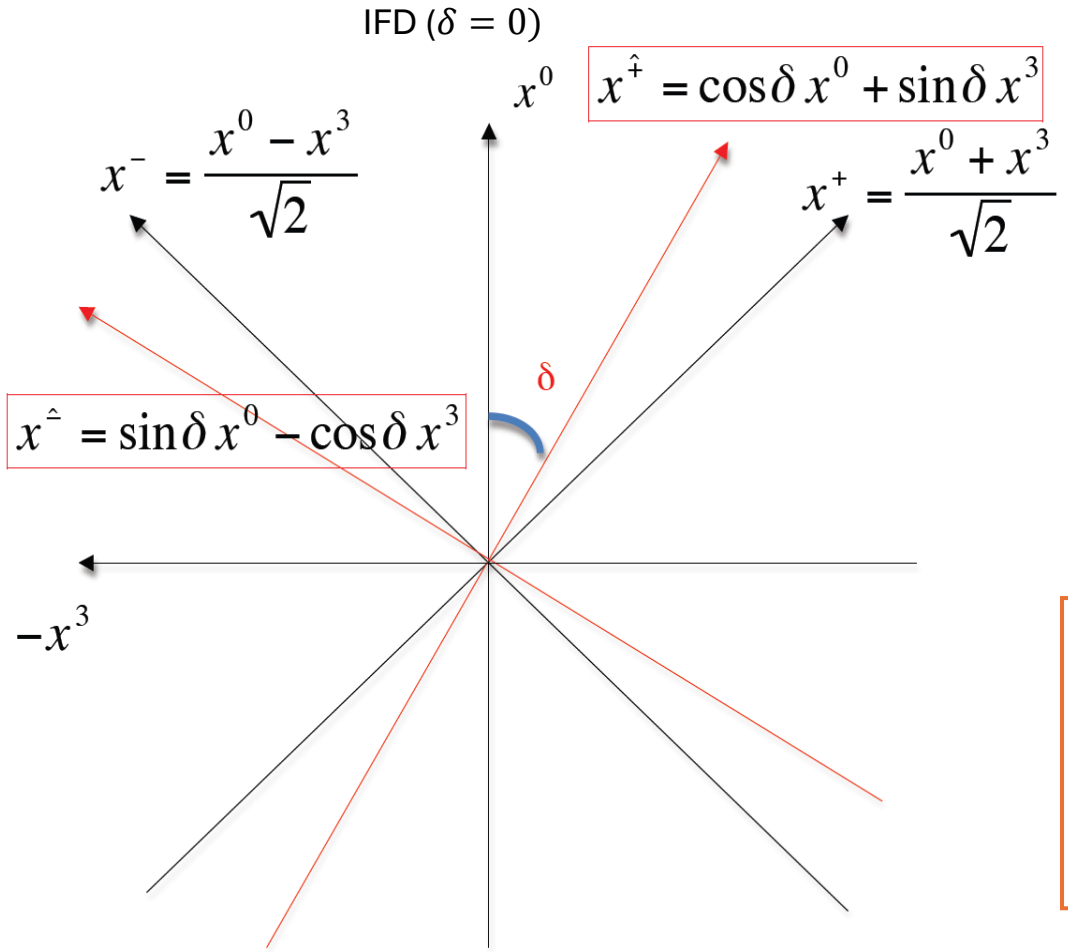
M. Kreshchuk, W. M. Kirby, G. Goldstein, H. Beau-chemin, and P. J. Love, Quantum simulation of quantum field theory in the light-front formulation, Phys. Rev. A105, 032418 (2022).

W. Qian, R. Basili, S. Pal, G. Luecke, and J. P. Vary, Solving hadron structures using the basis light-front quantization approach on quantum computers, Phys.Rev. Res. 4, 043193 (2022).

LF QFT is highly appealing as the starting point for quantum simulation,

- LF facilitate lower qubit counts
- Readily available good initial states

# Interpolating between Instant form dynamic (IFD) and light-front dynamic (LFD)




LFD ($\delta = \frac{\pi}{4}$)

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} cos(\delta) & 0 & 0 & \sin(\delta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\delta) & 0 & 0 & -cos(\delta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolation space time matrix

$$g_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{bmatrix} \qquad \mathbb{S} = Sin(2\delta) \quad \mathbb{C} = Cos(2\delta)$$

- Relate IFD and LFD and show the whole landscape in between.
- Clarify any conceivable confusion between the Infinite momentum frame (IMF) and LFD
- Magnify the vicinity of zero-mode .


