

# Interpolating Relativistic Spacetimes and Quantum Fields between Instant Form Dynamics and Light-Front Dynamics.

Department of Physics and Astronomy  
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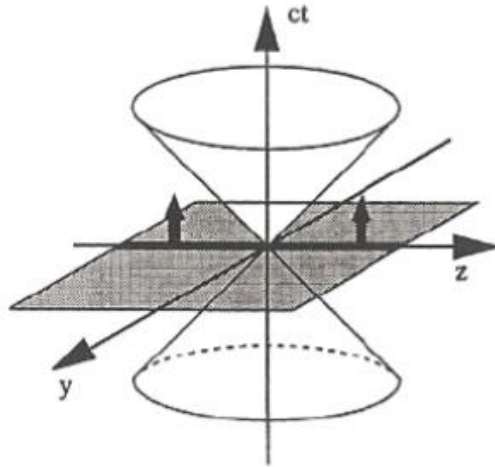
Final Exam  
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03-20-2026



# Outline

- ❑ Introduction
  - Dirac's proposition
  - interpolation between Instant Form Dynamics (IFD) and Light-front Form Dynamics (LFD)
  
- ❑ Quantum orientation entanglement analysis of the interpolating helicity states
  - Interpolating Wigner-d type of matrices of helicity states
  - Angular distribution of interpolating helicity amplitudes  
(D.Dayananda , C.-R. Ji arXiv:2603.18208 [hep-th])
  
- ❑ Manifestation of quantum correlation in the interpolating helicity amplitudes
  - Correlation between orientation entanglement and longitudinal boost.
  - Distinguished features of the LFD  
(forthcoming arXiv)
  
- ❑ Interpolating five-dimensional space-time
  - Interpolating De Sitter and anti-de Sitter spaces and group algebra
  - Contraction of five-dimensional space –time
  
- ❑ Future projects
  
- ❑ Summary and conclusion

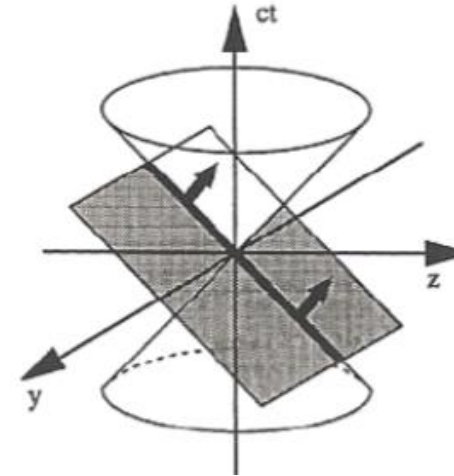
## Dirac's Proposition



The Instant Form



1949



The Front Form

P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949)

### ❑ Instant Form Dynamics

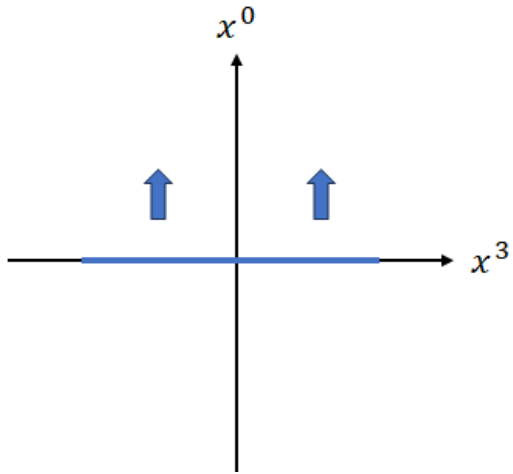
- Traditional approach evolved from non-relativistic dynamics
- Closely related to the Euclidian space
- Application in - LQCD, T-dept QFT, IMF ..

### ❑ Light Front Dynamics

- Innovative approach for relativistic dynamics
- Strictly in Minkowski space
- Application in -PDFs, DIS, GPDs ...

## Instant Form Dynamics vs. Light-Front Dynamics

### Energy-momentum dispersion relations

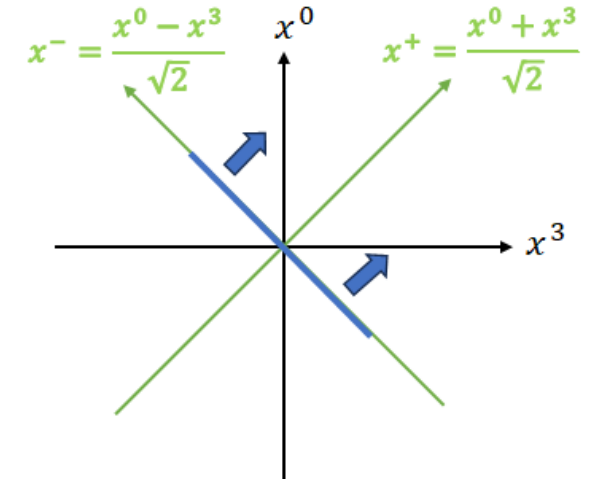


- Instant-form dynamics

$$P^0 = \sqrt{\vec{P}^2 + m^2}$$

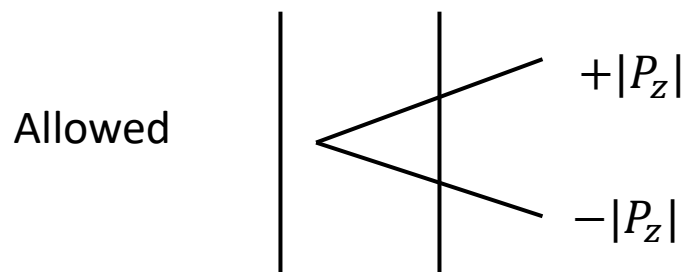
- Light-front dynamics

$$P^- = \frac{P_{\perp}^2 + m^2}{2P^+}$$

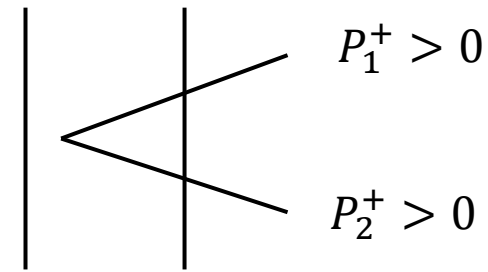


### Time-ordered diagrams : Vacuum fluctuation

Instant time  $x^0 \rightarrow$

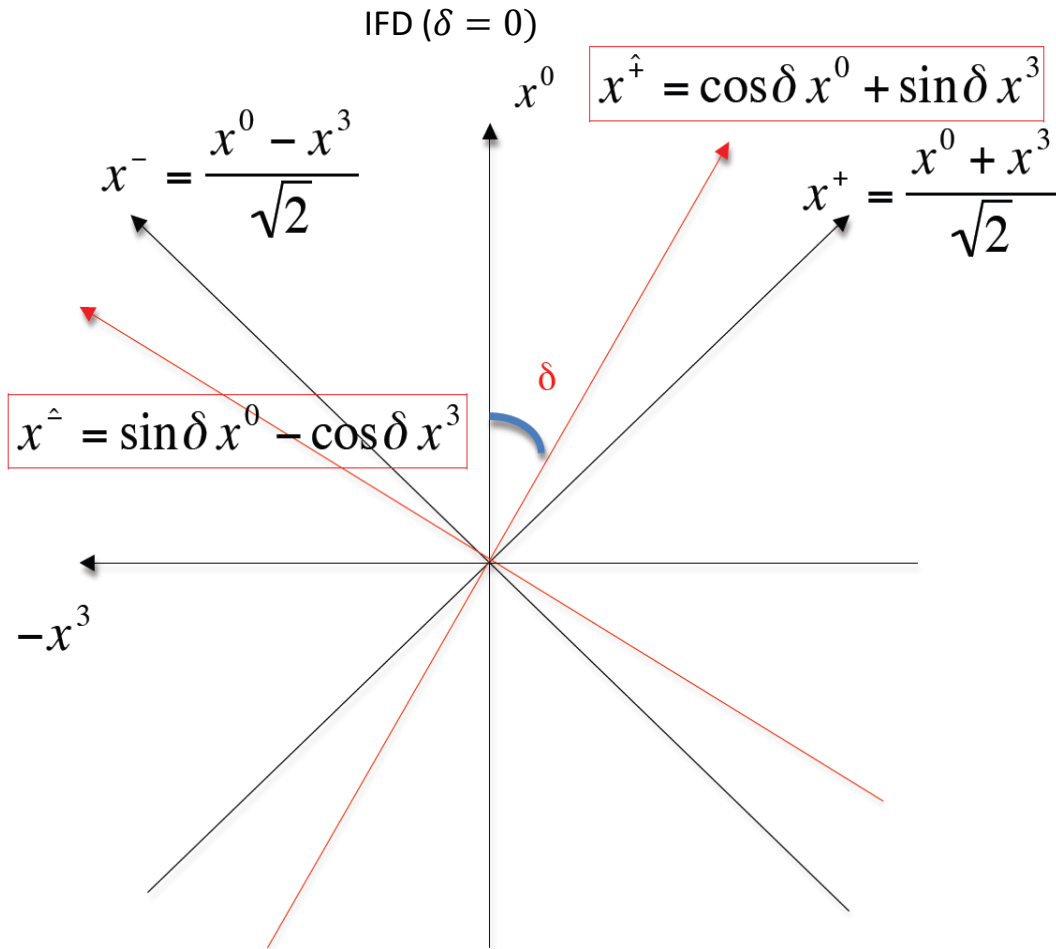


Light-front time  $x^+ \rightarrow$



Not Allowed  
Unless  $P_1^+ = P_2^+ = 0$   
(LF zero-modes)

## Interpolating between Instant form dynamic (IFD) and light-front dynamic (LFD)



$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos(\delta) & 0 & 0 & \sin(\delta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\delta) & 0 & 0 & -\cos(\delta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolation space time matrix

$$g_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \mathbf{C} & 0 & 0 & \mathbf{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbf{S} & 0 & 0 & -\mathbf{C} \end{bmatrix} \quad \begin{aligned} \mathbf{S} &= \text{Sin}(2\delta) \\ \mathbf{C} &= \text{Cos}(2\delta) \end{aligned}$$

- Relate IFD and LFD and show the whole landscape in between.
- Clarify any conceivable confusion between the Infinite momentum frame (IMF) and LFD
- Magnify the vicinity of zero-mode .

