QCD-Inspired Baryon Models and PMMs

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Motivation of Eigenvector Continuation

- Suppose we have a parameter-dependent Hamiltonian:

$$H(c) = H_0 + cH_1$$

- In practice, direct diagonalization of $H(c^*)$ at the target value c^* may be too expensive (e.g. large lattice).

Eigenvector Continuation (EC): Core Idea

- Explicitly diagonalize $H(c_i)$ at several tractable sample points $\{c_i\}$.
- Collect the corresponding eigenvectors $\{|v_i\rangle\}$.
- Approximate the target state $|v(c^*)\rangle$ as a linear combination:

$$|v(c^*)\rangle \approx \sum_i c_i |v_i\rangle$$

- Project $H(c^*)$ onto the subspace spanned by $\{|v_i\rangle\}$:

$$\sum_j \langle v_i|H(c^*)|v_j\rangle c_j = E \sum_j \langle v_i|v_j\rangle c_j$$

- Solve the generalized eigenvalue problem:

$$H_{\text{proj}} c = ESc$$

- This only assumes that the eigenvectors at the target parameter c lie in the span of the sampled eigenvectors.
- If $H(c)$ is analytic in c, its eigenvalues and eigenvectors are also analytic (from perturbation theory), so locally: $|\psi(c)\rangle = |\psi(c_0)\rangle + \frac{d|\psi\rangle}{dc}\Big|_{c_0}(c - c_0) + \dots$
- Approximating the derivative via finite differences between neighboring points:

$$\frac{d|\psi\rangle}{dc}\bigg|_{c_i} \approx \frac{|\psi(c_{i+1})\rangle - |\psi(c_i)\rangle}{c_{i+1} - c_i}$$

- Interpretation: $|\psi(c)\rangle$ is approximately a linear combination of eigenvectors at neighboring sampled points.
- Hence, as c varies, low-lying eigenvectors trace out only a small subspace of Hilbert space. with coefficients α_i determined by interpolation between neighboring points.
- Interpretation: the low-lying eigenvectors explore only a small portion of Hilbert space as c varies; they trace out a low-dimensional subspace.

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Computational Gain

- If the full Hamiltonian acts on a space of dimension 10^6 (e.g., 100^3 lattice points), EC reduces the problem to diagonalizing a much smaller matrix—say 100×100.
- By solving a few smaller problems (for several c_i), we can predict eigenvalues and eigenvectors across the full parameter range of $H(c)$.
- EC thus provides a non-perturbative interpolation scheme for eigenstates of large Hamiltonians.

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Extending EC to Finite Volume

- Consider simulations in finite periodic boxes of volumes $\{V_i\}$.
- Eigenvalues and eigenvectors vary smoothly with volume, encoding information about the infinite-volume limit.
- FVEC generalizes EC by treating the parameter as the box volume V instead of c.

Technical Subtleties of FVEC

- The inner product depends on V:

$$\langle v_i | v_j \rangle_V = \int_V d^3x\, v_i^*(x) v_j(x)$$

  so we cannot directly mix vectors from different volumes.
- Define an enlarged Hilbert space:

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{V_i}$$

- States are dilated to a common reference volume V_*, allowing projections like

$$\langle v_i | H(V^*) | v_j \rangle_*,$$

  where the subscript $*$ denotes the chosen common inner product.
- The EC algorithm then proceeds identically.

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Projected Matrix Models: The Big Idea

- In EC, the projected Hamiltonian H_{proj} acts as an effective Hamiltonian in the sample subspace.
- PMMs generalize this: instead of explicitly constructing $\{|v_i\rangle\}$, we train an effective matrix that reproduces the same local behavior of $H(c)$.
- We use gradient descent to learn matrices A, B in $H(c) = A + cB$ that minimizes some loss function: in our case, the mean squared error between sample eigenvalues predicted eigenvalues.
- The trained matrix then serves as a surrogate Hamiltonian in the relevant parameter region.
- Further, $H(c)$ doesn't need to have an affine dependence. We're effectively learning a low-order expansion of H around c.

Conceptual Motivation

- The full Hilbert space is often infinite-dimensional and contains much more information than needed.
- PMMs aim to isolate and learn the “active” region of Hilbert space relevant to the dynamics.
- Once trained, the PMM allows efficient prediction of eigenvalues and observables in that restricted region—without constructing or storing the full basis.

Challenges in Describing Baryons from QCD

- No direct analytic path from QCD to the rich structure of hadrons.
- Lattice QCD successfully reproduces ground-state masses, but excited-state spectra and transition dynamics remain difficult.
- Phenomenological models aim to capture QCD mechanisms in an effective form.