K. Hornbostel Phys. Rev. D **45**, 3781 (1992)-RQFT

C.-R. Ji, and S. Rey, Phys. Rev. D **53**, 5815 (1996)-Chiral Anomaly

C.-R. Ji, and A. Suzuki , Phys. Rev. D **87**, 065015 (2013)- Scattering Amps

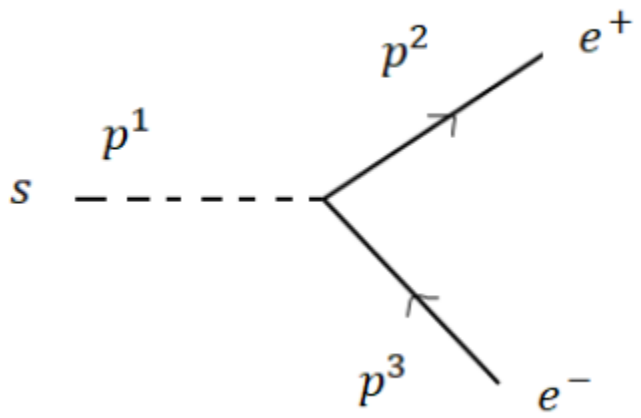
C.-R. Ji, Z. Li, and A. T. Suzuki, Phys. Rev. D **91**, 065020 (2015)-EM Gauges

C.-R. Ji, Z. Li, and B. Ma Phys. Rev. D **98**, 036017 (2018)- QED

B. Ma, C.-R. Ji Phys. Rev. D **104**, 036004 (2021)-$QCD_{1+1}$

## Scalar particle decays into a fermion and an anti-fermion

$$\phi \longrightarrow \psi + \bar{\psi}$$




Feynman digram

## Scalar Field

The scalar Lagrangian is

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2}m^2\phi^2.$$

We can write $\phi$ as a linear sum of an infinite number of creation and annihilation operators $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$ indexed by the 3-momentum $\vec{p}$

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\pi_\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

## Commutation relations

The commutation relations $\phi$ for and $\pi_\phi$ are equivalent to the following commutation relations $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$,

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \Leftrightarrow \quad [a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0$$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad \Leftrightarrow \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

## Hamiltonian for scalar field

We can compute the Hamiltonian for a scalar field in terms of $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$,

$$H_\phi = \frac{1}{2} \int d^3x \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$$H_\phi = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left[a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}\right] = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0)\right].$$

## Fermion Field

Take the Dirac Lagrangian

$$\mathcal{L}_\psi = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - m\,\bar{\psi}\psi.$$

We write the free fermion field as a sum of plane waves,

$$\psi(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right],$$

$$\psi^\dagger(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^{s}_{\vec{p}}v^s(\vec{p})^\dagger e^{+i\vec{p}\cdot\vec{x}}\right],$$

where the operators $b^{s\dagger}_{\vec{p}}$ create particles associated to the spinors $u^s(\vec{p})$, while $c^{s\dagger}_{\vec{p}}$ create particles associated to $v^s(\vec{p})$. As with the scalars, the commutation relations of the fields imply commutation relations for the annihilation and creation operators,

To quantize the theory, we promote the field $\psi$ and its momentum $\psi^\dagger$ to operators, satisfying the canonical commutation relations, which read

$$[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0,$$

$$[\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{y}).$$

The field commutation relations are equivalent to

$$\left[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}), \quad \left[c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\right] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}),$$

with all other commutators vanishing.

## Hamiltonian for fermion field

$$H_\psi = \int d^3x \; \bar{\psi}(-i\gamma^i \partial_i + m)\psi$$

$$H_\psi = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left( b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} - c^s_{\vec{p}} c^{s\dagger}_{\vec{p}} \right) = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left( b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}} c^s_{\vec{p}} + (2\pi)^3 \delta^{(3)}(0) \right)$$

## Interaction field

A minimal interaction for a real scalar field $\phi$ coupled to a Dirac fermion $\psi$ is:

$$\mathcal{L}_{\text{int}} = -g \, \phi \, \bar{\psi}\psi$$

where $g$ is the coupling constant.

## Interaction Hamiltonian

Similarly we can calculate the interaction Hamiltonian, $H_{int} = \int d^3x (g\phi\bar{\psi}\psi)$