K. Hornbostel Phys. Rev. D **45**, 3781 (1992)-RQFT

C.-R. Ji, and S. Rey, Phys. Rev. D **53**, 5815 (1996)-Chiral Anomaly

C.-R. Ji, and A. Suzuki, Phys. Rev. D **87**, 065015 (2013)- Scattering Amps

C.-R. Ji, Z. Li, and A. T. Suzuki, Phys. Rev. D **91**, 065020 (2015)-EM Gauges

C.-R. Ji, Z. Li, and B. Ma Phys. Rev. D **98**, 036017 (2018)- QED

B. Ma, C.-R. Ji Phys. Rev. D **104**, 036004 (2021)-QCD<sub>1+1</sub>

# Quantum Orientation Entanglement Analysis in the Interpolating Helicity States

# Motivation : Orientation Entanglement of Spins

## □ Singlet and triplet states

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \quad \longrightarrow$$

For triplet states: Symmetric wave functions ( $j = 1, m = 1, 0, -1$ )

$$|1, 1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$$

$$|1, -1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$$

For singlet state: Anti-Symmetric wave function ( $j = 0, m = 0$ )

$$|0, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$$

## □ Wigner-d matrix of spin-1/2

$$R_y^{1/2}(\theta_s) = \begin{pmatrix} \cos\left(\frac{\theta_s}{2}\right) & -\sin\left(\frac{\theta_s}{2}\right) \\ \sin\left(\frac{\theta_s}{2}\right) & \cos\left(\frac{\theta_s}{2}\right) \end{pmatrix} \quad \longrightarrow$$

$$R_y^{1/2}(\pi) |\uparrow\rangle = |\downarrow\rangle$$

$$R_y^{1/2}(\pi) |\downarrow\rangle = -|\uparrow\rangle$$

# Motivation : Orientation Entanglement of Spins

□ Tensor product of rotated states

$$R_y^1(\theta_s)|1, 1\rangle = R_y^{1/2}(\theta_s)|\uparrow\rangle \otimes R_y^{1/2}(\theta_s)|\uparrow\rangle,$$

$$R_y^1(\theta_s)|1, 0\rangle = \frac{1}{\sqrt{2}}(R_y^{1/2}(\theta_s)|\uparrow\rangle \otimes R_y^{1/2}(\theta_s)|\downarrow\rangle + R_y^{1/2}(\theta_s)|\downarrow\rangle \otimes R_y^{1/2}(\theta_s)|\uparrow\rangle),$$

$$R_y^1(\theta_s)|1, -1\rangle = R_y^{1/2}(\theta_s)|\downarrow\rangle \otimes R_y^{1/2}(\theta_s)|\downarrow\rangle,$$

$$R_y^0(\theta_s)|0, 0\rangle = \frac{1}{\sqrt{2}}(R_y^{1/2}(\theta_s)|\uparrow\rangle \otimes R_y^{1/2}(\theta_s)|\downarrow\rangle - R_y^{1/2}(\theta_s)|\downarrow\rangle \otimes R_y^{1/2}(\theta_s)|\uparrow\rangle).$$



□ Quantum-orientation entangled states

$$R_y^1(\theta_s)|1, 1\rangle = \frac{(1 + \cos \theta_s)}{2}|1, 1\rangle + \frac{\sin \theta_s}{\sqrt{2}}|1, 0\rangle + \frac{(1 - \cos \theta_s)}{2}|1, -1\rangle,$$

$$R_y^1(\theta_s)|1, 0\rangle = -\frac{\sin \theta_s}{\sqrt{2}}|1, 1\rangle + \cos \theta_s|1, 0\rangle + \frac{\sin \theta_s}{\sqrt{2}}|1, -1\rangle,$$

$$R_y^1(\theta_s)|1, -1\rangle = \frac{(1 - \cos \theta_s)}{2}|1, 1\rangle - \frac{\sin \theta_s}{\sqrt{2}}|1, 0\rangle + \frac{(1 + \cos \theta_s)}{2}|1, -1\rangle,$$

$$R_y^0(\theta_s)|0, 0\rangle = |0, 0\rangle.$$



$$R_y^1(\pi)|1, 1\rangle = |1, -1\rangle$$

$$R_y^1(\pi)|1, 0\rangle = -|1, 0\rangle$$

$$R_y^1(\pi)|1, -1\rangle = |1, 1\rangle$$

- How orientation entanglement work in the relativistic helicity states?
- Can helicity differ across different forms of relativistic dynamics (IFD and LFD)?
- How can we see the orientation entanglement in different forms of relativistic dynamics?

# Interpolating Helicity States

- To obtain the interpolating helicity states with arbitrary momentum, we apply the T transformation on a spinor/ polarization vector in its rest frame.

$$T = T_{12}T_3 = e^{i\beta_1\mathcal{K}^1+i\beta_2\mathcal{K}^2} e^{-i\beta_3K^3}$$

$$\begin{aligned} \mathcal{K}^{\hat{1}} &= -K^1 \sin \delta - J^2 \cos \delta, & (\delta \rightarrow 0), \mathcal{K}^{\hat{1}} &\rightarrow -J^2, \mathcal{K}^{\hat{2}} \rightarrow J^1 \\ \mathcal{K}^{\hat{2}} &= J^1 \cos \delta - K^2 \sin \delta, & (\delta \rightarrow \pi/4), \mathcal{K}^{\hat{1}} &\rightarrow -E_1, \mathcal{K}^{\hat{2}} \rightarrow -E_2 \end{aligned}$$

M. Jacob and G. Wick, Ann. Phys.7, 404 (1959)

H. Leutwyler and J. Stern, Ann Phys. (N.Y.) 112 94 (1978)

Z. Li, M. An, and C.-R. Ji, , Phys. Rev. D 92, 105014 (2015)

- Generalized helicity operator

$$\mathcal{J}_3|p; j, m\rangle_\delta = TJ_3T^{-1}T|0; j, m\rangle = m|p; j, m\rangle_\delta,$$

$$\mathcal{J}_3 = \frac{1}{\mathbb{P}}(P^{\hat{+}}J_3 + P^1\mathcal{K}^{\hat{2}} - P^2\mathcal{K}^{\hat{1}})$$

$\delta \rightarrow 0$

$\delta \rightarrow \frac{\pi}{4}$

Helicity defined in IFD

(Jacob and Wick helicity)

$$\mathcal{J}_3 = \frac{\mathbf{P} \cdot \mathbf{J}}{|\mathbf{P}|}$$

$$\mathcal{J}_3 = J_3 + \frac{(P^2 E_1 - P^1 E_2)}{P^+} \text{ Helicity defined in LFD}$$

# Spin Orientation in Interpolating Helicity States

$$T = B(\eta)D(\hat{\mathbf{m}}, \theta_s) = e^{-i\eta \cdot \mathbf{K}} e^{-i\hat{\mathbf{m}} \cdot \mathbf{J}\theta_s},$$

Boosts to momentum  $\mathbf{P}$

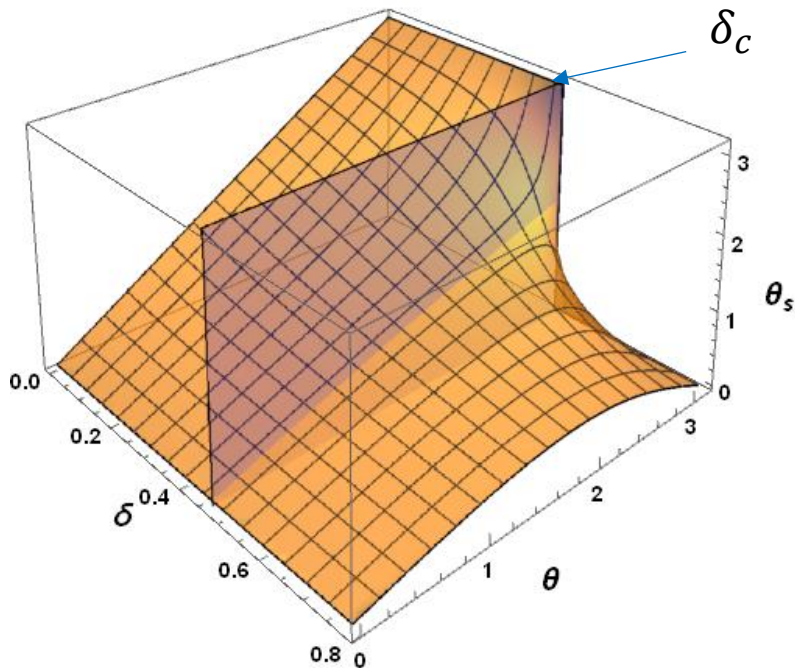
$$\hat{\mathbf{n}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Rotates the spin by angle  $\theta_s$  around the axis along the unit vector  $\hat{\mathbf{m}} = (-\sin\varphi_s, \cos\varphi_s, 0)$

- Comparing two  $T$  transformations

$$\begin{aligned} \cos \theta_s &= \frac{\cos \alpha(1 + \cosh \beta_3) + \cosh \beta_3 - \cosh \eta}{1 + \cosh \eta} \\ &= \frac{P_{\parallel}}{\mathbb{P}} - \frac{\mathbf{P}_{\perp}^2 \sin \delta}{\mathbb{P}(E + M)} \end{aligned}$$

$$\begin{aligned} \cos \phi_s &= \frac{\beta_1^2}{\sqrt{\beta_1^2 + \beta_2^2}} = \cos \phi = \frac{P^1}{\sqrt{\mathbf{P}_{\perp}^2}} \\ \sin \phi_s &= \frac{\beta_2^2}{\sqrt{\beta_1^2 + \beta_2^2}} = \sin \phi = \frac{P^2}{\sqrt{\mathbf{P}_{\perp}^2}} \end{aligned}$$



Spin orientation changes for a positive helicity

A unique relationship for the spin orientation changes with the particle's momentum direction between IFD and LFD for spin-1/2 and spin-1 spinors, as well as polarization vectors.

