Singular Lagrangians and Constrained Hamiltonian Systems (Part 2)

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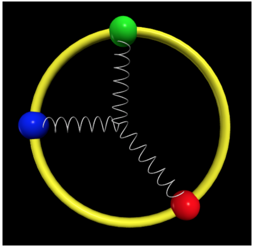

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Review: Dirac–Bergmann Algorithm

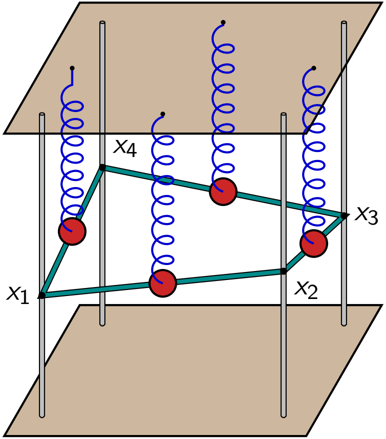
1. From the Lagrangian $L(q, \dot{q})$, define the conjugate momenta by $p_i = \partial L / \partial \dot{q}_i$. Identify the primary constraints $\phi_a(q, p) = 0$.
2. Define the canonical Hamiltonian $H_C(q, p)$ as $p_i \dot{q}_i - L$ written in terms of q's and p's.
3. Define the primary Hamiltonian by $H_P = H_C + \lambda^a \phi_a$, where λ^a are Lagrange multipliers.
4. Impose the consistency conditions $\{\phi_a, H_P\} = 0$ on the primary constraints. This can yield restrictions on the q's and p's, which are secondary constraints $\psi_m(q, p) = 0$, and/or restrictions on the λ's.
5. Apply the consistency conditions to the secondary constraints, $\{\psi_m, H_P\} = 0$. This can yield restrictions on the q's and p's, which are tertiary constraints, and/or restrictions on the λ's. Iterate as necessary. Let $\psi_m(q, p) = 0$ denote all secondary, tertiary, *etc* constraints.

Review: Dirac–Bergmann Algorithm

6. Define the total Hamiltonian H_T by incorporating restrictions on Lagrange multipiers into the primary Hamiltonian H_P.
7. Separate all constraints into first class $\mathcal{C}_\alpha^{(fc)}$ and second class $\mathcal{C}_\mu^{(sc)}$. The $\mathcal{C}_\alpha^{(fc)}$'s have vanishing Poisson brackets with all constraints. The $\mathcal{C}_\mu^{(sc)}$'s have nonvanishing Poisson brackets other second class constraints.
8. The total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$, where H_{fc} is the first class Hamiltonian and the Λ_A's are arbitrary. The primary first class constraints $\mathcal{C}_A^{(pfc)}$ generate gauge transformations.
9. The Dirac conjecture says that all first class constraints generate gauge transformations. Define the extended Hamiltonian by $H_E = H_{fc} + \Lambda^\alpha \mathcal{C}_\alpha^{(fc)}$.
10. Use constraints (either second class or first class plus second class plus gauge conditions) to construct the Dirac bracket. Eliminate variables from H_T or H_E to obtain a reduced Hamiltonian H_R. Time evolution uses H_R and the Dirac bracket.

Example: Masses, rods and springs

Four masses fixed at the midpoints of massless, freely extensible rods. (Rods expand and contract as needed to span the distance between vertical posts.)



Example: Masses, rods and springs

$$T = \frac{m}{2}\left[\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\left(\frac{\dot{x}_2+\dot{x}_3}{2}\right)^2+\left(\frac{\dot{x}_3+\dot{x}_4}{2}\right)^2+\left(\frac{\dot{x}_4+\dot{x}_1}{2}\right)^2\right]$$

$$V = mg\left[x_1+x_2+x_3+x_4\right] + \frac{k}{2}\left[\left(\frac{x_1+x_2}{2}-a\right)^2+\left(\frac{x_2+x_3}{2}-a\right)^2+\left(\frac{x_3+x_4}{2}-a\right)^2+\left(\frac{x_4+x_1}{2}-a\right)^2\right]$$

where $a =$ ceiling height $-$ relaxed spring length $=$ const.

