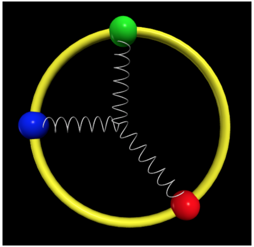
Singular Lagrangians and Constrained Hamiltonian Systems (Part 1)

David Brown
Department of Physics
North Carolina State University



NCSU, January 23, 2026

Singular Lagrangian Systems and Constrained Hamiltonian Systems

- Mass matrix is singular (Hessian, second dervatives of Lagrangian wrt velocities)

Singular Lagrangian Systems and Constrained Hamiltonian Systems

- Mass matrix is singular (Hessian, second dervatives of Lagrangian wrt velocities)
- Includes gauge theories

Singular Lagrangian Systems and Constrained Hamiltonian Systems

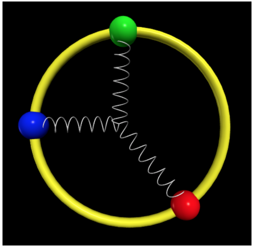
Examples:

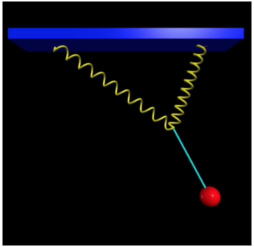
- electromagnetism
- Yang–Mills
- relativistic particle
- string theory
- general relativity

Singular Lagrangian Systems and Constrained Hamiltonian Systems

Examples:

- various (idealized) systems in classical mechanics





Singular Lagrangian Systems and Constrained Hamiltonian Systems

History . . .

Review: Lagrangian/Hamiltonian mechanics

The Lagrangian $L(q, \dot{q})$ is a function of the generalized coordinates $q_i(t)$ and the velocities $\dot{q}_i \equiv \frac{\partial q_i}{\partial t}$ (with $i = 1, 2, 3, \ldots$).

Review: Lagrangian/Hamiltonian mechanics

The Lagrangian $L(q, \dot{q})$ is a function of the generalized coordinates $q_i(t)$ and the velocities $\dot{q}_i \equiv \frac{\partial q_i}{\partial t}$ (with $i = 1, 2, 3, \ldots$).

The action

$$S[q] = \int_0^T dt\, L(q, \dot{q})$$

Extremize the action $\delta S = 0 \Longrightarrow$ Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Review: Lagrangian/Hamiltonian mechanics

The Lagrangian $L(q, \dot{q})$ is a function of the generalized coordinates $q_i(t)$ and the velocities $\dot{q}_i \equiv \frac{\partial q_i}{\partial t}$ (with $i = 1, 2, 3, \ldots$).

The action

$$S[q] = \int_0^T dt\, L(q, \dot{q})$$

Extremize the action $\delta S = 0 \Longrightarrow$ Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\Longrightarrow \qquad \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j = \frac{\partial L}{\partial q_i}$$

Review: Lagrangian/Hamiltonian mechanics

Define the mass matrix $M_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$, then

$$M_{ij} \ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j$$

This is Newton's second law in generalized coordinates.

Review: Lagrangian/Hamiltonian mechanics

Define the mass matrix $M_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$, then

$$M_{ij} \ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j$$

This is Newton's second law in generalized coordinates. If M_{ij} is invertible, solve for accelerations:

$$\ddot{q}_i = (M^{-1})_{ij} \left[\frac{\partial L}{\partial q_j} - \frac{\partial^2 L}{\partial \dot{q}_j \partial q_k} \dot{q}_k \right]$$

Initial value problem: Given initial conditions $q_i(0)$ and $\dot{q}_i(0)$, the future evolution of the system is determined uniquely.

Review: Lagrangian/Hamiltonian mechanics

Define the mass matrix $M_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$, then

$$M_{ij}\ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j$$

This is Newton's second law in generalized coordinates. If M_{ij} is invertible, solve for accelerations:

$$\ddot{q}_i = (M^{-1})_{ij}\left[\frac{\partial L}{\partial q_j} - \frac{\partial^2 L}{\partial \dot{q}_j \partial q_k}\dot{q}_k\right]$$

Initial value problem: Given initial conditions $q_i(0)$ and $\dot{q}_i(0)$, the future evolution of the system is determined uniquely.

If M_{ij} is not invertible, the Lagrangian is *singular*.

Review: Lagrangian/Hamiltonian mechanics

Define the conjugate momenta by

$$p_i = \mathcal{P}_i(q, \dot{q})$$

Review: Lagrangian/Hamiltonian mechanics

Define the conjugate momenta by

$$p_i = \mathcal{P}_i(q, \dot{q})$$

where $\mathcal{P}_i \equiv \partial L / \partial \dot{q}_i$. Invert for $\dot{q}_i = \mathcal{V}_i(q, p)$ and construct the Hamiltonian:

$$H(q, p) = p_i \mathcal{V}_i - L(q, \mathcal{V})$$

Review: Lagrangian/Hamiltonian mechanics

Define the conjugate momenta by

$$p_i = \mathcal{P}_i(q, \dot{q})$$

where $\mathcal{P}_i \equiv \partial L / \partial \dot{q}_i$. Invert for $\dot{q}_i = \mathcal{V}_i(q, p)$ and construct the Hamiltonian:

$$H(q, p) = p_i \mathcal{V}_i - L(q, \mathcal{V})$$

However: If the Lagrangian is singular, the mass matrix

$$M_{ij} = \frac{\partial \mathcal{P}_i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

cannot be inverted. We cannot solve $p_i = \mathcal{P}_i(q, \dot{q})$ for $\dot{q}_i = \mathcal{V}_i(q, p)$. The usual Hamiltonian construction fails.