# Expansion of Interpolating Helicity States

□ Interpolating Helicity states in the basis of instant form helicity states ( Jacob and Wick helicity states)

➤ Interpolating spin- ½ spinors

$$U_\delta(P, \frac{1}{2}) = \cos\left(\frac{\theta_h}{2}\right)U_I(P, \frac{1}{2}) + \sin\left(\frac{\theta_h}{2}\right)U_I(P, -\frac{1}{2})$$
$$U_\delta(P, -\frac{1}{2}) = -\sin\left(\frac{\theta_h}{2}\right)U_I(P, \frac{1}{2}) + \cos\left(\frac{\theta_h}{2}\right)U_I(P, -\frac{1}{2})$$

$$\theta_h = (\theta_s - \theta)$$

$$\mathbf{H}^{1/2} = \begin{pmatrix} \cos\left(\frac{\theta_h}{2}\right) & -\sin\left(\frac{\theta_h}{2}\right) \\ \sin\left(\frac{\theta_h}{2}\right) & \cos\left(\frac{\theta_h}{2}\right) \end{pmatrix}$$

➤ Interpolating spin- 1 spinors

$$U_\delta(P, +) = \left(\frac{1 + \cos\theta_h}{2}\right)U_I(P, +) + \frac{\sin\theta_h}{\sqrt{2}}U_I(P, 0) + \left(\frac{1 - \cos\theta_h}{2}\right)U_I(P, -)$$
$$U_\delta(P, 0) = -\frac{\sin\theta_h}{\sqrt{2}}U_I(P, +) + \cos\theta_h U_I(P, 0) + \frac{\sin\theta_h}{\sqrt{2}}U_I(P, -)$$
$$U_\delta(P, -) = \left(\frac{1 - \cos\theta_h}{2}\right)U_I(P, +) - \frac{\sin\theta_h}{\sqrt{2}}U_I(P, 0) + \left(\frac{1 + \cos\theta_h}{2}\right)U_I(P, -)$$

Valid for spin-1 polarization vectors as well

□ For any Interpolating spin- j spinor

$$U_\delta(P, \lambda) = \sum_{\lambda'=-j}^j H_{\lambda', \lambda}^j U_I(P, \lambda')$$

$\lambda$  = Interpolating helicity state value

$\lambda'$  = Instant form helicity state value

# Matrix functions of Interpolating Polarization Vectors

$$\mathbf{H}^1 = \begin{pmatrix} \frac{(1+\cos \theta_h)}{2} & -\frac{\sin \theta_h}{\sqrt{2}} & \frac{(1-\cos \theta_h)}{2} \\ \frac{\sin \theta_h}{\sqrt{2}} & \cos \theta_h & -\frac{\sin \theta_h}{\sqrt{2}} \\ \frac{(1-\cos \theta_h)}{2} & \frac{\sin \theta_h}{\sqrt{2}} & \frac{(1+\cos \theta_h)}{2} \end{pmatrix}$$

$$\cos \theta_h = \frac{\mathbb{P}^2 \cos \delta - P^+ P_- \sin \delta}{\mathbb{P}C|\mathbf{P}|}$$

$$\sin \theta_h = \frac{M|\mathbf{P}_\perp| \sin \delta}{\sqrt{2}|\mathbf{P}|\mathbb{P}}$$

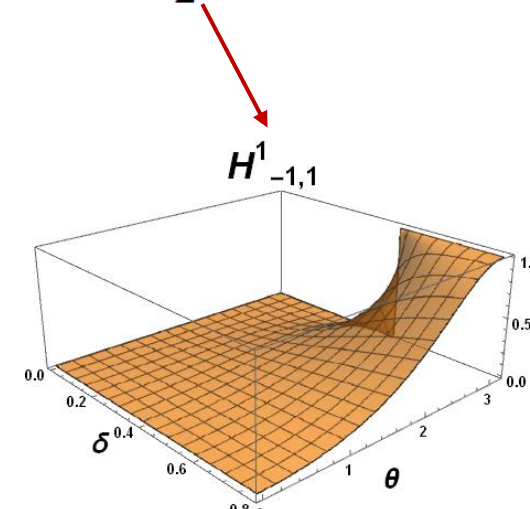
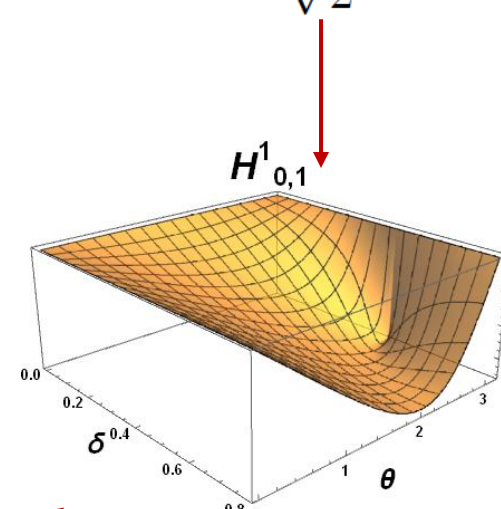
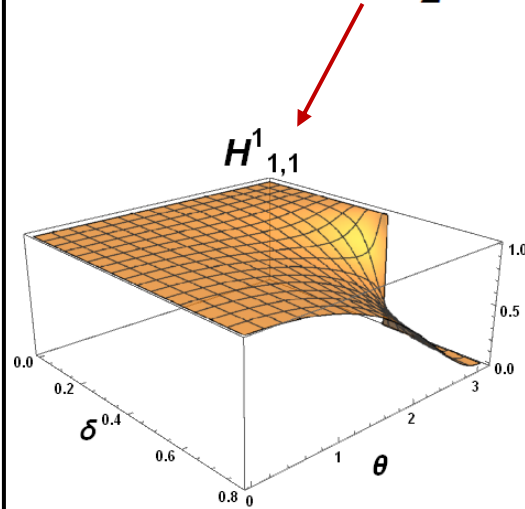
□ Critical interpolation angle effect

$$\theta = \pi$$

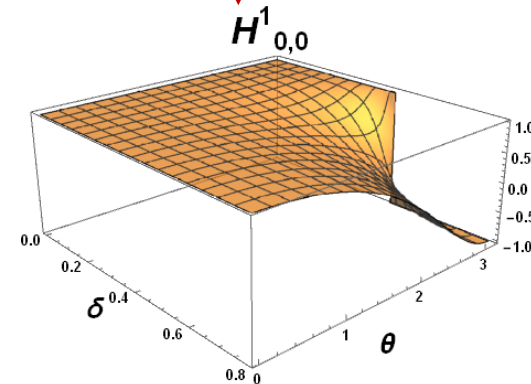
$$0 \leq \delta < \delta_c \qquad \delta_c < \delta \leq \pi/4$$

$$\mathbf{H}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{H}^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\epsilon_\delta^\mu(P, +) = \left(\frac{1 + \cos \theta_h}{2}\right) \epsilon_I^\mu(P, +) + \frac{\sin \theta_h}{\sqrt{2}} \epsilon_I^\mu(P, 0) + \left(\frac{1 - \cos \theta_h}{2}\right) \epsilon_I^\mu(P, -)$$

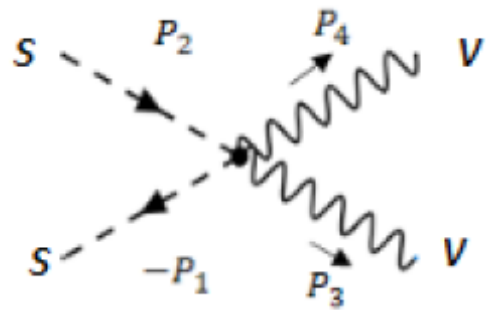


$$\epsilon_\delta^\mu(P, 0) = -\frac{\sin \theta_h}{\sqrt{2}} \epsilon_I^\mu(P, +) + \cos \theta_h \epsilon_I^\mu(P, 0) + \frac{\sin \theta_h}{\sqrt{2}} \epsilon_I^\mu(P, -)$$



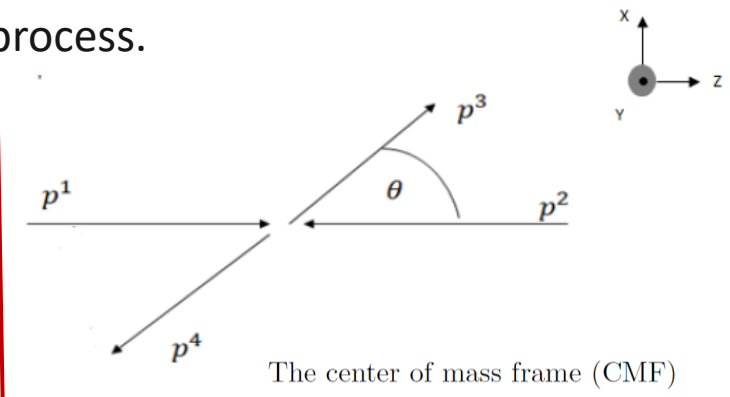
# Interpolating Helicity Amplitudes

Two massive spin-1 polarization vectors production by scalar particle and its anti-particle process.