Goal: Capture the essential physics of baryon structure and spectroscopy within a QCD-inspired framework.

Open Questions in Baryon Spectroscopy

- **The “Missing Baryons” Problem**
    - Quark models predict more excited states than have been observed.
    - Possible explanations:
        - Baryons may effectively behave as quark–diquark systems, reducing the number of effective degrees of freedom.
        - Some states couple weakly to production channels and so aren't as seen in cross sections.
- **Hybrid Baryons**
    - QCD allows excitations of the gluon field within baryons.
    - These “hybrids” do not have exotic J^P, so identifying them requires new observables.

Aims of Refined Models

- Clarify why certain predicted states go unobserved.
- Predict distinctive signatures of hybrid baryons.
- Connect effective quark models with lattice QCD results.
- Provide consistent picture of baryon structure as an effective description of QCD

A First Attempt: Mass Spectroscopy from Symmetry

- The interactions of quarks depends weakly on the masses of the quarks and very weakly on their electric charges.
- The QCD mass term:

$$\mathcal{L}_{\text{mass}} = \bar{q} M_q q, \quad M_q = \mathrm{diag}(m_u, m_d, m_s).$$

- Explicitly breaks $SU(3)_{\text{flavor}}$ since $m_u \neq m_d \neq m_s$.
- But since m_u, m_d, m_s are close, $SU(3)_{\text{flavor}}$ approximately holds. The difference between m_u, m_d is much smaller than the mass difference between light and strange quarks.
- Thus, the properties of baryon states that differ only by interchange of light and strange quarks are similar, and the properties of baryon states that differ only by interchange of light up and down quarks are very similar.

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Baryon Mass Relations

- Isospin symmetry breaking is much weaker than the strange-light quark mass difference.
- Thus, charge-averaged mass differences between baryons in the octet $J^P = \frac{1}{2}$ should depend $\approx$ only on mass differences between up / down and strange quarks:

$$\begin{aligned} M_N &= M_0 + 3m_{u,d} \\ M_\Lambda = M_\Sigma &= M_0 + 2m_{u,d} + m_s \\ M_\Xi &= M_0 + m_{u,d} + 2m_s, \end{aligned}$$

where M_0 is the interaction energy assuming full flavor symmetry.

Gell-Mann-Okubo Mass Formula

These expressions can be used to find mass splittings:

Mass Relation

$$\frac{1}{2}(M_\Sigma + 3M_\Lambda) = M_N + M_\Xi$$

This particular linear combination is most accurate as it cancels the first-order $SU(3)_f$ breaking effects introduced when considering an effective field theory for baryons.
An analysis of the $J^P = \frac{3}{2}^+$ decuplet baryons similarly leads to

$$M_{\Sigma^*} - M_\Delta = M_{\Xi^*} - M_{\Sigma^*} = M_\Omega - M_{\Xi^*}.$$

(the actual difference: 153, 149, 139 MeV, respectively)

Isgur–Karl Nonrelativistic Model

- Discrepancies in these formulae require a more delicate model. Symmetry-based mass formulae can't explain the sign of these differences. More detailed framework needed.
- Individual quarks have $m_u \approx 2.2$ MeV, $m_d \approx 4.7$ MeV, and $m_s \approx 96$ MeV, but dressed constituent quarks have $M_u, M_d \approx 200 - 350$ MeV, $M_s \approx M_{u,d} + 150 - 200$ MeV. These effective masses are on the order of Λ_{QCD} and so can be treated non-relativistically approximately.
- A (mostly) non-relativistic reduction of one-gluon exchange motivates:

$$H = \sum_i \left(m_i + \frac{\mathbf{p}_i^2}{2m_i} \right) + \sum_{i<j} \left(V^{ij} + H^{ij}_{\text{hyp}} \right)$$

- $V^{ij} = C_{qqq} + br_{ij} - \frac{2\alpha_s}{3r_{ij}}$ (Coulomb + linear confinement)
- H^{ij}_{hyp}: color-magnetic hyperfine term

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Hyperfine Interaction

$$H^{ij}_{\text{hyp}} = \frac{2\alpha_s}{3m_i m_j}\left[\frac{8\pi}{3}\mathbf{S}_i \cdot \mathbf{S}_j\, \delta^3(\mathbf{r}_{ij}) + \frac{1}{r^3_{ij}}\left(\frac{3(\mathbf{S}_i \cdot \mathbf{r}_{ij})(\mathbf{S}_j \cdot \mathbf{r}_{ij})}{r^2_{ij}} - \mathbf{S}_i \cdot \mathbf{S}_j\right)\right]$$

- Neglects spin–orbit terms (Thomas precession + OGE contribution) because their inclusion spoils predictions' agreement with experiment.
- Treat perturbatively: treat V^{ij} as a harmonic-oscillator potential $Kr^2_{ij}/2$ with some anharmonicity U^{ij}, and treat H^{ij}_{hyp} and U^{ij} as perturbations.
- $m_u \approx m_d$, so treat these as equal-mass quarks.