Review: Lagrangian/Hamiltonian mechanics

For nonsingular $L(q, \dot{q})$, the phase space form of the action is:

$$S[q,p] = \int_0^T dt\, \{p_i \dot{q}_i - H(q,p)\}$$

Review: Lagrangian/Hamiltonian mechanics

For nonsingular $L(q, \dot{q})$, the phase space form of the action is:

$$S[q, p] = \int_0^T dt \, \{p_i \dot{q}_i - H(q, p)\}$$

Extremize $\delta S = 0 \Longrightarrow$ Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \, , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Initial value problem: Given initial conditions $q_i(0)$, $p_i(0)$, the future evolution is uniquely determined.

Review: Lagrangian/Hamiltonian mechanics

For nonsingular $L(q, \dot{q})$, the phase space form of the action is:

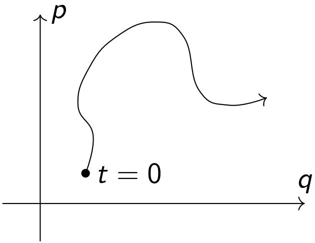
$$S[q,p] = \int_0^T dt \, \{p_i \dot{q}_i - H(q,p)\}$$

Extremize $\delta S = 0 \implies$ Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \, , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Initial value problem: Given initial conditions $q_i(0)$, $p_i(0)$, the future evolution is uniquely determined.

Phase space:



Review: Lagrangian/Hamiltonian mechanics

Recall: **Poisson bracket** of phase space functions $F(q, p)$ and $G(q, p)$ is defined by

$$\{F, G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

The "fundamental" Poisson bracket relations are $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$. Time evolution of any phase space function:

$$\dot{F} = \{F, H\}$$

Review: Lagrangian/Hamiltonian mechanics

Recall: **Poisson bracket** of phase space functions $F(q,p)$ and $G(q,p)$ is defined by

$$\{F,G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

The "fundamental" Poisson bracket relations are $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$. Time evolution of any phase space function:

$$\dot{F} = \{F, H\}$$

Hamilton's equations become

$$\dot{q}_i = \{q_i, H\}\ , \qquad \dot{p}_i = \{p_i, H\}$$

Dirac–Bergmann algorithm

If the Lagrangian is singular:

1. Let $\mathcal{P}_i(q, \dot{q}) \equiv \partial L / \partial \dot{q}_i$ so the momenta are defined by $p_i = \mathcal{P}_i(q, \dot{q})$. Since the Lagrangian is singular, we cannot invert to obtain the $\dot{q}$'s as functions of q's and p's.

Dirac–Bergmann algorithm

If the Lagrangian is singular:

1. Let $\mathcal{P}_i(q, \dot{q}) \equiv \partial L / \partial \dot{q}_i$ so the momenta are defined by $p_i = \mathcal{P}_i(q, \dot{q})$. Since the Lagrangian is singular, we cannot invert to obtain the $\dot{q}$'s as functions of q's and p's.
    - Observe:
    $$\delta p_i = \frac{\partial \mathcal{P}_i}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial \mathcal{P}_i}{\partial q_j} \delta q_j \Longrightarrow \delta p_i - \frac{\partial \mathcal{P}_i}{\partial q_j} \delta q_j = \mathcal{M}_{ij} \delta \dot{q}_j$$
    Since $\mathcal{M}_{ij}$ is not invertible, there are one or more vectors V^i with $V^i \mathcal{M}_{ij} = 0$. Then
    $$V^i \left(\delta p_i - \frac{\partial \mathcal{P}_i}{\partial q_j} \delta q_j \right) = 0$$

   This implies one or more relations among the p's and q's, called primary constraints. Denote the primary constraints by $\phi_a(q, p) = 0$.

Dirac–Bergmann algorithm

2. Define the canonical Hamiltonian H_C as $\mathcal{P}_i\dot{q}_i - L$ written in terms of q's and p's. This is always possible because:

$$\begin{aligned}\delta\left(\mathcal{P}_i\dot{q}_i - L\right) &= \mathcal{P}_i\delta\dot{q}_i + \dot{q}_i\delta\mathcal{P}_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i \\ &= \dot{q}_i\delta\mathcal{P}_i - \frac{\partial L}{\partial q_i}\delta q_i\end{aligned}$$

implies $\mathcal{P}_i\dot{q}_i - L = H_C(q,\mathcal{P})$. Then $H_C = H_C(q,p)$.