$$
\begin{aligned}
H_{\rm int} &= g \int d^3x\ \phi(\vec{x})\, \bar{\psi}(\vec{x})\psi(\vec{x}) \\
&= g \sum_{r,s} \int \frac{d^3\vec{k}\, d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{2E_{\vec{k}}\, 2E_{\vec{p}}\, 2E_{\vec{q}}}} \times \\
&\quad \Big\{ a_{\vec{k}}\ b^{r\dagger}_{\vec{q}} b^{s}_{\vec{p}}\ \bar{u}^{\,r}(\vec{q}) u^{\,s}(\vec{p})\ \delta^{(3)}(\vec{k}+\vec{p}-\vec{q}) \\
&\qquad + \ a_{\vec{k}}\ b^{r\dagger}_{\vec{q}} c^{s\dagger}_{\vec{p}}\ \bar{u}^{\,r}(\vec{q}) v^{\,s}(\vec{p})\ \delta^{(3)}(\vec{k}-\vec{p}-\vec{q}) \\
&\qquad + \ a_{\vec{k}}\ c^{r}_{\vec{q}} b^{s}_{\vec{p}}\ \bar{v}^{\,r}(\vec{q}) u^{\,s}(\vec{p})\ \delta^{(3)}(\vec{k}+\vec{p}+\vec{q}) \\
&\qquad + \ a_{\vec{k}}\ c^{r}_{\vec{q}} c^{s\dagger}_{\vec{p}}\ \bar{v}^{\,r}(\vec{q}) v^{\,s}(\vec{p})\ \delta^{(3)}(\vec{k}+\vec{q}-\vec{p}) \\
&\qquad + \ a^{\dagger}_{\vec{k}}\ b^{r\dagger}_{\vec{q}} b^{s}_{\vec{p}}\ \bar{u}^{\,r}(\vec{q}) u^{\,s}(\vec{p})\ \delta^{(3)}(-\vec{k}+\vec{p}-\vec{q}) \\
&\qquad + \ a^{\dagger}_{\vec{k}}\ b^{r\dagger}_{\vec{q}} c^{s\dagger}_{\vec{p}}\ \bar{u}^{\,r}(\vec{q}) v^{\,s}(\vec{p})\ \delta^{(3)}(-\vec{k}-\vec{p}-\vec{q}) \\
&\qquad + \ a^{\dagger}_{\vec{k}}\ c^{r}_{\vec{q}} b^{s}_{\vec{p}}\ \bar{v}^{\,r}(\vec{q}) u^{\,s}(\vec{p})\ \delta^{(3)}(-\vec{k}+\vec{p}+\vec{q}) \\
&\qquad + \ a^{\dagger}_{\vec{k}}\ c^{r}_{\vec{q}} c^{s\dagger}_{\vec{p}}\ \bar{v}^{\,r}(\vec{q}) v^{\,s}(\vec{p})\ \delta^{(3)}(-\vec{k}+\vec{q}-\vec{p}) \Big\}.
\end{aligned}
$$

Consider the relativistically normalized initial and final states for the scalar particle decays into a fermion and an anti-fermion

$$|i\rangle = \sqrt{2E_{\vec{k}}}\, a^{\dagger}_{\vec{k}}|0\rangle$$
$$|f\rangle = \sqrt{4E_{\vec{q}}E_{\vec{p}}}\, b^{\dagger}_{\vec{q}} c^{\dagger}_{\vec{p}}|0\rangle$$

$$H_{\text{decay}} = g\sum_{r,s}\int \frac{d^3p\, d^3q\, d^3k}{(2\pi)^6}\frac{1}{\sqrt{2E_{\vec{p}}\, 2E_{\vec{q}}\, 2E_{\vec{k}}}}\,\delta^{(3)}(\vec{k}-\vec{q}-\vec{p})\; a_{\vec{k}}\; b^{r\dagger}_{\vec{q}}\; c^{s\dagger}_{\vec{p}}\; \bar{u}^{\,r}(\vec{q})\, v^{\,s}(\vec{p}).$$

$$M^{\lambda_1,\lambda_2} = g.\bar{u}(p^2,\lambda_1)v(p^3,\lambda_2).$$

“qumodes” (short for *quantum modes*) are the basic units of information in continuous-variable (CV) quantum computing, analogous to qubits in standard (discrete-variable) quantum computing.

# Lattice formulation makes QFT finite and discrete

- Continuum field theories have infinitely many degrees of freedom.
- The lattice discretization replaces space(-time) by a finite grid, turning the field into a finite set of variables (one per site).

This makes it possible to:

- Store the field values in a finite number of qubits
- Represent the Hamiltonian as a finite matrix acting on this quantum register.

**The lattice version of QFT is the bridge between continuum field theory and a quantum circuit implementable on hardware.**

In quantum algorithms for simulating a Hamiltonian $H = \sum_j H_j$, we use Trotterization to approximate:

$$U(t) = e^{-iHt} \approx \left( \prod_j e^{-iH_j \Delta t} \right)^n, \quad \text{where } \Delta t = t/n$$

Each exponential $e^{-iH_j \Delta t}$ becomes a **quantum gate sequence** that acts on a small number of qubits.

# Lattice scalar field theory calculations

Let lattice sites be labelled by an integer vector $n = (n1, n_2, .., n_d)$. Physical position, $x = na$, where $a$ is the lattice spacing.

Replace spatial derivatives by finite differences. A common choice is the symmetric nearest-neighbor difference.