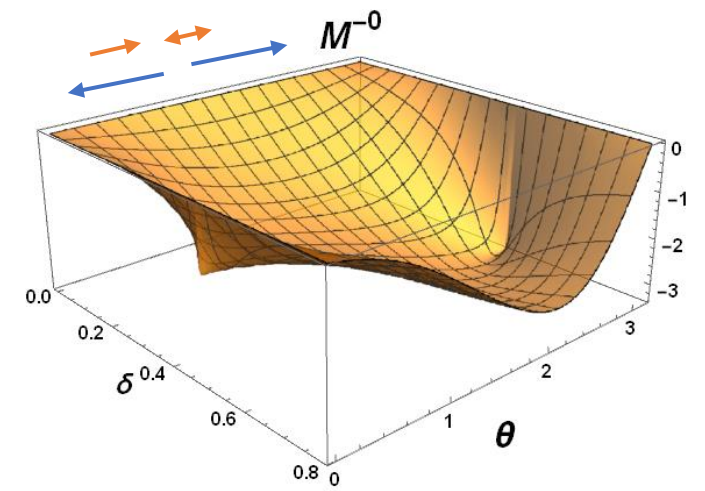
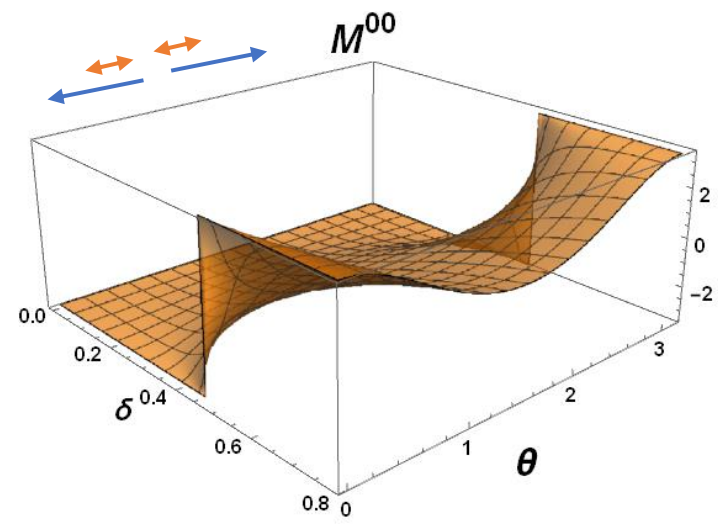
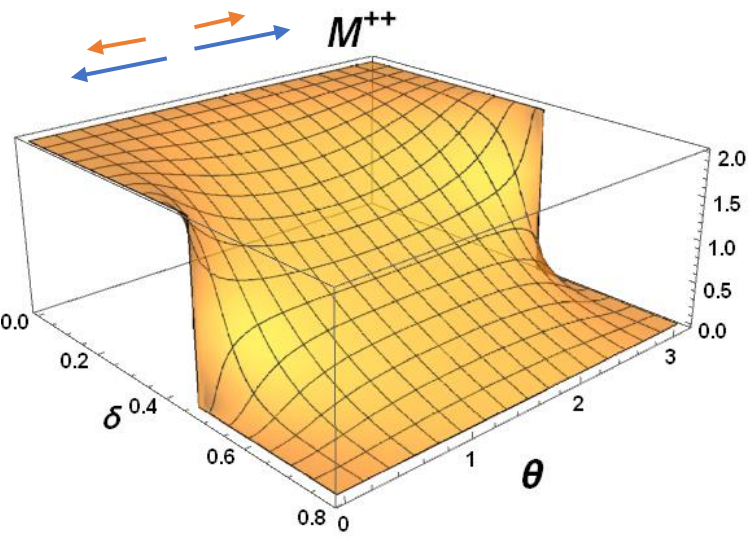


$$M_{\delta}^{\lambda_1, \lambda_2} = -2g_{\hat{\mu}\hat{\nu}}\epsilon^{*\hat{\mu}}(p^3, \lambda_1)\epsilon^{*\hat{\nu}}(p^4, \lambda_2)$$

Seagull channel: contact interaction  
 Angular momentum is conserved without involving orbital angular momentum.



$$\begin{aligned} p^1 &= (E_0, 0, 0, P_s), \\ p^2 &= (E_0, 0, 0, -P_s), \\ p^3 &= (E_0, P_v \sin \theta, 0, P_v \cos \theta), \\ p^4 &= (E_0, -P_v \sin \theta, 0, -P_v \cos \theta). \end{aligned}$$



The angular distribution of the interpolating helicity amplitudes depends on the orientation entanglement of the composite system formed by the final state interpolating polarization vectors.

# Interpolating Helicity Amplitudes

$$\begin{aligned}
 M_{\delta}^{\lambda_1, \lambda_2} &= -2g_{\hat{\mu}\hat{\nu}}\epsilon^{*\hat{\mu}}(p^3, \lambda_1)\epsilon^{*\hat{\nu}}(p^4, \lambda_2) \\
 &= -2g_{\mu\nu}\epsilon_{\delta}^{*\mu}(p^3, \lambda_1)\epsilon_{\delta}^{*\nu}(p^4, \lambda_2).
 \end{aligned}$$



$$\begin{aligned}
 M_{\delta}^{\lambda_1, \lambda_2} &= \sum_{\lambda'_1, \lambda'_2} H_{\lambda'_1, \lambda_1}^1(p^3) H_{\lambda'_2, \lambda_2}^1(p^4) M_I^{\lambda'_1, \lambda'_2} \\
 &= \sum_{\lambda'_1, \lambda'_2} M_{\lambda'_1, \lambda'_2}^{\lambda_1, \lambda_2}
 \end{aligned}$$



Projection helicity amplitudes

- IFD helicity amplitudes

$$\begin{aligned}
 M_I^{\lambda'_1, \lambda'_2} &= -2\epsilon_I^{*\mu}(p^3, \lambda'_1)\epsilon_{I\nu}^*(p^4, \lambda'_2), \\
 &= \begin{cases} 2 & \text{if } \lambda'_1 = \lambda'_2 = \pm, \\ \frac{-2(E_0^2 + P_v^2)}{m_v^2} & \text{if } \lambda'_1 = \lambda'_2 = 0, \\ 0 & \text{if } \lambda'_1 \neq \lambda'_2. \end{cases}
 \end{aligned}$$

Interpolation helicity amplitudes can be expressed in terms of IFD helicity amplitudes

- Symmetry based on parity conservation

$$M_{\delta}^{-\lambda_1, -\lambda_2} = (-1)^{\lambda_1 - \lambda_2} M_{\delta}^{\lambda_1, \lambda_2}$$

$$M_{-\lambda'_1, -\lambda'_2} = (-1)^{\lambda_1 - \lambda_2} M_{\lambda'_1, \lambda'_2}$$

# Interpolating Helicity Amplitudes

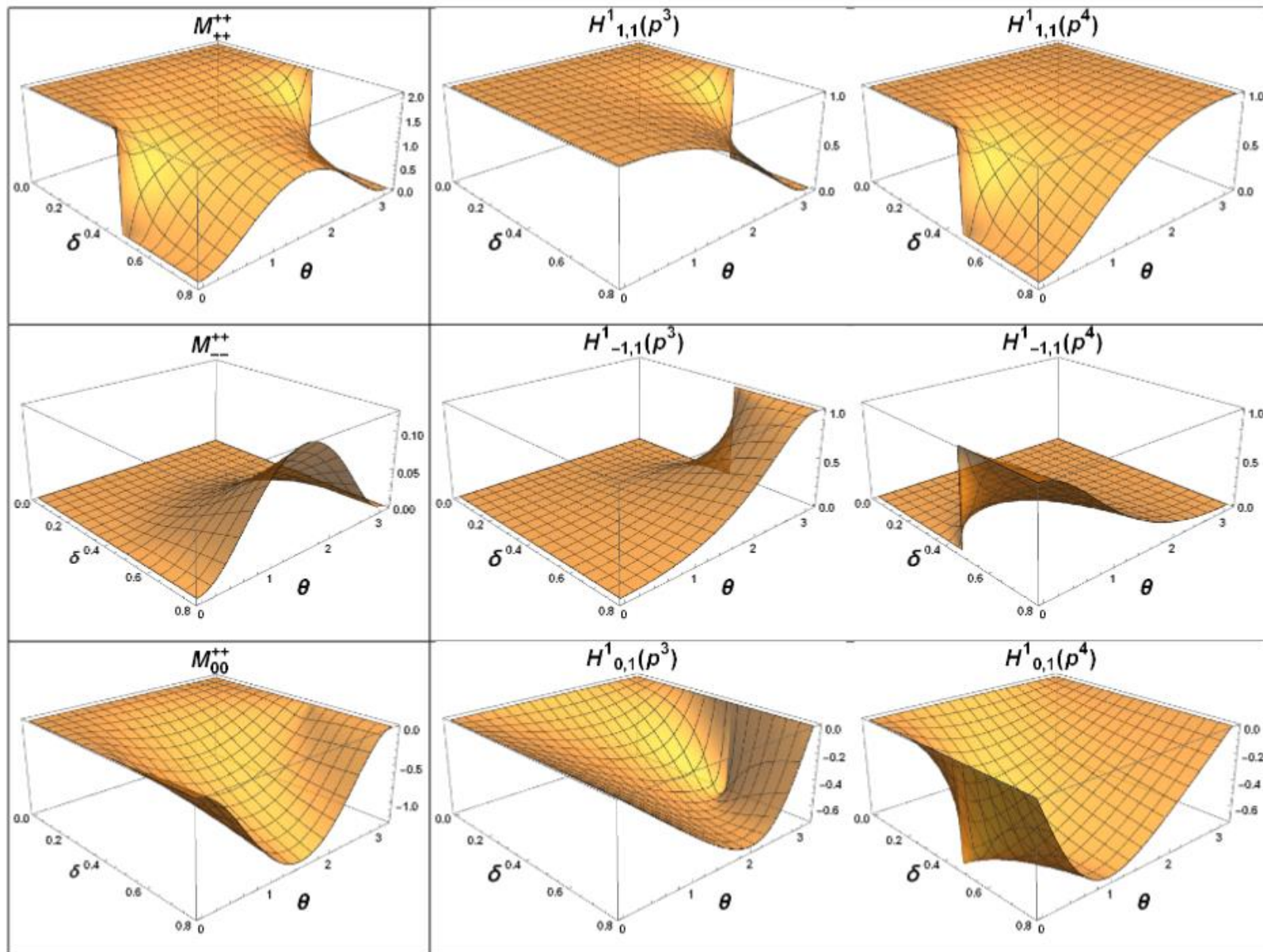
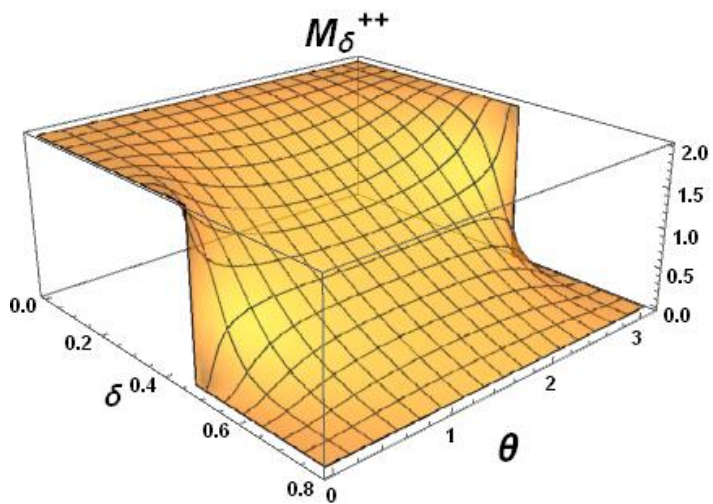
Example:

$$M_{\delta}^{++} = M_{++}^{++} + M_{--}^{++} + M_{00}^{++}$$

$$M_{++}^{++} = 2(H_{1,1}^1(p^3)H_{1,1}^1(p^4)),$$

$$M_{--}^{++} = 2(H_{-1,1}^1(p^3)H_{-1,1}^1(p^4)),$$

$$M_{00}^{++} = \frac{-2(E_0^2 + P_v^2)}{m_v^2} (H_{0,1}^1(p^3)H_{0,1}^1(p^4)).$$