Example: Masses, rods and springs

$$T = \frac{m}{2}\left[\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\left(\frac{\dot{x}_2+\dot{x}_3}{2}\right)^2+\left(\frac{\dot{x}_3+\dot{x}_4}{2}\right)^2+\left(\frac{\dot{x}_4+\dot{x}_1}{2}\right)^2\right]$$

$$V = mg\left[x_1+x_2+x_3+x_4\right] + \frac{k}{2}\left[\left(\frac{x_1+x_2}{2}-a\right)^2+\left(\frac{x_2+x_3}{2}-a\right)^2+\left(\frac{x_3+x_4}{2}-a\right)^2+\left(\frac{x_4+x_1}{2}-a\right)^2\right]$$

where a = ceiling height − relaxed spring length = const.
Lagrangian = $T - V$. Mass matrix (Hessian)

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} = \frac{1}{4}\begin{pmatrix} 2m & m & 0 & m \\ m & 2m & m & 0 \\ 0 & m & 2m & m \\ m & 0 & m & 2m \end{pmatrix}$$

is singular, $\det M = 0$.

Example: Masses, rods and springs

1. Conjugate momenta:

$$
\begin{aligned}
p_1 &= \frac{m}{4}(\dot{x}_4 + 2\dot{x}_1 + \dot{x}_2) \\
p_2 &= \frac{m}{4}(\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3) \\
p_3 &= \frac{m}{4}(\dot{x}_2 + 2\dot{x}_3 + \dot{x}_4) \\
p_4 &= \frac{m}{4}(\dot{x}_3 + 2\dot{x}_4 + \dot{x}_1)
\end{aligned}
$$

Example: Masses, rods and springs

1. Conjugate momenta:

$$p_1 = \frac{m}{4}(\dot{x}_4 + 2\dot{x}_1 + \dot{x}_2)$$
$$p_2 = \frac{m}{4}(\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3)$$
$$p_3 = \frac{m}{4}(\dot{x}_2 + 2\dot{x}_3 + \dot{x}_4)$$
$$p_4 = \frac{m}{4}(\dot{x}_3 + 2\dot{x}_4 + \dot{x}_1)$$

Primary constraint:

$$\phi = p_1 - p_2 + p_3 - p_4$$

Example: Masses, rods and springs

2. Canonical Hamiltonian:

$$
\begin{aligned}
H_C =& \frac{1}{2m}\left[\frac{5}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) - 2(p_1 p_2 + p_2 p_3 + p_3 p_4 + p_4 p_1) + p_1 p_3 + p_2 p_4\right] \\
&+ mg\left[x_1 + x_2 + x_3 + x_4\right] \\
&+ \frac{k}{2}\left[\left(\frac{x_1 + x_2}{2} - a\right)^2 + \left(\frac{x_2 + x_3}{2} - a\right)^2 + \left(\frac{x_3 + x_4}{2} - a\right)^2 + \left(\frac{x_4 + x_1}{2} - a\right)^2\right]
\end{aligned}
$$

Example: Masses, rods and springs

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&+ mg\left[x_1 + x_2 + x_3 + x_4\right] \\
&+ \frac{k}{2}\left[\left(\frac{x_1 + x_2}{2} - a\right)^2 + \left(\frac{x_2 + x_3}{2} - a\right)^2 + \left(\frac{x_3 + x_4}{2} - a\right)^2 + \left(\frac{x_4 + x_1}{2} - a\right)^2\right]
\end{aligned}
$$

3. Primary Hamiltonian:

$$H_P = H_C + \lambda\phi$$

Example: Masses, rods and springs

4. Consistency conditions:

$$\begin{aligned}\{\phi, H_P\} =& \{p_1 - p_2 + p_3 - p_4, H_P\} \\ =& \cdots \\ =& 0\end{aligned}$$

No secondary constraints, no restrictions on Lagrange multipliers.

Example: Masses, rods and springs

4. Consistency conditions:

$$\begin{aligned}\{\phi, H_P\} =& \{p_1 - p_2 + p_3 - p_4, H_P\} \\ =& \cdots \\ =& 0\end{aligned}$$

   No secondary constraints, no restrictions on Lagrange multipliers.

5. No more consistency conditions.

Example: Masses, rods and springs

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5. No more consistency conditions.
6. Total Hamiltonian

$$H_T = H_P$$

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   No secondary constraints, no restrictions on Lagrange multipliers.

5. No more consistency conditions.
6. Total Hamiltonian

$$H_T = H_P$$

7. Classify constraints:

$$\{\phi, \phi\} = 0$$

   so ϕ is first class. Since ϕ is also primary, it generates a gauge transformation.