Spatial gradient can be written as,

$$\partial_i \phi(x) \to \frac{\phi_{n+\hat{i}} - \phi_{n-\hat{i}}}{2a},$$

and the Laplacian by the second-difference,

$$\nabla^2 \phi(x) \longrightarrow \frac{1}{a^2} \sum_{i=1}^{d} (\phi_{n+\hat{i}} - 2\phi_n + \phi_{n-\hat{i}}).$$

Lattice Lagrangian density ( lagrangian per site),

$$\mathcal{L}_n = \frac{1}{2}\dot{\phi}_n^2 - \frac{1}{2a^2}\sum_{i=1}^{d}(\phi_{n+\hat{i}} - \phi_n)^2 - \frac{1}{2}m^2\phi_n^2.$$

Total Lagrangian can be found as,

$$L = \sum_n a^d \mathcal{L}_n = \sum_n a^d \left(\frac{1}{2}\dot{\phi}_n^2 - \frac{1}{2a^2}\sum_i (\phi_{n+\hat{i}} - \phi_n)^2 - \frac{1}{2}m^2\phi_n^2\right)$$

Here, $a^d$ is the lattice cell volume. We can also calculate the conjugate momentum at site n,

$$\pi_n(t) = \frac{\partial L}{\partial \dot{\phi}_n} = a^d \dot{\phi}_n.$$

Then the Hamiltonian for the lattice scalar field,

$$H = \sum_n \pi_n \dot{\phi}_n - L$$

$$= \sum_n \left( \frac{\pi_n^2}{2a^d} + a^d \left[ \frac{1}{2a^2} \sum_i (\phi_{n+\hat{i}} - \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 \right] \right)$$

# Lattice Dirac field theory calculations

Continuum Dirac Lagrangian density,

$$\mathcal{L}_\psi = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi = i\psi^\dagger \partial_0 \psi + i\psi^\dagger \gamma^0 \gamma^i \partial_i \psi - m\psi^\dagger \gamma^0 \psi.$$

Replace spatial derivatives by symmetric finite differences:

$$\partial_i \psi(x) \to \frac{\psi_{n+\hat{i}} - \psi_{n-\hat{i}}}{2a}.$$

Define the site Lagrangian density

$$\mathcal{L}_n = i\dot{\psi}_n^\dagger \psi_n + i\psi_n^\dagger \sum_i \gamma^0 \gamma^i \frac{\psi_{n+\hat{i}} - \psi_{n-\hat{i}}}{2a} - m\psi_n^\dagger \gamma^0 \psi_n.$$

Total Lagrangian,

$$L = \sum_n a^d \mathcal{L}_n(t).$$

Canonical momentum (per site)

$$\pi_n \equiv \frac{\partial L}{\partial \dot{\psi}_n} = a^d i \psi_n^\dagger.$$

Hamiltonian,

$$H = \sum_n \pi_n \dot{\psi}_n - L.$$

The time-derivative term cancels because $\pi_n \dot{\psi}_n = a^d i \psi_n^\dagger \dot{\psi}_n$ and $\mathcal{L}$ contains $+\sum_n a^d i \dot{\psi}_n^\dagger \psi_n$. So we are left with the negative of the spatial + mass pieces:

Hamiltonian of the lattice Dirac field.

$$
\begin{aligned}
H &= -\sum_n a^d \left[ i\psi_n^\dagger \sum_i \gamma^0\gamma^i \frac{\psi_{n+\hat{i}} - \psi_{n-\hat{i}}}{2a} - m\psi_n^\dagger \gamma^0 \psi_n \right] \\
&= \sum_n a^d \left[ -\frac{i}{2a}\psi_n^\dagger \sum_i \gamma^0\gamma^i (\psi_{n+\hat{i}} - \psi_{n-\hat{i}}) + m\psi_n^\dagger \gamma^0 \psi_n \right]
\end{aligned}
$$

# Lattice interaction field theory calculations

$$L_{\text{int}} = \sum_n a^d \mathcal{L}_n^{(\text{int})} = -g \sum_n a^d \phi_n \bar{\psi}_n \psi_n.$$

$$H_{\text{int}} = -L_{\text{int}} = +g \sum_n a^d \phi_n \bar{\psi}_n \psi_n.$$

# The total lattice Hamiltonian

$$H = \sum_n \left( \frac{\pi_n^2}{2a^d} + a^d \left[ \frac{1}{2a^2} \sum_i (\phi_{n+\hat{i}} - \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 \right] \right)$$
$$+ \sum_n a^d \left[ - \sum_i \frac{i}{2a} \psi_n^\dagger \gamma^0 \gamma^i (\psi_{n+\hat{i}} - \psi_{n-\hat{i}}) + m \psi_n^\dagger \gamma^0 \psi_n \right] + g \sum_n a^d \phi_n \bar{\psi}_n \psi_n .$$

When $d = 1$ ,

$$H = \sum_n \left[ \frac{\pi_n^2}{2a} + \frac{1}{2a} (\phi_{n+1} - \phi_n)^2 + \frac{a}{2} m^2 \phi_n^2 - \frac{i}{2} \psi_n^\dagger \gamma^0 \gamma^1 (\psi_{n+1} - \psi_{n-1}) + am \psi_n^\dagger \gamma^0 \psi_n + ag \phi_n \bar{\psi}_n \psi_n \right]$$