Dirac–Bergmann algorithm

3. Define the primary Hamiltonian as $H_P = H_C + \lambda^a \phi_a$, where λ^a are Lagrange multipliers.

Dirac–Bergmann algorithm

3. Define the primary Hamiltonian as $H_P = H_C + \lambda^a \phi_a$, where λ^a are Lagrange multipliers.
   - Extremization of the primary action

     $$S_P[q,p,\lambda] = \int_0^T \{p_i \dot{q}_i - H_P(q,p)\}$$

     yields the correct equations of motion:

     $$\frac{\delta S_P}{\delta p_i} = 0 \implies \dot{q}_i = \frac{\partial H_P}{\partial p_i}$$
     $$\frac{\delta S_P}{\delta q_i} = 0 \implies \dot{p}_i = -\frac{\partial H_P}{\partial q_i}$$
     $$\frac{\delta S_P}{\delta \lambda^a} = 0 \implies \phi_a(q,p) = 0$$

Dirac–Bergmann algorithm

3. Define the primary Hamiltonian as $H_P = H_C + \lambda^a \phi_a$, where λ^a are Lagrange multipliers.
   - Extremization of the primary action

     $$S_P[q, p, \lambda] = \int_0^T \{p_i \dot{q}_i - H_P(q, p)\}$$

     yields the correct equations of motion:

     $$\begin{aligned} \frac{\delta S_P}{\delta p_i} = 0 &\implies \dot{q}_i = \frac{\partial H_P}{\partial p_i} \\ \frac{\delta S_P}{\delta q_i} = 0 &\implies \dot{p}_i = -\frac{\partial H_P}{\partial q_i} \\ \frac{\delta S_P}{\delta \lambda^a} = 0 &\implies \phi_a(q, p) = 0 \end{aligned}$$

   - Are we finished?

Dirac–Bergmann algorithm

3. Define the primary Hamiltonian as $H_P = H_C + \lambda^a \phi_a$, where λ^a are Lagrange multipliers.
   - Extremization of the primary action

     $$S_P[q, p, \lambda] = \int_0^T \{p_i \dot{q}_i - H_P(q, p)\}$$

     yields the correct equations of motion:

     $$\begin{aligned} \frac{\delta S_P}{\delta p_i} = 0 &\Longrightarrow \dot{q}_i = \frac{\partial H_P}{\partial p_i} \\ \frac{\delta S_P}{\delta q_i} = 0 &\Longrightarrow \dot{p}_i = -\frac{\partial H_P}{\partial q_i} \\ \frac{\delta S_P}{\delta \lambda^a} = 0 &\Longrightarrow \phi_a(q, p) = 0 \end{aligned}$$

   - Are we finished? No, not if we want to interpret the equations as an initial value problem.

Dirac–Bergmann algorithm

4. Impose the consistency conditions $\{\phi_a, H_P\} = 0$ to ensure that the primary constraints are preserved under time evolution. These conditions will reduce to a combination of (i) identities when the primary constraints hold, (ii) restrictions on the Lagrange multipliers, and/or (iii) restrictions among the phase space variables q_i and p_i. Restrictions on q's and p's are secondary constraints $\psi_m(q, p) = 0$.

Dirac–Bergmann algorithm

4. Impose the consistency conditions $\{\phi_a, H_P\} = 0$ to ensure that the primary constraints are preserved under time evolution. These conditions will reduce to a combination of (i) identities when the primary constraints hold, (ii) restrictions on the Lagrange multipliers, and/or (iii) restrictions among the phase space variables q_i and p_i. Restrictions on q's and p's are secondary constraints $\psi_m(q, p) = 0$.
5. Apply the consistency conditions to the secondary constraints, $\{\psi_m, H_P\} = 0$. These reduce to a combination of (i) identities when the constraints hold, (ii) restrictions on the Lagrange multipliers, and/or (iii) restrictions among the q's and p's. Restrictions on q's and p's are tertiary constraints. Continue to apply the consistency conditions to generate higher–order constraints. Let $\psi_m(q, p) = 0$ denote all secondary, tertiary, *etc* constraints.

Dirac–Bergmann algorithm

4. Impose the consistency conditions $\{\phi_a, H_P\} = 0$ to ensure that the primary constraints are preserved under time evolution. These conditions will reduce to a combination of (i) identities when the primary constraints hold, (ii) restrictions on the Lagrange multipliers, and/or (iii) restrictions among the phase space variables q_i and p_i. Restrictions on q's and p's are secondary constraints $\psi_m(q, p) = 0$.
5. Apply the consistency conditions to the secondary constraints, $\{\psi_m, H_P\} = 0$. These reduce to a combination of (i) identities when the constraints hold, (ii) restrictions on the Lagrange multipliers, and/or (iii) restrictions among the q's and p's. Restrictions on q's and p's are tertiary constraints. Continue to apply the consistency conditions to generate higher–order constraints. Let $\psi_m(q, p) = 0$ denote all secondary, tertiary, *etc* constraints.
6. Define the total Hamiltonian H_T by incorporating restrictions on Lagrange multipiers into the primary Hamiltonian H_P.