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Wavefunction Construction

- Total baryon wavefunction:

$$\Psi_{\text{total}} = C_A \otimes \sum \psi \otimes \chi \otimes \phi$$

- C_A: totally antisymmetric color wavefunction
- ψ, χ, ϕ: spatial, spin, and flavor wavefunctions
- Fermions $\Rightarrow \sum \psi \otimes \chi \otimes \phi$ symmetric under exchange of identical quarks

Spin Wavefunction Sectors

Quarks are spin-$\frac{1}{2}$, so

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}=\frac{3}{2}_S\oplus\frac{1}{2}_M\oplus\frac{1}{2}_M$$

$$\chi^{S}_{\frac{3}{2},\frac{3}{2}}=|\uparrow\uparrow\uparrow\rangle$$

$$\chi^{M_1}_{\frac{1}{2},\frac{1}{2}}=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle-|\downarrow\uparrow\uparrow\rangle)$$

$$\chi^{M_2}_{\frac{1}{2},\frac{1}{2}}=-\frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle+|\uparrow\downarrow\uparrow\rangle-2|\uparrow\uparrow\downarrow\rangle)$$

Lowering operator $S_-=S_-^{(1)}+S_-^{(2)}+S_-^{(3)}$ generates other projections

Flavor Wavefunctions (Non-Strange)

For non-strange baryons, isospin admits the same irreps as spin (total isospin 3/2 or 1/2)

$$\phi^S_{\Delta^{++}} = |uuu\rangle, \quad \phi^S_{\Delta^{+}} = \frac{1}{\sqrt{3}}(|uud\rangle + |udu\rangle + |duu\rangle), \quad etc.$$

$$\phi^{M_1}_p = \frac{1}{\sqrt{2}}(|udu\rangle - |duu\rangle), \quad \phi^{M_1}_n = \frac{1}{\sqrt{2}}(|udd\rangle + |dud\rangle), \quad etc.$$

$$\phi^{M_2}_p = -\frac{1}{\sqrt{6}}(|duu\rangle + |udu\rangle - 2|uud\rangle), \quad \phi^{M_2}_n = \frac{1}{\sqrt{6}}(|udd\rangle + |dud\rangle - 2|ddu\rangle), \quad etc.$$

- Non-strange baryons: $SU(2)_{\text{iso}}$ limit ($m_u = m_d$)
- Different isospin projections obtained via isospin lowering operator

Flavor Wavefunctions (Strange)

We only need to impose symmetry under exchange of equal mass quarks,

$$
\begin{aligned}
\phi_\Lambda &= \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \otimes |s\rangle \\
\phi_\Sigma &= |uus\rangle, \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \otimes |s\rangle, |dds\rangle \\
\phi_\Xi &= |ssu\rangle, |ssd\rangle \\
\phi_{\Omega^-} &= |sss\rangle
\end{aligned}
$$

- Symmetry only imposed under (12) exchange.
- This is the *uds* basis

Spatial Wavefunction

- Near minimum of V_{ij}: expand as SHO

$$V_{ij} \approx \frac{1}{2} K r_{ij}^2$$

- Treat anharmonicity U_{ij} and hyperfine H_{hyp}^{ij} perturbatively
- Zeroth-order spatial wavefunction:

$$\Psi_{NLM}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi_{n_1,l_1,m_1} \otimes \psi_{n_2,l_2,m_2} \otimes \psi_{n_3,l_3,m_3}$$

Jacobi Coordinates

$$\rho = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}, \quad \lambda = \frac{\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3}{\sqrt{6}}, \quad \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3}{3}$$

- Hamiltonian factorizes: $H_0 = H_{CM}(\mathbf{R}) + H_\rho(\rho) + H_\lambda(\lambda)$
- Wavefunction: $\Psi_{NLM} = \psi_{CM} \otimes \psi_\rho \otimes \psi_\lambda$ with parity $P = (-1)^{l_\rho + l_\lambda}$.
- $N = (2n_\rho + l_\rho) + (2n_\lambda + l_\lambda)$, $L = l_\rho + l_\lambda$, $M = m_\rho + m_\lambda$
- For non-strange baryons, the zeroth-order energies are $E = (N + 3/2)\omega$ with $\omega^2 = 3K/m$, and the symmetric ground state ($N = 0$) is

$$\psi^S_{000} = \frac{\alpha^3}{\pi^{3/2}} \exp\left[-\frac{\alpha^2}{2}(\rho^2 + \lambda^2)\right]$$