# Rescaled Lattice Hamiltonian for Scalar, Fermion, and Interaction Fields

To eliminate explicit factors of $a^d$, define dimensionless (lattice–normalized) fields:

$$\tilde{\phi}_n \equiv a^{\frac{d}{2}} \phi_n, \qquad \tilde{\pi}_n \equiv a^{-\frac{d}{2}} \pi_n.$$

Equivalently,

$$\phi_n = a^{-\frac{d}{2}} \tilde{\phi}_n, \qquad \pi_n = a^{\frac{d}{2}} \tilde{\pi}_n.$$

For a scalar (bosonic) field on the lattice, the canonical commutator is

$$[\phi_n, \pi_m] = i\delta_{nm},$$

with all other equal-time commutators vanishing.

Under the rescaling,

$$[\tilde{\phi}_n, \tilde{\pi}_m] = [a^{d/2}\phi_n, a^{-d/2}\pi_m] = a^{d/2}a^{-d/2}[\phi_n, \pi_m] = i\delta_{nm}.$$

Hence the canonical commutation relation is preserved:

Substituting these into $H$, we obtain

$$H_{\text{scalar}} = \sum_n \left[ \frac{\tilde{\pi}_n^2}{2} + \frac{1}{2a^2} \sum_i (\tilde{\phi}_{n+\hat{i}} - \tilde{\phi}_n)^2 + \frac{1}{2} m^2 \tilde{\phi}_n^2 \right].$$

Thus, the Hamiltonian in terms of the rescaled fields has no explicit powers of $a^d$.

## Rescaling of Fermionic Fields

Define dimensionless (rescaled) lattice fermion fields:

$$\chi_n \equiv a^{\frac{d}{2}} \psi_n \qquad \Longleftrightarrow \qquad \psi_n = a^{-\frac{d}{2}} \chi_n. \qquad\qquad \bar{\chi}_n = \chi_n^\dagger \gamma^0.$$

On the lattice, the discrete equal-time canonical anticommutator is

$$\{\psi_n^\alpha, \psi_m^{\beta\dagger}\} = a^{-d}\, \delta_{nm}\, \delta^{\alpha\beta}, \qquad \text{where } \alpha, \beta \text{ are spinor indices.}$$

Under the rescaling $\chi_n = a^{d/2} \psi_n$,

$$\begin{aligned} \{\chi_n^\alpha, \chi_m^{\beta\dagger}\} &= a^{d/2} a^{d/2} \{\psi_n^\alpha, \psi_m^{\beta\dagger}\} \\ &= a^d \cdot a^{-d} \delta_{nm} \delta^{\alpha\beta} = \delta_{nm} \delta^{\alpha\beta} \end{aligned}$$

Substituting into $H$ removes the explicit factor $a^d$:

$$H_{\text{fermion}} = \sum_n \left[ -i\,\chi_n^\dagger \sum_i \gamma^0\gamma^i \frac{\chi_{n+\hat{i}} - \chi_{n-\hat{i}}}{2a} + m\,\chi_n^\dagger \gamma^0 \chi_n \right]$$

The scalar–fermion interaction Hamiltonian (Yukawa-type) is

$$H_{\text{int}} = -L_{\text{int}} = +g \sum_n a^d\,\phi_n\,\bar{\psi}_n\psi_n,$$

$$\phi_n = a^{-\frac{d}{2}}\tilde{\phi}_n, \qquad \psi_n = a^{-\frac{d}{2}}\chi_n, \qquad \bar{\psi}_n = a^{-\frac{d}{2}}\bar{\chi}_n.$$

$$a^d\,\phi_n\,\bar{\psi}_n\psi_n = a^d\left(a^{-\frac{d}{2}}\tilde{\phi}_n\right)\left(a^{-\frac{d}{2}}\bar{\chi}_n\right)\left(a^{-\frac{d}{2}}\chi_n\right)$$
$$= a^{d-3\frac{d}{2}}\tilde{\phi}_n\,\bar{\chi}_n\chi_n = a^{-\frac{d}{2}}\tilde{\phi}_n\,\bar{\chi}_n\chi_n.$$

Hence,

$$H_{\text{int}} = g\sum_n a^{-\frac{d}{2}}\,\tilde{\phi}_n\,\bar{\chi}_n\chi_n = \left(g\,a^{-\frac{d}{2}}\right)\sum_n \tilde{\phi}_n\,\bar{\chi}_n\chi_n.$$

$$\tilde{g} \equiv g\,a^{-\frac{d}{2}}$$

Then the rescaled interaction Hamiltonian becomes

$$H_{\text{int}} = \tilde{g}\sum \tilde{\phi}_n\,\bar{\chi}_n\chi_n.$$

Combining all contributions, the full rescaled lattice Hamiltonian is

$$
\begin{aligned}
H_{\text{total}} &= H_{\text{scalar}} + H_{\text{fermion}} + H_{\text{int}} \\
&= \sum_n \Bigg[ \frac{\tilde{\pi}_n^2}{2} + \frac{1}{2a^2} \sum_i (\tilde{\phi}_{n+\hat{i}} - \tilde{\phi}_n)^2 + \frac{1}{2} m^2 \tilde{\phi}_n^2 \\
&\quad - i\, \chi_n^\dagger \sum_i \gamma^0 \gamma^i \frac{\chi_{n+\hat{i}} - \chi_{n-\hat{i}}}{2a} + m\, \chi_n^\dagger \gamma^0 \chi_n + \tilde{g}\, \tilde{\phi}_n\, \bar{\chi}_n \chi_n \Bigg].
\end{aligned}
$$