# Interpolating Helicity Amplitudes

$$M_{\delta}^{00} = M_{++}^{00} + M_{--}^{00} + M_{00}^{00}$$

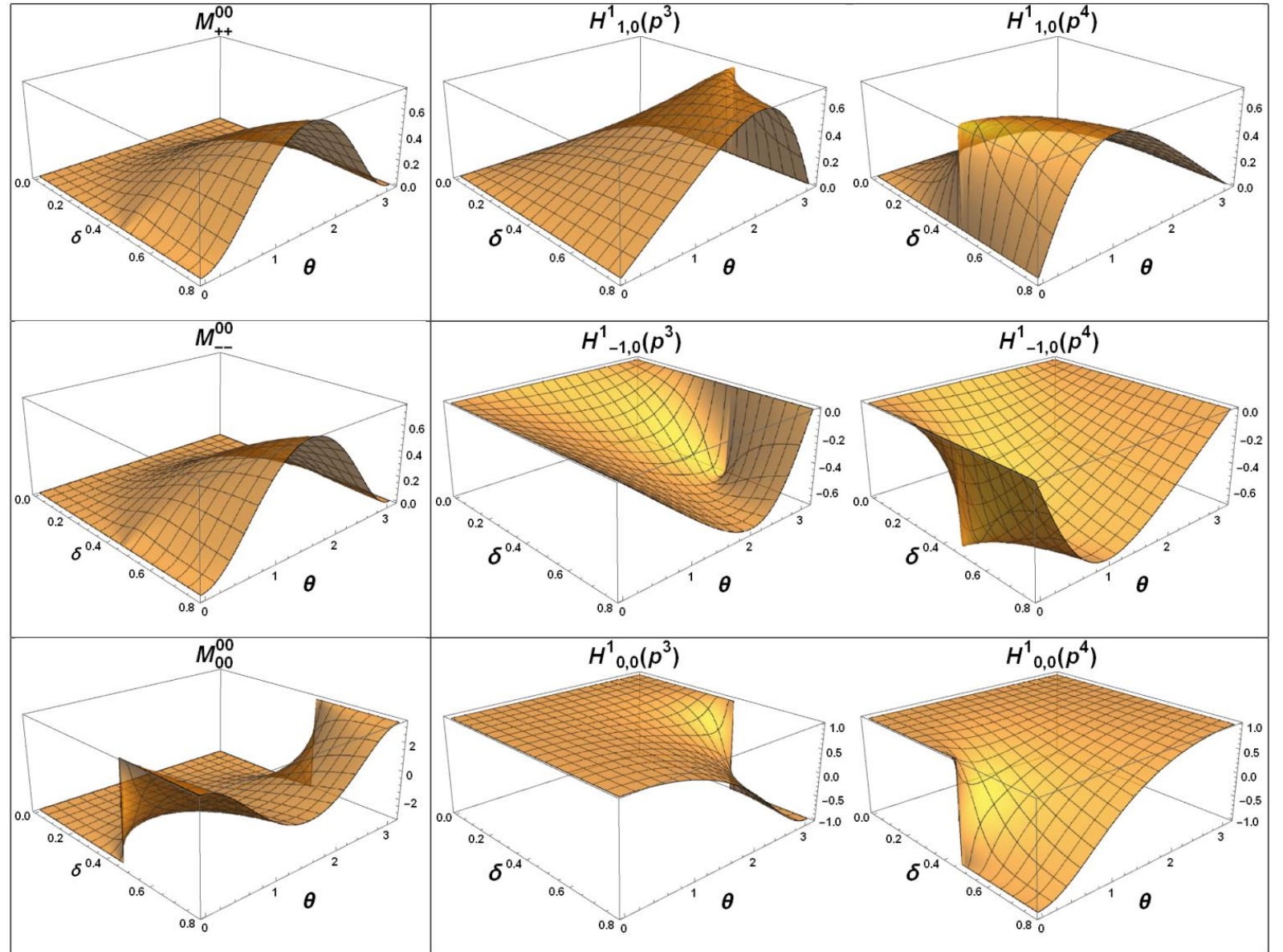
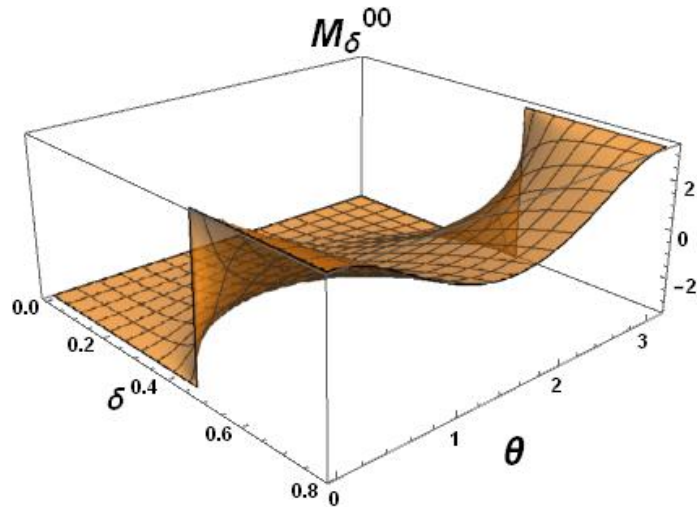
$$M_{++}^{00} = 2(H_{1,0}^1(p^3)H_{1,0}^1(p^4)),$$

$$M_{--}^{00} = 2(H_{-1,0}^1(p^3)H_{-1,0}^1(p^4)),$$

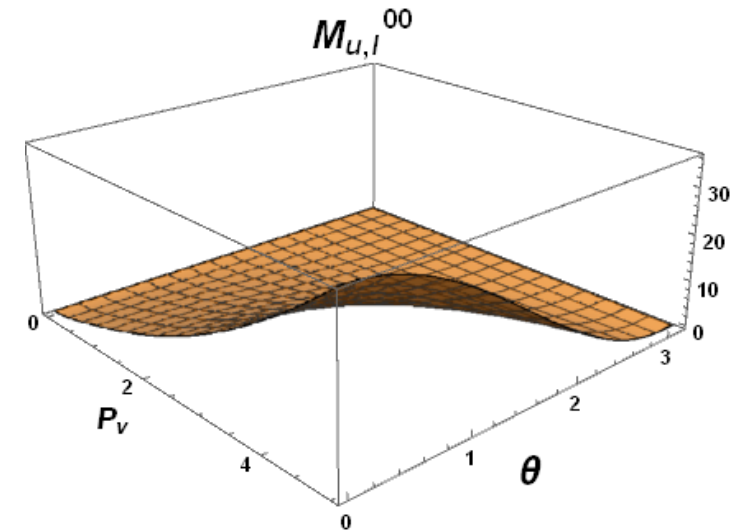
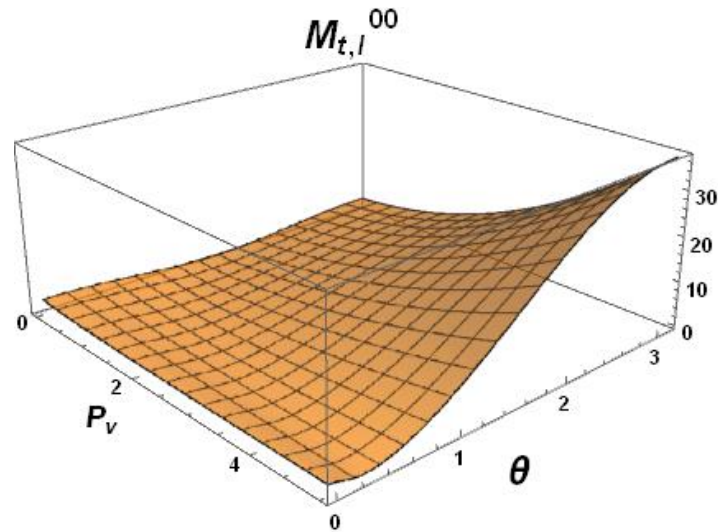
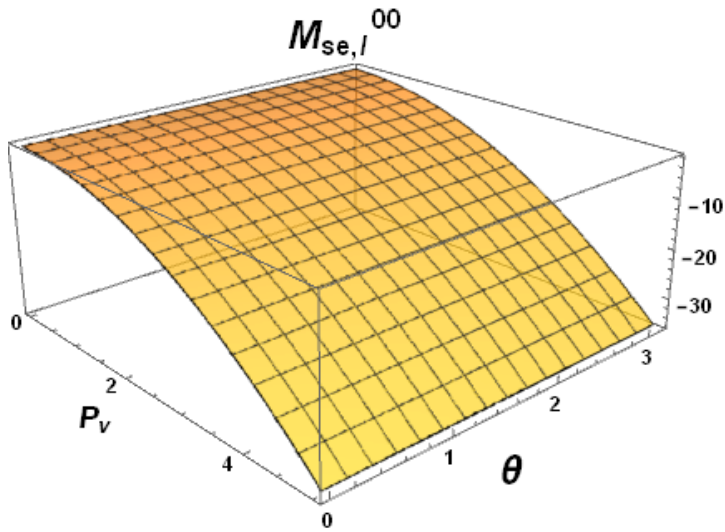
$$M_{00}^{00} = \frac{-2(E_0^2 + P_v^2)}{m_v^2} (H_{0,0}^1(p^3)H_{0,0}^1(p^4)).$$



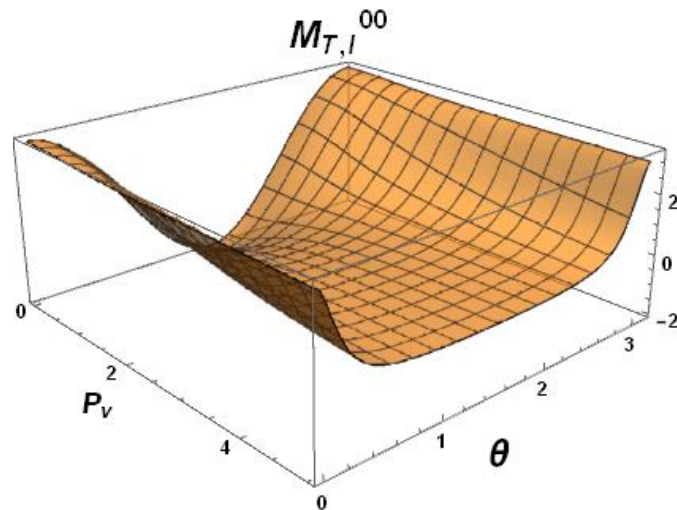
$$M_I^{00} = -2 - \left(\frac{P_v}{m_v}\right)^2$$



# Relativistic Effect of Longitudinal Polarization Vectors in the Helicity Amplitudes



The Seagull channel by itself is not gauge invariant; gauge invariance is restored only when the Seagull  $t$  and  $u$  channels are combined.