Dirac–Bergmann algorithm

7. Separate the primary, secondary, tertiary, *etc* constraints (ϕ_a and ψ_m) into first class constraints $\mathcal{C}_{\alpha}^{(fc)}$ and second class constraints $\mathcal{C}_{\mu}^{(sc)}$. First class constraints have the property that their Poisson bracket with *all* constraints vanish when the constraints hold.

Dirac–Bergmann algorithm

7. Separate the primary, secondary, tertiary, *etc* constraints (ϕ_a and ψ_m) into first class constraints $\mathcal{C}_\alpha^{(fc)}$ and second class constraints $\mathcal{C}_\mu^{(sc)}$. First class constraints have the property that their Poisson bracket with *all* constraints vanish when the constraints hold.
8. The subset of primary constraints ϕ_a that are first class are called primary first class constraints, $\mathcal{C}_A^{(pfc)}$. The total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$, where the first class Hamiltonian H_{fc} has vanishing Poisson brackets with all constraints.

Dirac–Bergmann algorithm

7. Separate the primary, secondary, tertiary, *etc* constraints (ϕ_a and ψ_m) into first class constraints $\mathcal{C}_\alpha^{(fc)}$ and second class constraints $\mathcal{C}_\mu^{(sc)}$. First class constraints have the property that their Poisson bracket with *all* constraints vanish when the constraints hold.
8. The subset of primary constraints ϕ_a that are first class are called primary first class constraints, $\mathcal{C}_A^{(pfc)}$. The total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$, where the first class Hamiltonian H_{fc} has vanishing Poisson brackets with all constraints.
   - The equations of motion from H_T are equivalent to Lagrange's equations. Since Λ_A's are arbitrary, primary first class constraints $\mathcal{C}_A^{(pfc)}$ generate gauge transformations. These transformations do not change the physical state of the system.

Dirac–Bergmann algorithm

7. Separate the primary, secondary, tertiary, *etc* constraints (ϕ_a and ψ_m) into first class constraints $\mathcal{C}_\alpha^{(fc)}$ and second class constraints $\mathcal{C}_\mu^{(sc)}$. First class constraints have the property that their Poisson bracket with *all* constraints vanish when the constraints hold.
8. The subset of primary constraints ϕ_a that are first class are called primary first class constraints, $\mathcal{C}_A^{(pfc)}$. The total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$, where the first class Hamiltonian H_{fc} has vanishing Poisson brackets with all constraints.
    - The equations of motion from H_T are equivalent to Lagrange's equations. Since Λ_A's are arbitrary, primary first class constraints $\mathcal{C}_A^{(pfc)}$ generate gauge transformations. These transformations do not change the physical state of the system.
    - The Dirac conjecture says that *all* first class constraints $\mathcal{C}_\alpha^{(fc)}$ generate gauge transformations.

Dirac–Bergmann algorithm

7. Separate the primary, secondary, tertiary, *etc* constraints (ϕ_a and ψ_m) into first class constraints $\mathcal{C}_\alpha^{(fc)}$ and second class constraints $\mathcal{C}_\mu^{(sc)}$. First class constraints have the property that their Poisson bracket with *all* constraints vanish when the constraints hold.
8. The subset of primary constraints ϕ_a that are first class are called primary first class constraints, $\mathcal{C}_A^{(pfc)}$. The total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$, where the first class Hamiltonian H_{fc} has vanishing Poisson brackets with all constraints.
   - The equations of motion from H_T are equivalent to Lagrange's equations. Since Λ_A's are arbitrary, primary first class constraints $\mathcal{C}_A^{(pfc)}$ generate gauge transformations. These transformations do not change the physical state of the system.
   - The Dirac conjecture says that *all* first class constraints $\mathcal{C}_\alpha^{(fc)}$ generate gauge transformations.
9. Assuming the Dirac conjecture holds, define the extended Hamiltonian $H_E = H_{fc} + \Lambda^\alpha \mathcal{C}_\alpha^{(fc)}$.

Dirac–Bergmann algorithm

10. Phase space functions are evolved in time using $\dot{F} = \{F, H_E\}$ or $\dot{F} = \{F, H_T\}$. The description can be reduced by replacing the Poisson bracket with the Dirac bracket. Options:
    - Eliminate second class constraints leaving gauge freedom intact. Let $\mathcal{M}_{\mu\nu} = \{\mathcal{C}_\mu^{(sc)}, \mathcal{C}_\nu^{(sc)}\}$ and define

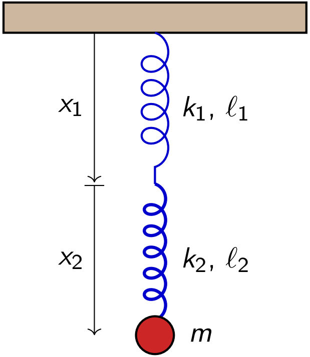
      $$\{F, G\}^* = \{F, G\} - \{F, \mathcal{C}_\mu^{(sc)}\}\mathcal{M}^{\mu\nu}\{\mathcal{C}_\nu^{(sc)}, G\}$$