# Canonical energy–momentum tensor

We start from the *canonical* energy–momentum tensor for a Lagrangian $\mathcal{L}(\Phi_a, \partial_\mu \Phi_a)$:

$$T^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \, \partial^\nu \Phi_a - g^{\mu\nu} \, \mathcal{L},$$

where the sum runs over independent field components $\Phi_a$. The spatial momentum operator is

$$P^i = \int d^3x \, T^{0i}(x).$$

# Real scalar field.

The scalar Lagrangian is

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} m^2 \phi^2.$$

Compute the derivative with respect to $\partial_\mu \phi$:

$$\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

$$T^{\mu\nu}_\phi = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \, \mathcal{L}_\phi.$$

Since $g^{0i} = 0$ for $i = 1, 2, 3$,

$$T^{0i}_\phi = \partial^0 \phi \, \partial^i \phi.$$

Using the canonical momentum $\pi(x) = \dfrac{\partial \mathcal{L}_\phi}{\partial(\partial_0 \phi)} = \partial^0 \phi = \partial_0 \phi$ and $\partial^i = -\partial_i$

$$T_\phi^{0i} = \pi\, \partial^i \phi = \pi\, (-\partial_i \phi) = -\,\pi\, \partial_i \phi.$$

Hence the spatial momentum operator is

$$P_\phi^i = \int d^3x\, T_\phi^{0i} = -\int d^3x\, \pi(\mathbf{x})\, \partial_i \phi(\mathbf{x}).$$

In particular, the longitudinal momentum is

$$\boxed{P_\phi^3 = -\int d^3x\, \pi(\mathbf{x})\, \partial_3 \phi(\mathbf{x})}.$$

# Fermionic Fields

Take the Dirac Lagrangian

$$\mathcal{L}_\psi = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - m\,\bar{\psi}\psi.$$

Treat $\psi$ and $\bar{\psi}$ as independent fields. The only term containing derivatives is $i\bar{\psi}\gamma^\mu\partial_\mu\psi$, so

$$\frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\psi)} = i\,\bar{\psi}\gamma^\mu, \qquad \frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\bar{\psi})} = 0.$$

$$T^{\mu\nu}_\psi = \frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\psi)}\,\partial^\nu\psi + \frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\bar{\psi})}\,\partial^\nu\bar{\psi} - g^{\mu\nu}\,\mathcal{L}_\psi.$$

Only the first term contributes, so

$$T^{\mu\nu}_{\psi} = i\,\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - g^{\mu\nu}\mathcal{L}_{\psi}.$$

The momentum density is the $0i$ component:

$$T^{0i}_{\psi} = i\,\bar{\psi}\gamma^{0}\partial^{i}\psi - g^{0i}\mathcal{L}_{\psi}.$$

With $g^{0i} = 0$ and $\bar{\psi}\gamma^{0} = \psi^{\dagger}$,

$$T^{0i}_{\psi} = i\,\psi^{\dagger}\partial^{i}\psi.$$

Using $\partial^{i} = -\partial_{i}$ we can rewrite

$$T^{0i}_{\psi} = -\,i\,\psi^{\dagger}\partial_{i}\psi.$$

Therefore the spatial momentum operator becomes

$$P^{i}_{\psi} = \int d^{3}x\, T^{0i}_{\psi} = -i\int d^{3}x\, \psi^{\dagger}(\mathbf{x})\,\partial_{i}\psi(\mathbf{x}).$$

In particular,

$$\boxed{P^{3}_{\psi} = -i\int d^{3}x\, \psi^{\dagger}(\mathbf{x})\,\partial_{3}\psi(\mathbf{x})}.$$

# Interaction field

Take the interaction Lagrangian $\mathcal{L}_{\rm int} = -g\phi\bar{\psi}\psi$

$$T^{\mu\nu}_{\rm int} = \sum_a \frac{\partial \mathcal{L}_{\rm int}}{\partial(\partial_\mu \Phi_a)}\,\partial^\nu \Phi_a - g^{\mu\nu}\mathcal{L}_{\rm int}.$$