8. Total Hamiltonian has the form $H_T = H_{fc} + \Lambda\mathcal{C}^{(pfc)}$ where (in this example) $H_{fc} = H_C$ and $\Lambda = \lambda$ and $\mathcal{C}^{(pfc)} = \phi$ is the generator of gauge transformations.

Example: Masses, rods and springs

8. Total Hamiltonian has the form $H_T = H_{fc} + \Lambda \mathcal{C}^{(pfc)}$ where (in this example) $H_{fc} = H_C$ and $\Lambda = \lambda$ and $\mathcal{C}^{(pfc)} = \phi$ is the generator of gauge transformations.
9. There are no secondary (or tertiary or ...) first class constraints, so the Dirac conjecture does not apply. The extended Hamiltonian coincides with H_T, which coincides with H_P:

$$
\begin{aligned}
H_P =& \frac{1}{2m}\left[\frac{5}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) - 2(p_1 p_2 + p_2 p_3 + p_3 p_4 + p_4 p_1) + p_1 p_3 + p_2 p_4\right] \\
&+ mg\left[y_1 + y_2 + y_3 + y_4\right] \\
&+ \frac{k}{2}\left[\left(\frac{y_1 + y_2}{2} - a\right)^2 + \left(\frac{y_2 + y_3}{2} - a\right)^2 + \left(\frac{y_3 + y_4}{2} - a\right)^2 + \left(\frac{y_4 + y_1}{2} - a\right)^2\right] \\
&+ \lambda(p_1 - p_2 + p_3 - p_4)
\end{aligned}
$$

Example: Masses, rods and springs

Hamilton's equations $\dot{x}_i = \{x_i, H_P\}$ and $\dot{p}_i = \{p_i, H_P\}$:

$$\dot{x}_1 = \frac{1}{2m}[5p_1 - 2p_2 + p_3 - 2p_4] + \lambda$$

$$\dot{x}_2 = \frac{1}{2m}[-2p_1 + 5p_2 - 2p_3 + p_4] - \lambda$$

$$\dot{x}_3 = \frac{1}{2m}[p_1 - 2p_2 + 5p_3 - 2p_4] + \lambda$$

$$\dot{x}_4 = \frac{1}{2m}[-2p_1 + p_2 - 2p_3 + 5p_4] - \lambda$$

$$\dot{p}_1 = -\frac{k}{4}[2x_1 + x_2 + x_4 - 4a] - mg$$

$$\dot{p}_2 = -\frac{k}{4}[x_1 + 2x_2 + x_3 - 4a] - mg$$

$$\dot{p}_3 = -\frac{k}{4}[x_2 + 2x_3 + x_4 - 4a] - mg$$

$$\dot{p}_4 = -\frac{k}{4}[x_1 + x_3 + 2x_4 - 4a] - mg$$

Example: Masses, rods and springs

Gauge transformations generated by $G = \epsilon\phi = \epsilon(p_1 - p_2 + p_3 - p_4)$ do not change the physical state of the system.

$$\delta x_1 = \{x_1, G\} = \epsilon$$
$$\delta x_2 = \{x_2, G\} = -\epsilon$$
$$\delta x_3 = \{x_3, G\} = \epsilon$$
$$\delta x_4 = \{x_4, G\} = -\epsilon$$
$$\delta p_1 = \delta p_2 = \delta p_3 = \delta p_4 = 0$$

Example: Masses, rods and springs

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\delta p_1 &= \delta p_2 = \delta p_3 = \delta p_4 = 0
\end{aligned}$$

The phase space point $x_i + \delta x_i$, $p_i + \delta p_i$ represents the same physical state as x_i, p_i. Integrate to obtain gauge orbits.

Example: Masses, rods and springs

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\delta x_4 &= \{x_4, G\} = -\epsilon \\
\delta p_1 &= \delta p_2 = \delta p_3 = \delta p_4 = 0
\end{aligned}$$

The phase space point $x_i + \delta x_i,\ p_i + \delta p_i$ represents the same physical state as $x_i,\ p_i$.
Integrate to obtain gauge orbits.
Observables are gauge invariant functions such as the locations of masses. For example:

$$\delta\left((x_1 + x_2)/2\right) = \{(x_1 + x_2)/2, G\} = \epsilon/2 - \epsilon/2 = 0$$