      Use $\mathcal{C}_\mu^{(sc)} = 0$ to eliminate variables in H_E or H_T, which yields the partially reduced Hamiltonian H_{PR}. Time evolution is $\dot{F} = \{F, H_{PR}\}^*$.
    - Impose gauge conditions and eliminate both first and second class constraints. Let $\mathcal{C}_M^{(all)}$ denote all constraints and gauge conditions and define $\mathcal{M}_{MN} = \{\mathcal{C}_M^{(all)}, \mathcal{C}_N^{(all)}\}$. Dirac bracket is

      $$\{F, G\}^* = \{F, G\} - \{F, \mathcal{C}_M^{(all)}\}\mathcal{M}^{MN}\{\mathcal{C}_N^{(all)}, G\}$$

      Use $\mathcal{C}_M^{(all)} = 0$ to eliminate variables, yielding the fully reduced Hamiltonian H_{FR}. Time evolution is $\dot{F} = \{F, H_{FR}\}^*$.

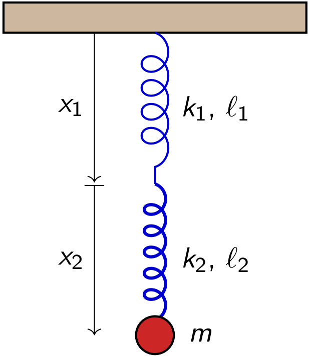
Example: Compound spring



Lagrangian:

$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2$$

Example: Compound spring



Lagrangian:

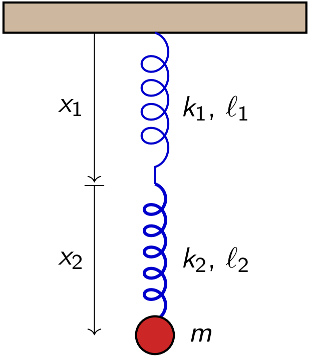
$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2$$

Mass matrix

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \begin{pmatrix} m & m \\ m & m \end{pmatrix}$$

is singular.

Example: Compound spring

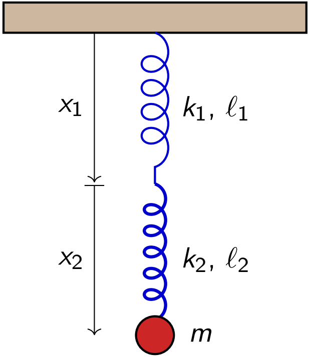


Lagrange's equations:

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= mg - k_1(x_1 - \ell_1) \\ m(\ddot{x}_1 + \ddot{x}_2) &= mg - k_2(x_2 - \ell_2) \end{aligned}$$

Can't solve for accelerations. Is the initial value problem well defined?

Example: Compound spring



Lagrange's equations:

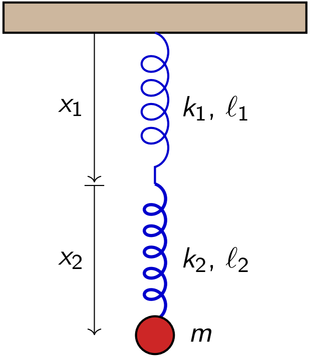
$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= mg - k_1(x_1 - \ell_1) \\ m(\ddot{x}_1 + \ddot{x}_2) &= mg - k_2(x_2 - \ell_2) \end{aligned}$$

Can't solve for accelerations. Is the initial value problem well defined?

Lagrange's equations imply Newton's third law:

$$\underbrace{k_1(x_1 - \ell_1)}_{\text{force of 1 on 2}} = \underbrace{k_2(x_2 - \ell_2)}_{-\text{force of 2 on 1}}$$

Example: Compound spring



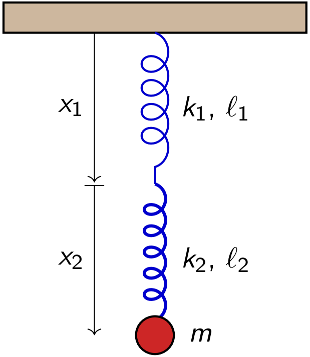
Solve for x_1:

$$x_1 = \frac{k_2 x_2 - k_2 \ell_2 + k_1 \ell_1}{k_1}$$

and insert into one of Lagrange's equations:

$$\ddot{x}_2 = -\frac{k_1 k_2}{m(k_1 + k_2)} x_2 + \frac{k_1}{k_1 + k_2} (g + k_2 \ell_2 / m)$$

Example: Compound spring



Solve for x_1:

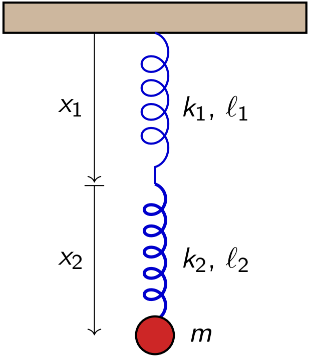
$$x_1 = \frac{k_2 x_2 - k_2 \ell_2 + k_1 \ell_1}{k_1}$$

and insert into one of Lagrange's equations:

$$\ddot{x}_2 = -\frac{k_1 k_2}{m(k_1 + k_2)} x_2 + \frac{k_1}{k_1 + k_2} \left(g + k_2 \ell_2 / m\right)$$

The evolution of the system is uniquely determined from initial data $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$, $\dot{x}_2(0)$, as long as the data satisfies Newton's third law.