# Manifestation of Quantum Correlation in the Interpolating Helicity Amplitudes

# Motivation : Correlation between Spin Orientation and Longitudinal Boost

- Momentum direction
- Spin direction

Helicity Plus



Longitudinal boost in the opposite direction of initial momentum.

☐ Instant Form Dynamics

- Longitudinal boost is a dynamic operator



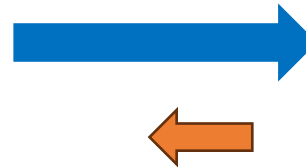
Helicity minus

☐ Light-front Dynamics

- Light-front longitudinal boost is a kinematic operator

Spin rotation in  $180^\circ$

- Transverse rotation is a kinematic operator



Helicity minus

- Light-front transverse rotation is a dynamic operator

How the helicity 0 states of scalar and vector particles transform under a longitudinal boost.

# Correlation between Spin Orientation and Longitudinal Boost

- The relationship we obtained in the analysis of quantum orientation entanglement between IFD and LFD

$$\cos \theta_s = \frac{\cos \alpha (1 + \cosh \beta_3) + \cosh \beta_3 - \cosh \eta}{1 + \cosh \eta}$$

$$= \frac{P_{\hat{z}}}{\mathbb{P}} - \frac{\mathbf{P}_{\perp}^2 \sin \delta}{\mathbb{P}(E + M)}$$

- When we fix the particle's initial momentum direction as +z ,

$$\theta_s = 0, \quad \cos \alpha = P_{\hat{z}} / |P_{\hat{z}}| \rightarrow 1 \quad P_{\hat{z}} > 0$$

$$\theta_s = \pi, \quad \cos \alpha = P_{\hat{z}} / |P_{\hat{z}}| \rightarrow -1 \quad P_{\hat{z}} < 0$$

Interpolating longitudinal momentum ( $P_{\hat{z}}$ )

- Initial direction of particles' momentum
- Interpolation angle
- Boost of the frame

- Interpolating Longitudinal Boost

$$\begin{pmatrix} P'_{\hat{z}} \\ P'_1 \\ P'_2 \\ P'_{\hat{z}} \end{pmatrix} = \begin{pmatrix} \gamma(1 - \beta S) & 0 & 0 & \gamma \beta C \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta C & 0 & 0 & \gamma(1 + \beta S) \end{pmatrix} \begin{pmatrix} P_{\hat{z}} \\ P_1 \\ P_2 \\ P_{\hat{z}} \end{pmatrix}$$

$$P'_{\hat{z}} = \gamma \beta C P_{\hat{z}} + \gamma(1 + \beta S) P_{\hat{z}}$$

$$= \gamma(\beta P^{\hat{z}} + P_{\hat{z}}).$$

- Quantum correlation condition

$$\beta = -(P_{\hat{z}} / P^{\hat{z}}) \longrightarrow P'_{\hat{z}} = 0$$

# Quantum Correlated Interpolating Helicity States

Spin- $\frac{1}{2}$ spinors	Spin-1 spinors	Spin-1 polarization vectors
$U_\delta(P'_\pm > 0, \frac{1}{2}) \Rightarrow U_\delta(P'_\pm > 0, -\frac{1}{2})$	$U_\delta(P'_\pm > 0, +1) \Rightarrow U_\delta(P'_\pm > 0, -1)$	$\epsilon^{\hat{\mu}}(P'_\pm > 0, +1) \Rightarrow \epsilon^{\hat{\mu}}(P'_\pm > 0, -1)$
$U_\delta(P'_\pm > 0, -\frac{1}{2}) \Rightarrow -U_\delta(P'_\pm > 0, \frac{1}{2})$	$U_\delta(P'_\pm > 0, 0) \Rightarrow -U_\delta(P'_\pm > 0, 0)$	$\epsilon^{\hat{\mu}}(P'_\pm > 0, 0) \Rightarrow -\epsilon^{\hat{\mu}}(P'_\pm > 0, 0)$
$U_\delta(P'_\pm > 0, \frac{1}{2}) \Rightarrow -U_\delta(P'_\pm > 0, -\frac{1}{2})$	$U_\delta(P'_\pm > 0, -1) \Rightarrow U_\delta(P'_\pm > 0, +1)$	$\epsilon^{\hat{\mu}}(P'_\pm > 0, -1) \Rightarrow \epsilon^{\hat{\mu}}(P'_\pm > 0, +1)$

$\Rightarrow$  Indicates sign change of interpolating longitudinal momentum ( $P'_\pm$ )

# Scalar particle and its anti-particle production by two massive spin-1 polarization vectors ( $VV \rightarrow SS$ )

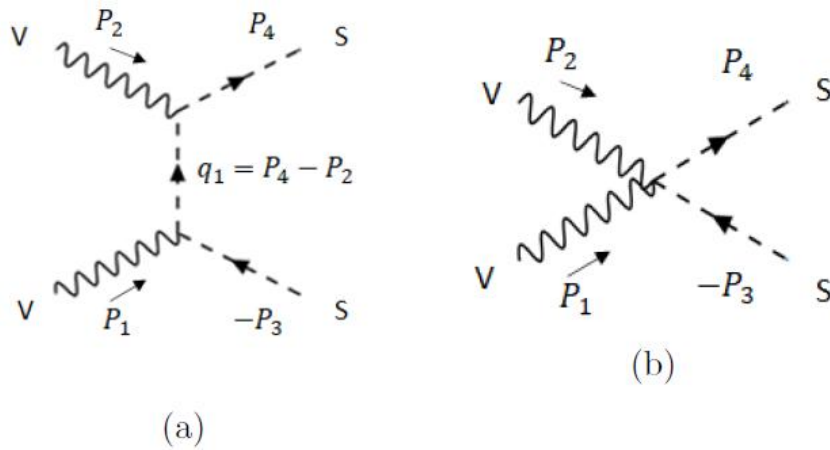


Fig. (a) t-channel Feynman diagram, the cross channel (u-channel) can be drawn by crossing the two final states' particles. Fig. (b) is drawn for the seagull channel.

V -> Vector particle (Spin-1)  
S -> Scalar particle (Spin-0)

## □ Interpolating helicity amplitudes

$$M_t^{\lambda_1 \lambda_2} = (-p_3 + q_1)^{\hat{\mu}} \varepsilon_{\hat{\mu}}(p_1, \lambda_1) \frac{1}{q_1^2 - m_s^2} (p_4 + q_1)^{\hat{\nu}} \varepsilon_{\hat{\nu}}(p_2, \lambda_2)$$

$$M_u^{\lambda_1 \lambda_2} = (-p_3 + q_2)^{\hat{\nu}} \varepsilon_{\hat{\nu}}(p_2, \lambda_2) \frac{1}{q_2^2 - m_s^2} (-p_4 + q_2)^{\hat{\mu}} \varepsilon_{\hat{\mu}}(p_1, \lambda_1)$$

$$M_{se}^{\lambda_1 \lambda_2} = -2g_{\hat{\mu}\hat{\nu}} \varepsilon^{\hat{\mu}}(p_1, \lambda_1) \varepsilon^{\hat{\nu}}(p_2, \lambda_2)$$

Where  $q_2 = p_3 - p_2$

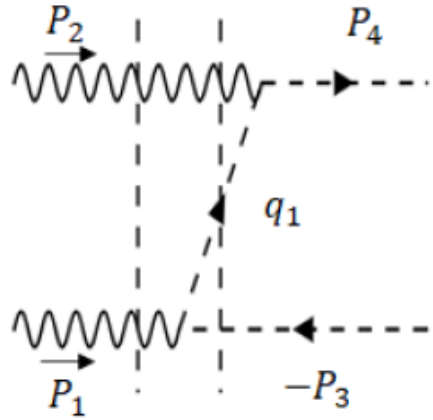
## □ $t$ and $u$ channels have time-ordered interpolating helicity amplitudes.

## □ Other processes

- Two massive spin-1 polarization vectors production by scalar particle and its anti-particle process. ( $SS \rightarrow VV$ )
- Scalar particle and its anti-particle production by real photons ( $\gamma\gamma \rightarrow SS$ ) and ( $SS \rightarrow \gamma\gamma$ ) process

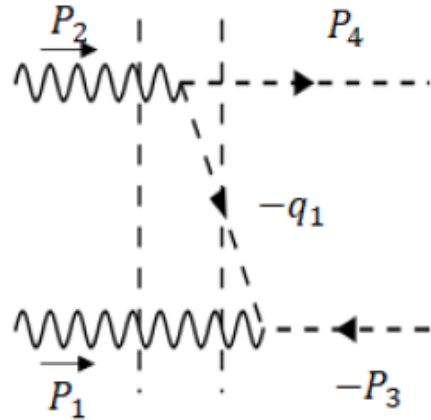
# Interpolating Time-ordered Helicity Amplitudes

## □ $t$ -channel process



(a)

- Forward-time ordered diagram



(b)

- Backward-time ordered diagram

The system evolves with the interpolating time  $x^{\hat{+}}$

- The interpolating time-ordered propagators of the intermediate virtual scalar

$$\Sigma_a^\delta = \frac{1}{2Q_1^{\hat{+}}} \left[ \frac{\mathbb{C}}{q_1^{\hat{+}} - Q_1^{\hat{+}}} \right] \quad + \quad \Sigma_b^\delta = -\frac{1}{2Q_1^{\hat{+}}} \left[ \frac{\mathbb{C}}{q_1^{\hat{+}} + Q_1^{\hat{+}}} \right]$$

$$\Sigma_t = \frac{1}{q_1^2 - m_s^2}$$

- Covariant Feynman propagator

- The  $u$ -channel time-ordered diagram can be obtained by crossing the final state particles.