- with $\alpha = (3Km_u)^{1/4}$

- For strange baryons, the difference in mass leads to harmonic oscillators with different frequencies:

$$\psi^S_{000} = \frac{\alpha_\rho^{3/2}\alpha_\lambda^{3/2}}{\pi^{3/2}} e^{-(\alpha_\rho^2\rho^2+\alpha_\lambda^2\lambda^2)/2},$$

- with $\alpha_\rho = (3Km)^{1/4}$ and $\alpha_\lambda = (3Km_\lambda)^{1/4}$ and $m = m_1 + m_2 = (m_u + m_d)/2$ and $m_\lambda = 3mm_s/(2m + m_s)$.

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Ground-State Non-strange Baryons

Now, we can combine the spatial, spin, and flavor wavefunctions to obtain the zeroth-order baryon kets. For the non-strange baryons, the ground states are:

$$|N^2S_S\frac{1}{2}^+\rangle = C_A\psi_{000}^S\frac{1}{\sqrt{2}}(\phi_p^{M_1}\chi_{1/2}^{M_1} + \phi_p^{M_2}\chi_{1/2}^{M_2})$$

$$|\Delta^4S_S\frac{3}{2}^+\rangle = C_A\psi_{000}^S\phi_\Delta^S\chi_{3/2}^S$$

- States labeled $|X^{2S+1}L_\pi J^P\rangle$
- $X = N, \Delta$, S is the total quark spin, $L = S, P, D$ is the total orbital angular momentum, $\pi = S, M, A$ is the permutational symmetry of spatial wavefunction, and J^P is the total angular momentum and parity.

P-wave Non-strange Excited States

$$\left|N^4 P_M\left(\frac{1}{2}^-,\frac{3}{2}^-,\frac{5}{2}^-\right)\right\rangle = C_A \chi^S_{\frac{3}{2}} \frac{1}{\sqrt{2}}\left(\phi_N^{M_1}\psi_{11}^{M_1}+\phi_N^{M_2}\psi_{11}^{M_2}\right)$$

$$\left|N^2 P_M\left(\frac{1}{2}^-,\frac{3}{2}^-\right)\right\rangle = C_A \frac{1}{2}\left(\phi_N^{M_1}[\psi_{11}^{M_1}\chi_{\frac{1}{2}}^{M_2}+\psi_{11}^{M_2}\chi_{\frac{1}{2}}^{M_1}]+\phi_N^{M_2}[\psi_{11}^{M_1}\chi_{\frac{1}{2}}^{M_1}-\psi_{11}^{M_2}\chi_{\frac{1}{2}}^{M_2}]\right)$$

$$\left|\Delta^2 P_M\left(\frac{1}{2}^-,\frac{3}{2}^-\right)\right\rangle = C_A \phi^S_\Delta \frac{1}{\sqrt{2}}\left(\psi_{11}^{M_1}\chi_{\frac{1}{2}}^{M_1}+\psi_{11}^{M_2}\chi_{\frac{1}{2}}^{M_2}\right)$$

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Hyperfine Interaction and Mixing

$$|(N^*\frac{1}{2}^-)_1\rangle = \cos\theta_S|N^2P_M\frac{1}{2}^-\rangle - \sin\theta_S|N^4P_M\frac{1}{2}^-\rangle$$

$$\tan 2\theta_S = \frac{2\langle N^4P_M\frac{1}{2}^-|H_{\rm hyp}|N^2P_M\frac{1}{2}^-\rangle}{\langle N^4P_M\frac{1}{2}^-|H_{\rm hyp}|N^4P_M\frac{1}{2}^-\rangle - \langle N^2P_M\frac{1}{2}^-|H_{\rm hyp}|N^2P_M\frac{1}{2}^-\rangle}$$

- Contact term sets basic splitting (Δ vs N); tensor term mixes degenerate states
- Mixing angles measurable experimentally. $N^*\frac{1}{2}^-$ sector: large mixture between $N^2P_M\frac{1}{2}^-$ and $N^4P_M\frac{1}{2}^-$ with $\theta_S \approx -32^\circ$. $N^*\frac{3}{2}^-$ sector: $\theta_S \approx 6^\circ$. Both in agreement with experiment.