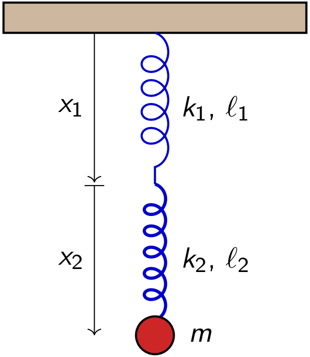
Example: Compound spring



Apply the Dirac–Bergmann algorithm. Lagrangian:

$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2$$

Example: Compound spring



Apply the Dirac–Bergmann algorithm. Lagrangian:

$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2$$

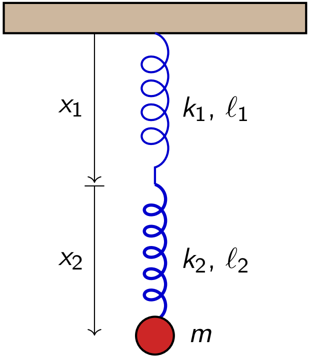
1. Conjugate momenta:

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m(\dot{x}_1 + \dot{x}_2)$$
$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m(\dot{x}_1 + \dot{x}_2)$$

Primary constraint:

$$\phi \equiv p_2 - p_1 = 0$$

Example: Compound spring

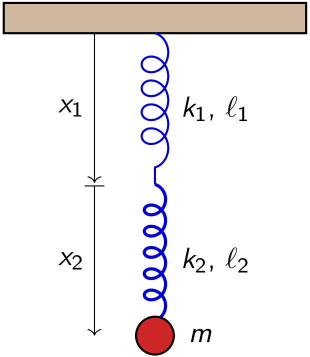


2. Construct the canonical Hamiltonian:

$$\begin{aligned} H_C &= p_1\dot{x}_1 + p_2\dot{x}_2 - L \\ &= \frac{1}{2m}p_1p_2 - mg(x_1+x_2) + \frac{k_1}{2}(x_1-\ell_1)^2 + \frac{k_2}{2}(x_2-\ell_2)^2 \end{aligned}$$

(H_C is not unique.)

Example: Compound spring



2. Construct the canonical Hamiltonian:

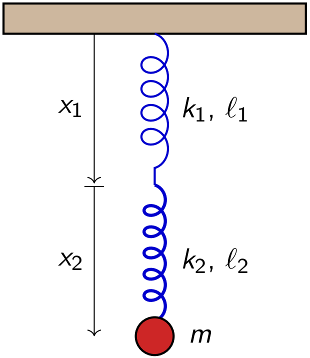
$$\begin{aligned} H_C &= p_1\dot{x}_1 + p_2\dot{x}_2 - L \\ &= \frac{1}{2m}p_1p_2 - mg(x_1+x_2) + \frac{k_1}{2}(x_1-\ell_1)^2 + \frac{k_2}{2}(x_2-\ell_2)^2 \end{aligned}$$

(H_C is not unique.)

3. Define the primary Hamiltonian:

$$\begin{aligned} H_P &= H_C + \lambda\phi \\ &= \frac{1}{2m}p_1p_2 - mg(x_1+x_2) + \frac{k_1}{2}(x_1-\ell_1)^2 \\ &\quad + \frac{k_2}{2}(x_2-\ell_2)^2 + \lambda(p_2-p_1) \end{aligned}$$

Example: Compound spring

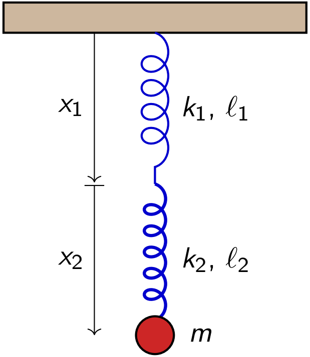


4. Apply consistency condition:

$$\{\phi, H_P\} = 0 \Longrightarrow \psi \equiv k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2) = 0$$

This is a secondary constraint (Newton's third law).

Example: Compound spring



4. Apply consistency condition:

$$\{\phi, H_P\} = 0 \Longrightarrow \psi \equiv k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2) = 0$$

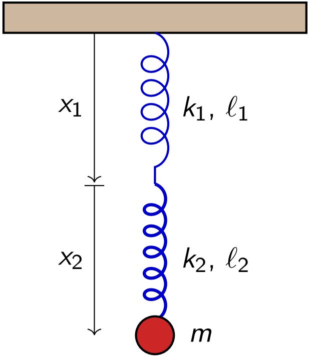
   This is a secondary constraint (Newton's third law).