# Symmetries between Covariant Helicity Amplitudes

## □ t – u Symmetry

$M_t^{++} = M_t^{--}$	$M_t^{+-} = M_t^{-+}$
$M_u^{++} = M_u^{--}$	$M_u^{+-} = M_u^{-+}$
$M_{se}^{++} = M_{se}^{--}$	$M_{se}^{+-} = M_{se}^{-+}$
$M_t^{0+} = -(M_t^{0-})$	$M_t^{+0} = -(M_t^{-0})$
$M_u^{0+} = -(M_u^{0-})$	$M_u^{+0} = -(M_u^{-0})$
$M_{se}^{0+} = -(M_{se}^{0-})$	$M_{se}^{+0} = -(M_{se}^{-0})$

$$\begin{aligned}
 M_t^{++}(\theta) &= M_u^{++}(\pi - \theta) \\
 M_t^{+-}(\theta) &= M_u^{-+}(\pi - \theta) \\
 M_t^{0+}(\theta) &= M_u^{+0}(\pi - \theta) \\
 M_t^{+0}(\theta) &= M_u^{0+}(\pi - \theta) \\
 M_t^{00}(\theta) &= M_u^{00}(\pi - \theta)
 \end{aligned}$$

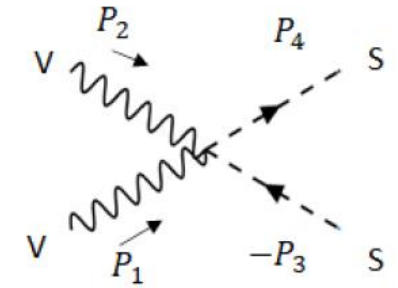
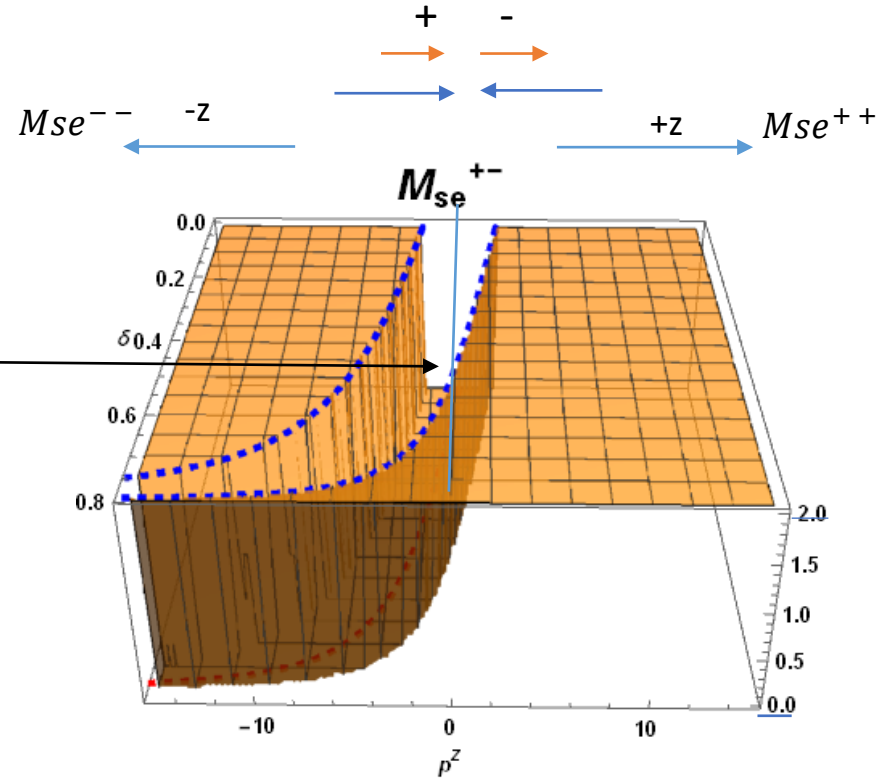
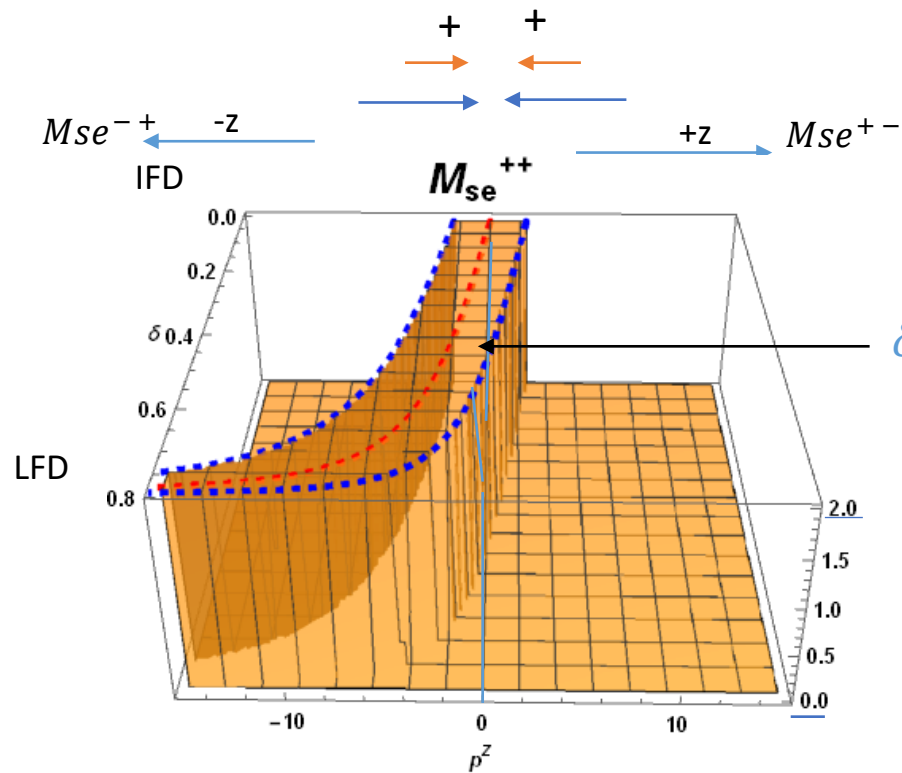
□ Helicity amplitudes satisfy symmetry based on parity conservation.

$$M(-\lambda', -\lambda) = (-1)^{\lambda' - \lambda} M(\lambda', \lambda)$$

C-R . Ji, B.L.G.Bakker, International Journal of Modern Physics .Vol.22, No 02, 133002 (2013)

- All time-ordered diagrams also satisfy the same symmetries mentioned above.

# Seagull channels with transverse polarization vectors



Parity conservation

$$Mse^{++} = Mse^{--}$$

$$Mse^{-+} = Mse^{-}$$

Quantum correlation condition  $\Rightarrow P'_{-,+z} = 0$

$$P'_{-,-z} = 0$$

$$\tan(\delta_{c,+z}) = -\frac{E_0 P^z + P_v \sqrt{\bar{E}^2 + (P^z)^2}}{P_v P^z + E_0 \sqrt{\bar{E}^2 + (P^z)^2}}$$

$$\tan(\delta_{c,-z}) = -\frac{E_0 P^z - P_v \sqrt{\bar{E}^2 + (P^z)^2}}{-P_v P^z + E_0 \sqrt{\bar{E}^2 + (P^z)^2}}$$

# Seagull channel with transverse polarization vectors

To see the QC in the zero-mode, we consider total interpolating longitudinal momentum of the system