Example: Masses, rods and springs

The phase space action (built from $H_E = H_T = H_P$) is

$$S[x, p, \lambda] = \int_0^T dt\, [p_i \dot{x}_i - H_P]$$

We can eliminate the p's using the equations of motion $\delta S / \delta p_i = 0$:

$$p_1 = \frac{m}{20}(11\dot{x}_1 + 4\dot{x}_2 + \dot{x}_3 + 4\dot{x}_4 - 4\lambda)$$
$$p_2 = \frac{m}{20}(4\dot{x}_1 + 11\dot{x}_2 + 4\dot{x}_3 + \dot{x}_4 + 4\lambda)$$
$$p_3 = \frac{m}{20}(\dot{x}_1 + 4\dot{x}_2 + 11\dot{x}_3 + 4\dot{x}_4 - 4\lambda)$$
$$p_4 = \frac{m}{20}(4\dot{x}_1 + \dot{x}_2 + 4\dot{x}_3 + 11\dot{x}_4 + 4\lambda)$$

Action becomes

$$S[x, \lambda] = \int_0^T dt\, L(x, \dot{x}, \lambda)$$

Example: Masses, rods and springs

with Lagrangian:

$$\begin{aligned}
L(x,\dot{x},\lambda) = & \frac{m}{40}\left[11(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)+8(\dot{x}_1\dot{x}_2+\dot{x}_2\dot{x}_3+\dot{x}_3\dot{x}_4+\dot{x}_4\dot{x}_1)\right. \\
& \left.+2(\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_4)\right]-\frac{m}{5}\lambda(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)+\frac{2m}{5}\lambda^2 \\
& -\frac{k}{8}\left[(x_1+x_2)^2+(x_2+x_3)^2+(x_3+x_4)^2+(x_4+x_1)^2\right] \\
& -(mg-ak)(x_1+x_2+x_3+x_4)-2a^2k
\end{aligned}$$

Example: Masses, rods and springs

with Lagrangian:

$$
\begin{aligned}
L(x,\dot{x},\lambda) =& \frac{m}{40}\big[11(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)+8(\dot{x}_1\dot{x}_2+\dot{x}_2\dot{x}_3+\dot{x}_3\dot{x}_4+\dot{x}_4\dot{x}_1)\\
&+2(\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_4)\big]-\frac{m}{5}\lambda(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)+\frac{2m}{5}\lambda^2\\
&-\frac{k}{8}\big[(x_1+x_2)^2+(x_2+x_3)^2+(x_3+x_4)^2+(x_4+x_1)^2\big]\\
&-(mg-ak)(x_1+x_2+x_3+x_4)-2a^2k
\end{aligned}
$$

Aside: We can eliminate λ using the equation of motion $\delta S/\delta\lambda = 0$:

$$
\lambda = \frac{1}{4}(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)
$$

Action becomes $S[x] = \int_0^T dt\, L(x,\dot{x})$ where $L(x,\dot{x})$ is the Lagrangian we started with.

Example: Masses, rods and springs

The Lagrangian

$$\begin{aligned} L(x,\dot{x},\lambda) =& \frac{m}{40}\Big[11(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)+8(\dot{x}_1\dot{x}_2+\dot{x}_2\dot{x}_3+\dot{x}_3\dot{x}_4+\dot{x}_4\dot{x}_1) \\ &+2(\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_4)\Big]-\frac{m}{5}\lambda(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)+\frac{2m}{5}\lambda^2 \\ &-\frac{k}{8}\Big[(x_1+x_2)^2+(x_2+x_3)^2+(x_3+x_4)^2+(x_4+x_1)^2\Big] \\ &-(mg-ak)(x_1+x_2+x_3+x_4)-2a^2k \end{aligned}$$

is analogous to many gauge field theories.

Example: Masses, rods and springs

The Lagrangian

$$\begin{aligned} L(x,\dot{x},\lambda) = & \frac{m}{40}\big[11(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)+8(\dot{x}_1\dot{x}_2+\dot{x}_2\dot{x}_3+\dot{x}_3\dot{x}_4+\dot{x}_4\dot{x}_1) \\ & +2(\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_4)\big] - \frac{m}{5}\lambda(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)+\frac{2m}{5}\lambda^2 \\ & -\frac{k}{8}\big[(x_1+x_2)^2+(x_2+x_3)^2+(x_3+x_4)^2+(x_4+x_1)^2\big] \\ & -(mg-ak)(x_1+x_2+x_3+x_4)-2a^2k \end{aligned}$$

is analogous to many gauge field theories. Maxwell Lagrangian ($F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$):

$$\begin{aligned} S[A] &= -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} \\ &= \int dt\, d^3x \left[\frac{1}{2}(\dot{A}_i-\partial_i A_0)(\dot{A}_i-\partial_i A_0)-\frac{1}{4}(\partial_i A_j-\partial_j A_i)(\partial_i A_j-\partial_j A_i)\right] \end{aligned}$$

where $i=1,2,3$. $\dot{A}_0$ doesn't appear because A_0 is a Lagrange multiplier.