But for the Yukawa interaction $\mathcal{L}_{\rm int} = -g\phi\bar{\psi}\psi$ there are *no derivatives* of the fields in $\mathcal{L}_{\rm int}$, so

$$\frac{\partial \mathcal{L}_{\rm int}}{\partial(\partial_\mu \Phi_a)} = 0 \qquad \text{for all fields } \Phi_a.$$

Hence

$$T^{\mu\nu}_{\rm int} = -\,g^{\mu\nu}\,\mathcal{L}_{\rm int}.$$

For the momentum density we need the $0i$ component. For spatial index $i$ the metric components $g^{0i} = 0$, therefore

$$T^{0i}_{\rm int} = -\,g^{0i}\,\mathcal{L}_{\rm int} = 0.$$

The longitudinal momentum coming from the interaction piece is

$$\boxed{P^3_{\rm int} = \int d^3x\, T^{03}_{\rm int} = 0}.$$

Combining all contributions, the total longitudinal momentum

$$\boxed{P^3 = -\int d^3x\, \pi(\mathbf{x})\, \partial_3\phi(\mathbf{x}) \; - i \int d^3x\, \psi^\dagger(\mathbf{x})\, \partial_3\psi(\mathbf{x})}.$$

**Lattice rescaled longitudinal momentum**

$$\boxed{P^3 \;= -\frac{1}{2a}\sum_n \tilde{\pi}_n(\tilde{\phi}_{n+\hat{3}} - \tilde{\phi}_{n-\hat{3}}) - \;\; i\sum_n \chi^\dagger_n \frac{\chi_{n+\hat{3}} - \chi_{n-\hat{3}}}{2a}}.$$

## Light-front Hamiltonian $P^-$

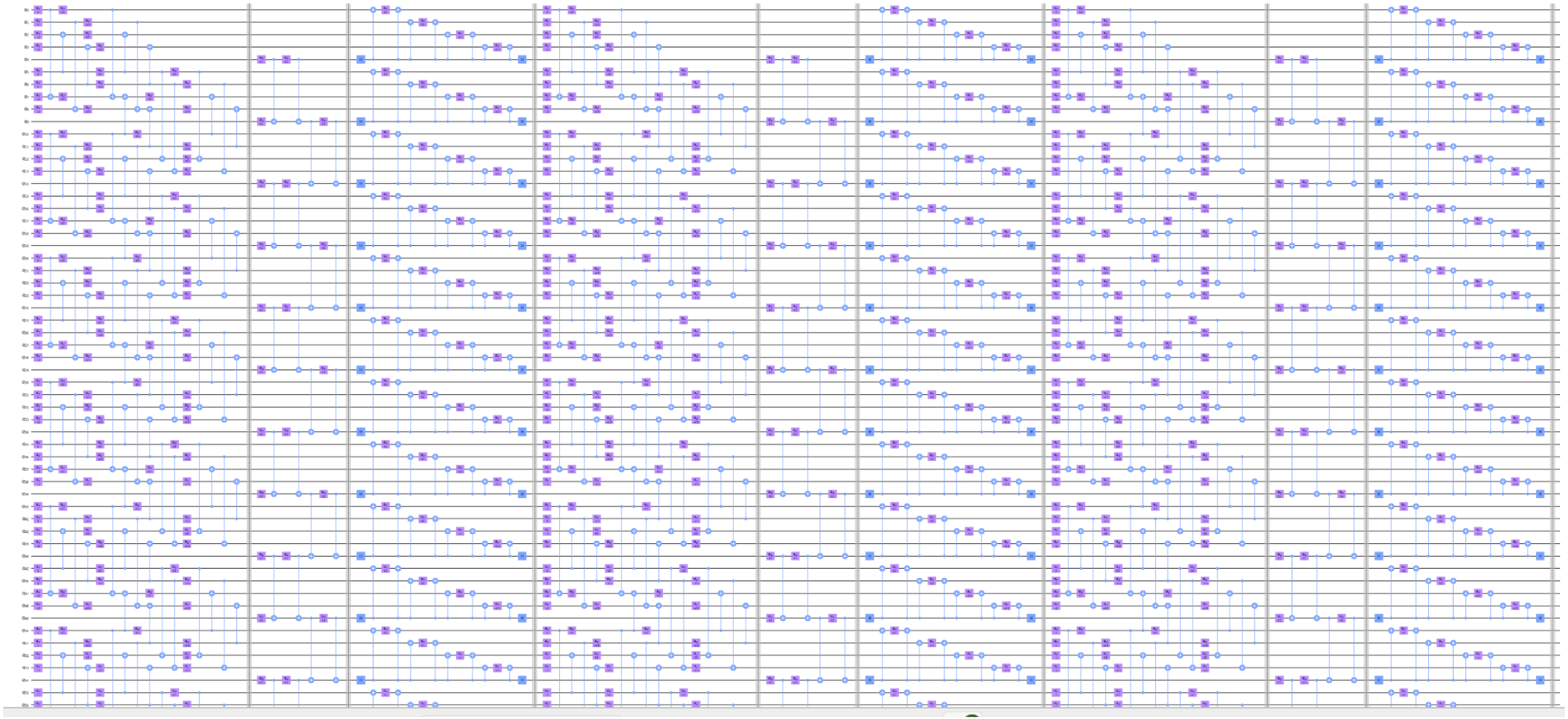
$$P^- = \frac{(P^0 - P^3)}{\sqrt{2}}$$

$$P^0 = \sum_n \left[ \frac{\tilde{\pi}_n^2}{2} + \frac{1}{2a^2} \sum_i (\tilde{\phi}_{n+\hat{i}} - \tilde{\phi}_n)^2 + \frac{1}{2} m^2 \tilde{\phi}_n^2 \right.$$