5. Apply consistency condition:

$$\{\psi, H_P\} = 0 \Longrightarrow \lambda = \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)}$$

   This is a restriction on the Lagrange multiplier.

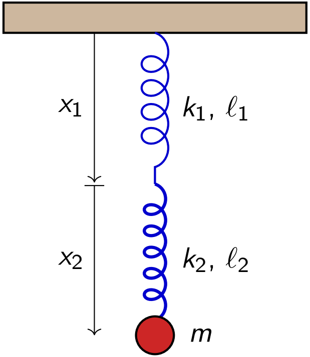
Example: Compound spring



6. Define the total Hamiltonian:

$$H_T = \frac{1}{2m}p_1p_2 - mg(x_1 + x_2) + \frac{k_1}{2}(x_1 - \ell_1)^2 + \frac{k_2}{2}(x_2 - \ell_2)^2 + \frac{k_1p_2 - k_2p_1}{2m(k_1 + k_2)}(p_2 - p_1)$$

Example: Compound spring



6. Define the total Hamiltonian:

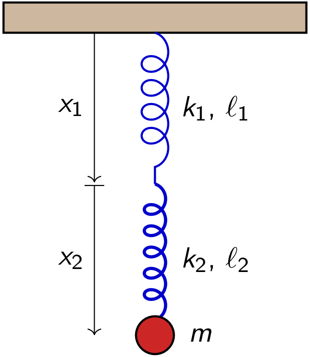
$$H_T = \frac{1}{2m} p_1 p_2 - mg(x_1 + x_2) + \frac{k_1}{2}(x_1 - \ell_1)^2 + \frac{k_2}{2}(x_2 - \ell_2)^2 + \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)}(p_2 - p_1)$$

7. Classify constraints as first or second class:

$$\{\phi, \psi\} = \{p2 - p1, k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2)\} = k_1 + k_2 \neq 0$$

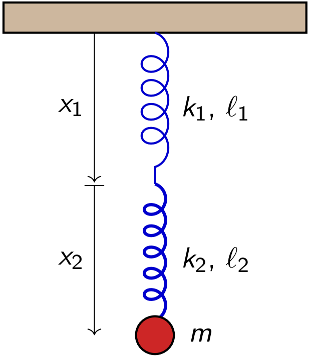
ϕ and ψ are both second class.

Example: Compound spring



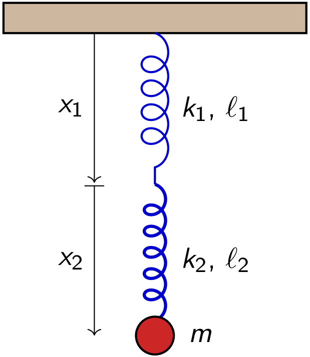
8. There are no first class constraints, so there is no gauge freedom. The first class Hamiltonian H_{fc} concides with the total Hamiltonian H_T.

Example: Compound spring



8. There are no first class constraints, so there is no gauge freedom. The first class Hamiltonian H_{fc} concides with the total Hamiltonian H_T.
9. The Dirac conjecture does not apply. The extended Hamiltonian H_E coincides with the total Hamiltonian H_T.

Example: Compound spring



10. Eliminate the second class constraints using the Dirac bracket. Define $\mathcal{C}_1^{(sc)} = \phi$, $\mathcal{C}_2^{(sc)} = \psi$ and

$$\mathcal{M}_{\mu\nu} = \{\mathcal{C}_\mu^{(sc)}, \mathcal{C}_\nu^{(sc)}\} = \begin{pmatrix} 0 & k_1 + k_2 \\ -k_1 - k_2 & 0 \end{pmatrix}$$

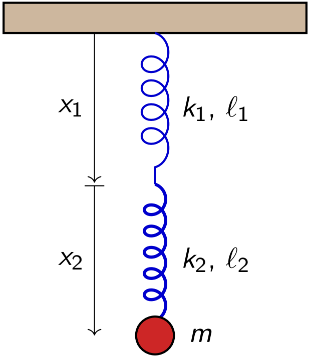
with inverse

$$\mathcal{M}^{\mu\nu} = \frac{1}{k_1 + k_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then $\{F, G\}^* = \{F, G\} - \{F, \mathcal{C}_\mu^{(sc)}\}\mathcal{M}^{\mu\nu}\{\mathcal{C}_\nu^{(sc)}, G\}$ gives

$$\{x_1, p_1\}^* = \{x_1, p_2\}^* = k_2/(k_1 + k_2)$$
$$\{x_2, p_1\}^* = \{x_2, p_2\}^* = k_1/(k_1 + k_2)$$

Example: Compound spring



10. (continued) Use the constraints $\phi = p_2 - p_1 = 0$ and $\psi = k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2) = 0$ to eliminate x_1 and p_1 from the total Hamiltonian H_T:

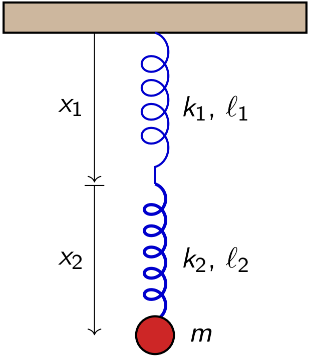
$$x_1 = \ell_1 + \frac{k_2}{k_1}(x_2 - \ell_2)$$

$$p_1 = p_2$$

The reduced Hamiltonian is

$$H_R = \frac{p_2^2}{2m} - \frac{mg}{k_1}[(k_1 + k_2)x_2 + k_1\ell_1 - k_2\ell_2] + \frac{1}{2}(k_2 + k_2^2/k_1)(x_2 - \ell_2)^2$$

Example: Compound spring

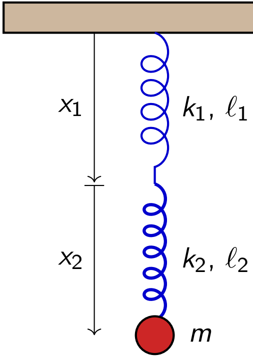


10. (continued) Equations of motion:

$$\dot{x}_2 = \{x_2, H_R\}^* = \frac{k_1}{m(k_1+k_2)} p_2$$

$$\dot{p}_2 = \{p_2, H_R\}^* = mg - k_2(x_2 - \ell_2)$$

Example: Compound spring



10. (continued) Equations of motion:

$$\dot{x}_2 = \{x_2, H_R\}^* = \frac{k_1}{m(k_1 + k_2)} p_2$$

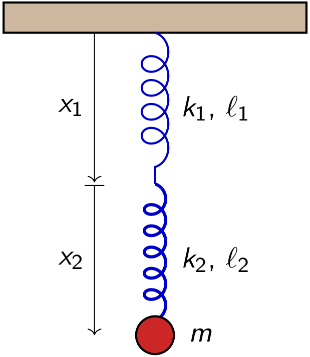
$$\dot{p}_2 = \{p_2, H_R\}^* = mg - k_2(x_2 - \ell_2)$$

Also:

$$\dot{x}_1 = \{x_1, H_R\}^* = \frac{k_2}{m(k_1 + k_2)} p_2$$

$$\dot{p}_1 = \{p_2, H_R\}^* = mg - k_2(x_2 - \ell_2)$$

Example: Compound spring

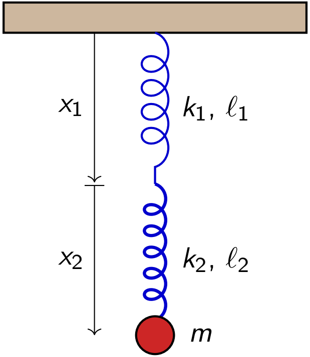


10. (continued) Distance from ceiling to mass:

$$\frac{d(x_1+x_2)}{dt} = \{x_1+x_2, H_R\}^* = \frac{p_2}{m}$$
$$\frac{d^2(x_1+x_2)}{dt^2} = \{p_2/m, H_R\}^* = g - \frac{k_2}{m}(x_2-\ell_2)$$

Agrees with Lagrange's equations.

Example: Compound spring



10. (continued) Distance from ceiling to mass:

$$\frac{d(x_1+x_2)}{dt} = \{x_1+x_2, H_R\}^* = \frac{p_2}{m}$$
$$\frac{d^2(x_1+x_2)}{dt^2} = \{p_2/m, H_R\}^* = g - \frac{k_2}{m}(x_2-\ell_2)$$

Agrees with Lagrange's equations. Use $\psi = 0$ to write as:

$$m(\ddot{x}_1+\ddot{x}_2) = mg - \frac{k_1 k_2}{k_1+k_2}(x_1+x_2-\ell_1-\ell_2)$$

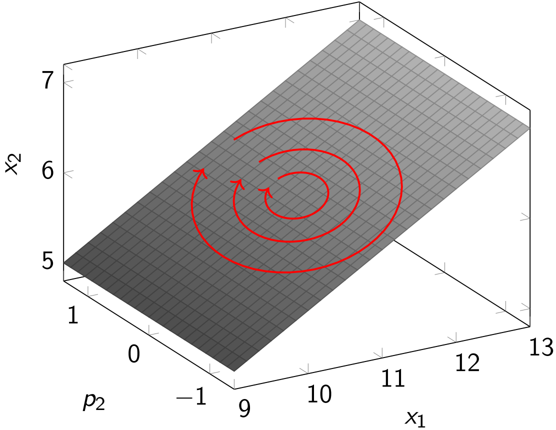
Example: Compound spring

The *constraint surface* is the subspace of phase space where the constraints hold.

$\phi = p_2 - p_1 = 0$

$\psi = k_2(x_2 - \ell_2) - k_1(x_1 - \ell_1) = 0$

$(k_1 = 1,\ k_2 = 2,\ \ell_1 = \ell_2 = 1,\ m = 1,\ g = 10)$



Next time: First class constraints and **gauge theories**