Example: Masses, rods and springs

The Lagrangian

$$\begin{aligned} L(x,\dot{x},\lambda) = &\frac{m}{40}\big[11(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)+8(\dot{x}_1\dot{x}_2+\dot{x}_2\dot{x}_3+\dot{x}_3\dot{x}_4+\dot{x}_4\dot{x}_1) \\ &+2(\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_4)\big]-\frac{m}{5}\lambda(\dot{x}_1-\dot{x}_2+\dot{x}_3-\dot{x}_4)+\frac{2m}{5}\lambda^2 \\ &-\frac{k}{8}\big[(x_1+x_2)^2+(x_2+x_3)^2+(x_3+x_4)^2+(x_4+x_1)^2\big] \\ &-(mg-ak)(x_1+x_2+x_3+x_4)-2a^2k \end{aligned}$$

is analogous to many gauge field theories.

Likewise for GR: Spacetime metric components $1/\sqrt{g^{00}}$ and g_{0i} are Lagrange multipliers (lapse function and shift vector).

Treating λ's as q's leads so new primary constraints! It works, but it's confusing...

Example: Masses, rods and springs

Return to the Hamiltonian:

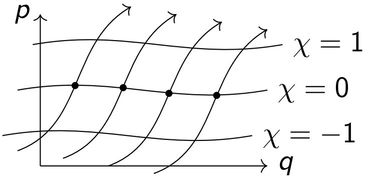
$$
\begin{aligned}
H_P =& \frac{1}{2m}\left[\frac{5}{2}(p_1^2+p_2^2+p_3^2+p_4^2) - 2(p_1p_2+p_2p_3+p_3p_4+p_4p_1) + p_1p_3 + p_2p_4\right] \\
&+ \frac{k}{2}\left[\left(\frac{x_1+x_2}{2}-a\right)^2 + \left(\frac{x_2+x_3}{2}-a\right)^2 + \left(\frac{x_3+x_4}{2}-a\right)^2 + \left(\frac{x_4+x_1}{2}-a\right)^2\right] \\
&+ mg\left[x_1+x_2+x_3+x_4\right] + \lambda(p_1-p_2+p_3-p_4)
\end{aligned}
$$

Example: Masses, rods and springs

Return to the Hamiltonian:

$$H_P = \frac{1}{2m}\left[\frac{5}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) - 2(p_1 p_2 + p_2 p_3 + p_3 p_4 + p_4 p_1) + p_1 p_3 + p_2 p_4\right]$$
$$+ \frac{k}{2}\left[\left(\frac{x_1 + x_2}{2} - a\right)^2 + \left(\frac{x_2 + x_3}{2} - a\right)^2 + \left(\frac{x_3 + x_4}{2} - a\right)^2 + \left(\frac{x_4 + x_1}{2} - a\right)^2\right]$$
$$+ mg\left[x_1 + x_2 + x_3 + x_4\right] + \lambda(p_1 - p_2 + p_3 - p_4)$$

10. Eliminate gauge freedom with a **gauge condition**. Physical states are curves ("gauge orbits") in phase space generated by $G = \epsilon\phi$. A gauge condition $\chi = 0$ selects a representative point on each curve. This requires $\delta\chi = \{\chi, G\} \neq 0$.



Example: Masses, rods and springs

10. (continuted) Choose gauge condition

$$\chi = x_1 - x_2 + x_3 - x_4$$

and check:

$$\delta\chi = \{\chi, G\} = \epsilon\{x_1 - x_2 + x_3 - x_4, p_1 - p_2 + p_3 - p_4\} = 4\epsilon \neq 0$$

Together, ϕ and χ are *second class*: $\{\chi, \phi\} \neq 0$.