Mass Splittings

- $\Delta - N$ and $\Sigma - \Lambda$ predicted fairly well
- Masses of low-lying non-strange states like N^*, Δ^* accurate
- P-wave spectrum correctly orders the Λ^* and Σ^* excitations
- $\Lambda(1520)$ lighter than $\Sigma(1670)$ due to hyperfine:
  - Λ: ud pair in spin-0, isospin-0 → strongly bound
  - Σ: ud pair in spin-1, isospin-1 → less bound
- Model predicts extra $\Lambda(1880)$ resonance not observed
- $\Lambda(1405)$ predicted too heavy ($\Lambda(1520)$)

Summary and Criticisms

- Reproduces many negative-parity states and decay patterns
- Explains mass splittings and mixing angles
- Predicts some resonances incorrectly (e.g., $\Lambda(1405)$)
- Neglects spin-orbit forces; ad hoc choice for fitting spectrum. Data nevertheless shows spin-orbit splittings ($\Lambda(1520) - \Lambda(1405)$ and $\Lambda(1700) - \Lambda(1620)$).
- Assumes non-relativistic treatment despite $p/m \approx 1$ for light quarks.

Relativized Quark Model

$$H = \sum_{i=1}^{3} \sqrt{p_i^2 + m_i^2} + V$$

$$V = V_{\text{string}} + V_{\text{Coul}} + V_{\text{hyp}} + V_{\text{so}}^{\text{OGE}} + V_{\text{so}}^{\text{Thomas}}$$

- V_{string} elevated to the more accurate $V_{\text{string}} = \sigma \sum |r_i - r_0|$.
- Restores spin–orbit and Thomas precession terms.
- Potentials are "relativized" to account for finite quark size and momentum dependence.

Relativistic Smearing and Energy Dependence

- In a relativistic model, quark wavefunctions are sharper at short distances, so $\langle \delta^3(\vec{r}) \rangle$ becomes ill-defined or unrealistically large.
- To account for the finite size of constituent quarks (dressed with gluons and sea quarks):

$$\delta^3(\vec{r}_{ij}) \;\rightarrow\; \rho_{ij}(r) = \frac{\sigma_{ij}^3}{\pi^{3/2}} e^{-\sigma_{ij}^2 r^2}$$

- To include relativistic kinematics, replace quark masses in denominators with energies:

$$E_i = \sqrt{m_i^2 + p_i^2}, \quad H_{\text{contact}}^{\text{hyp}} = -\frac{8\pi\alpha_s}{3} \frac{\vec{S}_i \cdot \vec{S}_j}{E_i E_j} \rho_{ij}(r_{ij})$$

- Otherwise, the model logic remains the same: diagonalize H in a harmonic oscillator basis to obtain baryon masses.

Photocouplings and Helicity Amplitudes

- Electromagnetic transitions ($\gamma N \to N^*$) probe the internal structure of baryon wavefunctions.
- The transition strength is expressed in terms of helicity amplitudes:

$$A_{1/2}, A_{3/2} = \sqrt{\frac{2\pi\alpha}{k}}\langle N^*|\epsilon\cdot\mathbf{J}(0)|N\rangle, \quad S_{1/2} = \sqrt{\frac{2\pi\alpha}{k}}\langle N^*|J^0(0)|N\rangle.$$

- $A_{1/2}$, $A_{3/2}$: transverse photon couplings (spin flip).
- $S_{1/2}$: longitudinal (Coulomb) coupling.
- These amplitudes are measured experimentally and compared to model predictions.

Current Operators and Results

$$\mathbf{J} = \sum_i e_i \boldsymbol{\alpha}_i e^{i\mathbf{k}\cdot\mathbf{r}_i} \quad \Rightarrow \quad H_{M1} \sim \sum_i \frac{e_i}{2m_i} \boldsymbol{\sigma}_i \cdot \mathbf{B}\, e^{i\mathbf{k}\cdot\mathbf{r}_i},$$

$$H_{E2} \sim \sum_i e_i (\mathbf{r}_i \cdot \epsilon) e^{i\mathbf{k}\cdot\mathbf{r}_i}.$$

- $M1$ transitions dominate (single-quark spin flip).
- Small $E2$ amplitudes indicate $SU(6)$ symmetry breaking and deformation.
- The $\gamma N \to \Delta$ transition is well reproduced qualitatively.
- Quantitatively, predicted photocoupling magnitudes are typically smaller than experiment.

Limitations and Relativistic Extensions

- Nonrelativistic current operators violate gauge invariance and current conservation.
- Predicted helicity amplitudes are systematically smaller than experiment.
- Relativized quark models and light-front approaches improve consistency and dynamics, but still underestimate helicity amplitudes by 30%-50%

Next Step

Incorporate fully relativistic quark currents and explicit meson–baryon coupling to achieve quantitative agreement.