$$\left. - i\, \chi_n^\dagger \sum_i \gamma^0 \gamma^i \frac{\chi_{n+\hat{i}} - \chi_{n-\hat{i}}}{2a} + m\, \chi_n^\dagger \gamma^0 \chi_n + \tilde{g}\, \tilde{\phi}_n\, \bar{\chi}_n \chi_n \right].$$

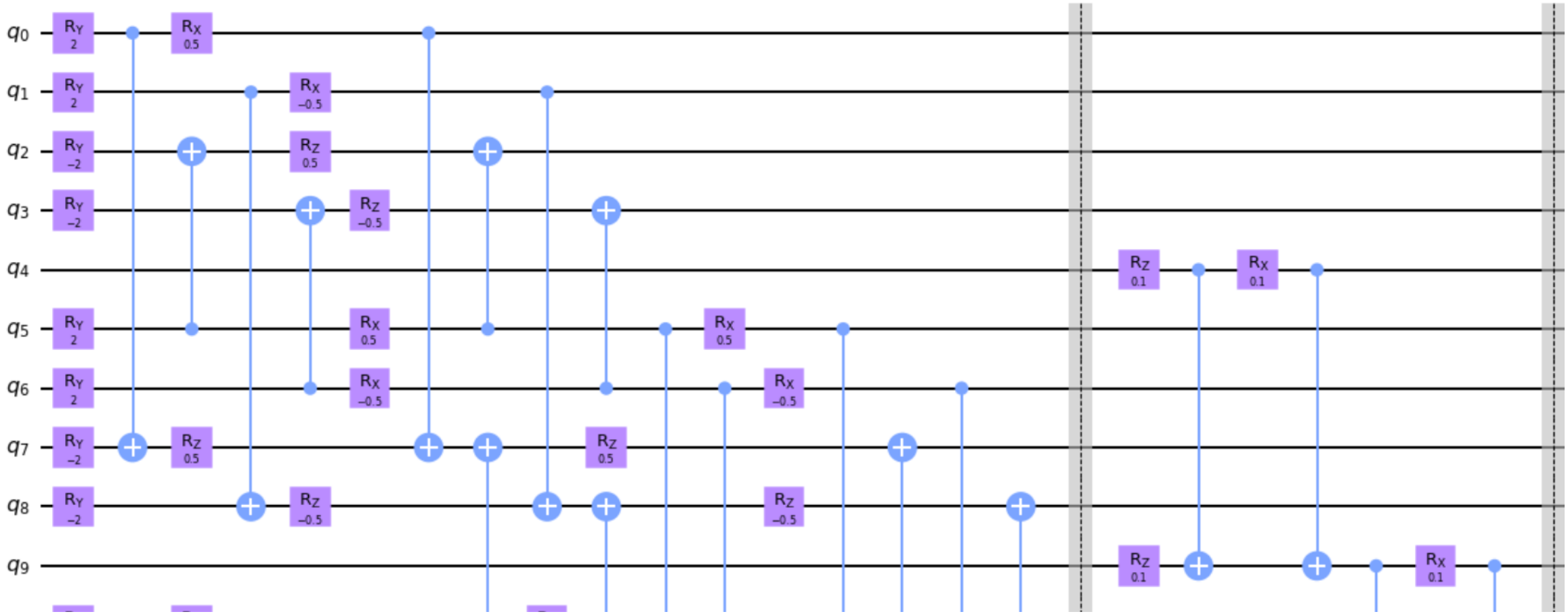
$$P^3 = -\frac{1}{2a} \sum_n \tilde{\pi}_n (\tilde{\phi}_{n+\hat{3}} - \tilde{\phi}_{n-\hat{3}}) - \quad i \sum_n \chi_n^\dagger \frac{\chi_{n+\hat{3}} - \chi_{n-\hat{3}}}{2a}$$

# Quantum circuit in progress



Efekan Kökcü

# Quantum circuit



# Conclusion

- LF QFT is highly appealing as the starting point for quantum simulation
- The lattice version of QFT is the bridge between continuum field theory and a quantum circuit implementable on hardware.

**Future :**

- ❑ Implementing interpolation quantum field theory between instant form dynamics and light front dynamics in the quantum computing.