10. (continuted) Choose gauge condition

$$\chi = x_1 - x_2 + x_3 - x_4$$

and check:

$$\delta\chi = \{\chi, G\} = \epsilon\{x_1 - x_2 + x_3 - x_4, p_1 - p_2 + p_3 - p_4\} = 4\epsilon \neq 0$$

Together, ϕ and χ are *second class*: $\{\chi, \phi\} \neq 0$.

Construct Dirac bracket: Let $\mathcal{C}_M^{(all)} = (\phi, \chi)$, and $M_{MN} = \{\mathcal{C}_M^{(all)}, \mathcal{C}_N^{(all)}\}$, then

$$\{F, G\}^* = \{F, G\} - \{F, \mathcal{C}_M^{(all)}\}(M^{-1})_{MN}\{\mathcal{C}_N^{(all)}, G\}$$

Example: Masses, rods and springs

Nonzero fundamental Dirac brackets:

$$
\begin{aligned}
3/4 &= \{x_1, p_1\}^* = \{x_2, p_2\}^* = \{x_3, p_3\}^* = \{x_4, p_4\}^* \\
1/4 &= \{x_1, p_2\}^* = \{x_1, p_4\}^* = \{x_2, p_3\}^* = \{x_2, p_1\}^* \\
&= \{x_3, p_4\}^* = \{x_3, p_2\}^* = \{x_4, p_1\}^* = \{x_4, p_3\}^* \\
-1/4 &= \{x_1, p_3\}^* = \{x_2, p_4\}^* = \{x_3, p_1\}^* = \{x_4, p_2\}^*
\end{aligned}
$$

Example: Masses, rods and springs

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3/4 &= \{x_1, p_1\}^* = \{x_2, p_2\}^* = \{x_3, p_3\}^* = \{x_4, p_4\}^* \\
1/4 &= \{x_1, p_2\}^* = \{x_1, p_4\}^* = \{x_2, p_3\}^* = \{x_2, p_1\}^* \\
&= \{x_3, p_4\}^* = \{x_3, p_2\}^* = \{x_4, p_1\}^* = \{x_4, p_3\}^* \\
-1/4 &= \{x_1, p_3\}^* = \{x_2, p_4\}^* = \{x_3, p_1\}^* = \{x_4, p_2\}^*
\end{aligned}$$

Eliminate variables

$$\phi = 0 \Longrightarrow p_4 = p_1 - p_2 + p_3$$

$$\chi = 0 \Longrightarrow x_4 = x_1 - x_2 + x_3$$

and reduce the Hamiltonian:

$$\begin{aligned}
H_R =& \frac{1}{2m}\left[3p_1^2 + 4p_2^2 + 3p_3^2 + 2p_1p_3 - 4p_1p_2 - 4p_2p_3\right] \\
&+ \frac{k}{4}\left[3x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3 - 2x_1x_2 - 2x_2x_3\right] \\
&+ 2(mg - ak)(x_1 + x_3) + 2a^2k
\end{aligned}$$

Example: Masses, rods and springs

Hamilton's equations:

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\dot{x}_1 &= \{y_1, H_R\}^* = (3p_1 - p_3)/(2m) \\
\dot{x}_2 &= \{y_2, H_R\}^* = (-p_1 + 4p_2 - p_3)/(2m) \\
\dot{x}_3 &= \{y_3, H_R\}^* = (-p_1 + 3p_3)/(2m) \\
\dot{x}_4 &= \{y_4, H_R\}^* = (3p_1 - 4p_2 + 3p_3)/(2m) \\
\dot{p}_1 &= \{p_1, H_R\}^* = -(k/4)(3x_1 + x_3) + ak - mg \\
\dot{p}_2 &= \{p_2, H_R\}^* = -(k/4)(x_1 + 2x_2 + x_3) + ak - mg \\
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Movie

Example: Dirac conjecture

The Dirac conjecture says that all first class constraints (not just primary first class constraints) generate gauge transformations.

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Lagrangian:

$$L = \frac{1}{2}(\dot{x}_1 - x_2)^2$$

Conjugate momenta:

$$p_1 = \dot{x}_1 - x_2$$
$$p_2 = 0$$

Primary constraint:

$$\phi = p_2$$

Canonical Hamiltonian:

$$\begin{aligned} H_C &= p_1\dot{x}_1 + p_2\dot{x}_2 - \frac{1}{2}(\dot{x}_1 - x_2)^2 \\ &= \frac{1}{2}p_1^2 + p_1 x_2 \end{aligned}$$

Example: Dirac conjecture

Primary Hamiltonian:

$$H_P = H_C + \lambda\phi = \frac{1}{2}p_1^2 + p_1 x_2 + \lambda p_2$$

Consistency condition:

$$\{\phi, H_P\} = 0 \Longrightarrow p_1 = 0$$

Secondary constraint:

$$\psi = p_1$$

Consistency condition:

$$\{\psi, H_P\} = 0 \Longrightarrow 0 = 0$$

Classify constraints:

$$\{\phi, \psi\} = 0$$

so ϕ and ψ are first class.

Example: Dirac conjecture

There are no restrictions on Lagrange multipliers and no second class constraints. Time evolution can be defined with either the total Hamiltonian H_T or the extended Hamiltonian H_E:

$$H_T = \frac{1}{2}p_1^2 + p_2 x_1 + \lambda p_2$$

$$H_E = \frac{1}{2}p_1^2 + p_2 x_1 + \lambda p_2 + \gamma p_1$$

Example: Dirac conjecture

With $\dot{F} = \{F, H_T\}$:

$$\begin{aligned} \dot{x}_1 &= p1 + x2 \\ \dot{x}_2 &= \lambda \\ \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{aligned}$$

With $\dot{F} = \{F, H_E\}$:

$$\begin{aligned} \dot{x}_1 &= p1 + x2 + \gamma \\ \dot{x}_2 &= \lambda \\ \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{aligned}$$

Example: Dirac conjecture

Evolution with H_E tells us that both x_1 and x_2 can be anything. That's already implicit in the evolution with H_T. Using H_T, the general solution of the equations of motion is

$$\begin{aligned} x_1(t) &= C_1 + C_2 t + \int_0^t ds \left(\int_0^s du\, \lambda(u) \right) \\ x_2(t) &= C_2 + \int_0^t ds\, \lambda(s) \\ p_1(t) &= 0 \\ p_2(t) &= 0 \end{aligned}$$

There is enough freedom in $\lambda(t)$ so that at any give time, say $t = T$, both x_1 and x_2 can be arbitrary.

Example: Dirac conjecture

For example, let $\lambda(t) = a + bt$ with

$$a = -\frac{2}{T^2}(3C_1 - 3K_1 + 2C_2T + K_2T)$$
$$b = \frac{6}{T^3}(2C_1 - 2K_1 + C_2T - K_2T)$$

Then the general solution is

$$x_1(t) = C_1 + C_2t + at^2/2 + bt^3/6$$
$$x_2(t) = C_2 + at + bt^2/2$$

which gives

$$x_1(0) = C_1 \ , \quad x_1(T) = K_1$$
$$x_2(0) = C_2 \ , \quad x_2(T) = K_2$$

Example: Dirac conjecture

Both x_1 and x_2 are "pure gauge" in the sense that they can evolve from initial values C_1, C_2 to arbitrary, independent values K_1, K_2 at any later time T.

The extended Hamiltonian makes the gauge freedom for x_1 explicit by including the arbitrary function $\gamma(t)$ in the equation of motion for x_1.

Example: Dirac conjecture

Examine the gauge freedom at the Lagrangian level. The Lagrangian $L = (\dot{x}_1 - x_2)^2/2$ is invariant under the gauge transformation

$$x_1 \to x_1 + \xi$$
$$x_2 \to x_2 + \dot{\xi}$$

where $\xi(t)$ is arbitrary.

Example: Dirac conjecture

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$$x_1 \to x_1 + \xi$$
$$x_2 \to x_2 + \dot{\xi}$$

where $\xi(t)$ is arbitrary. Lagrange's equations:

$$\ddot{x}_1 = \dot{x}_2$$
$$\dot{x}_1 = x_2$$

General solution:

$$x_1(t) = C_1 + \int_0^t ds\, x_2(s)$$
$$x_2(t) = \text{arbitrary}$$

Example: Dirac conjecture

Let $x_2(t) = at + bt^2/2$ with a and b defined as above. Then:

$$x_1(0) = C_1 \ , \quad x_1(T) = K_1$$
$$x_2(0) = C_2 \ , \quad x_2(T) = K_2$$

We can evolve from C_1, C_2 to any values K_1, K_2 at any later time T. Both x_1 and x_2 are "pure gauge."

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Does this explain in general how secondary first class constraints can generate gauge transformations?

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