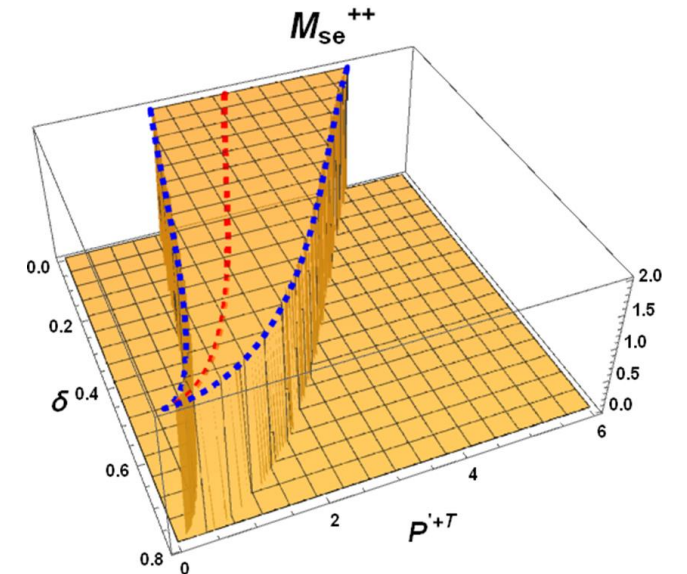
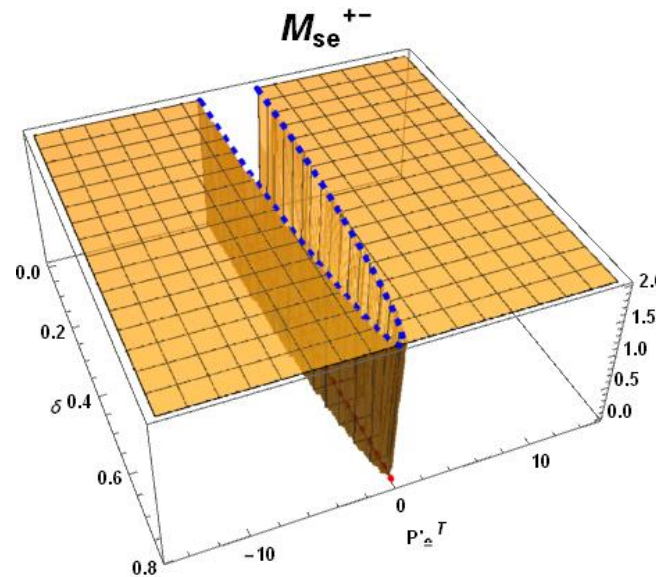
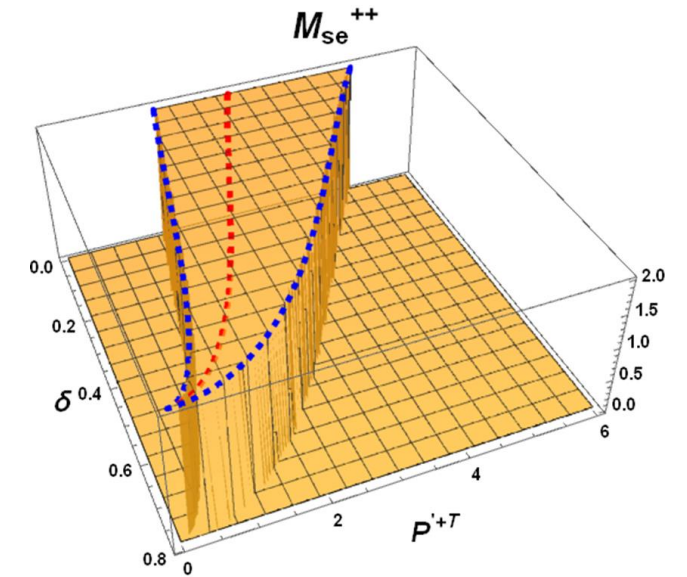
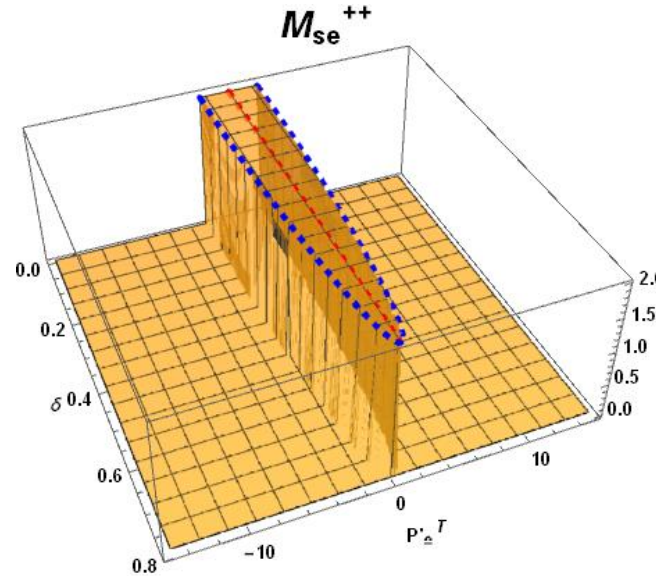
$$(P'_{\perp})^T = \sqrt{\bar{E}^2 + (P^z)^2} \sin \delta + P^z \cos \delta$$

To review the quantum correlation boundaries, from the light-front perspective, we consider the total light-front momentum of the system.

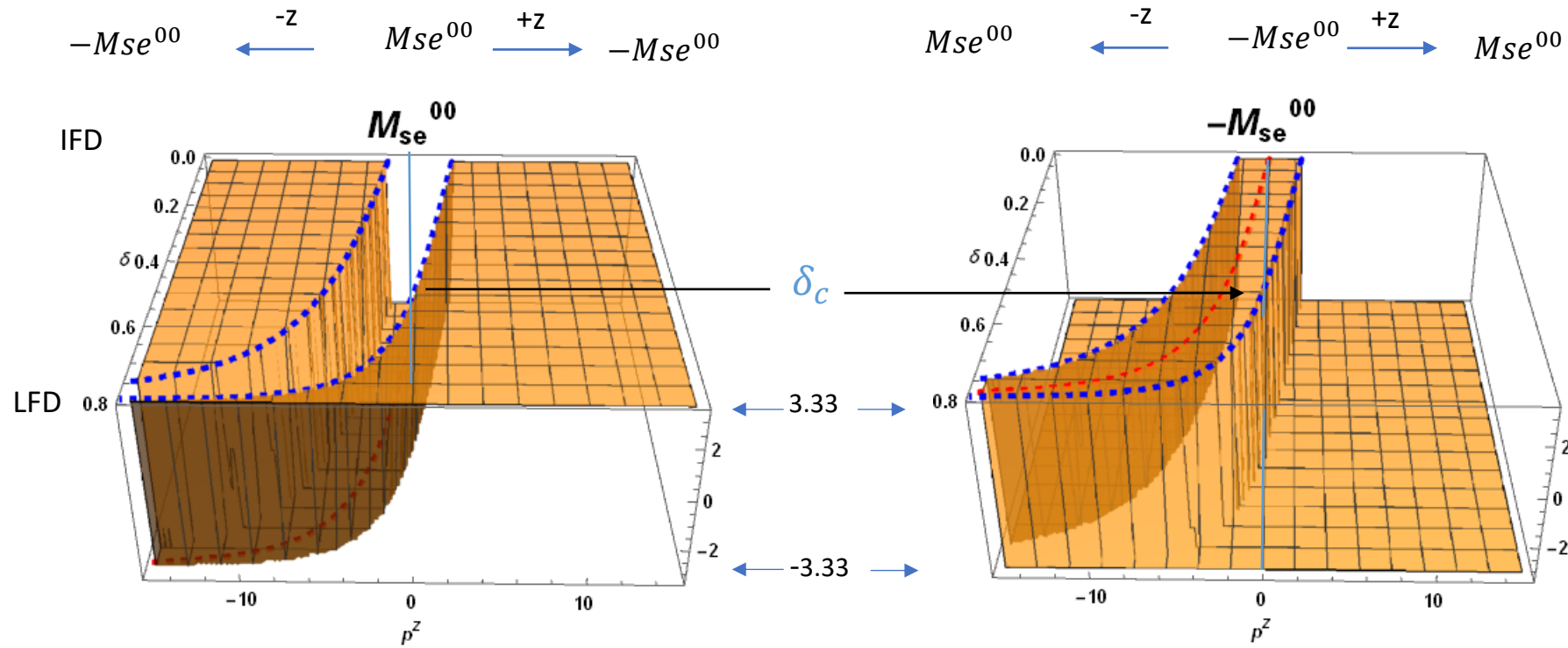
$$(P'^+)^T = \frac{1}{\sqrt{2}} \sqrt{\bar{E}^2 + (P^z)^2} + P^z$$

- Zero-mode

$$(P'^+)^T \rightarrow 0$$

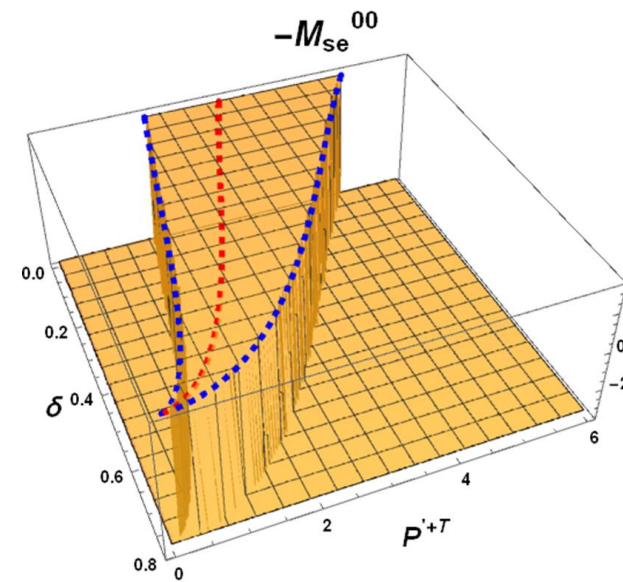
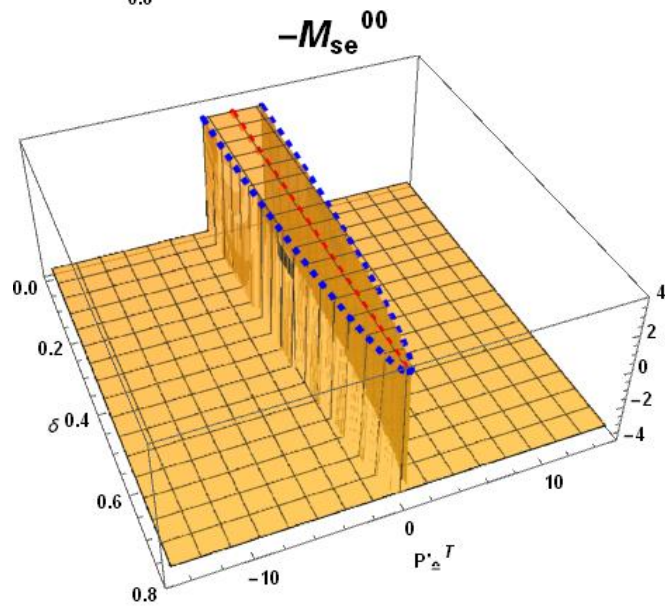
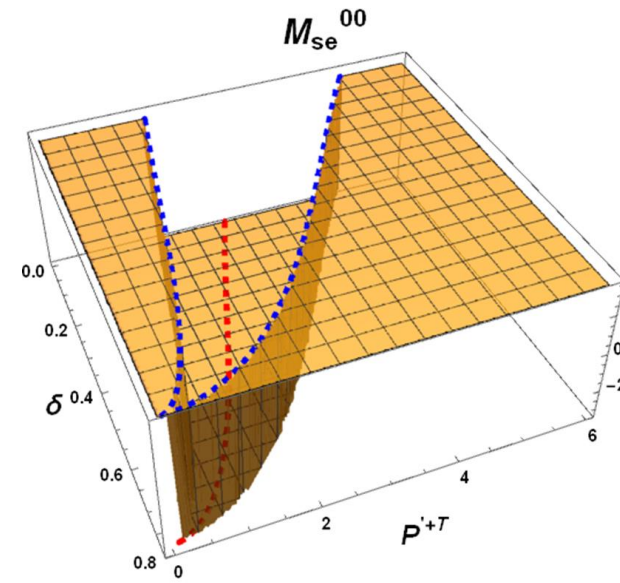
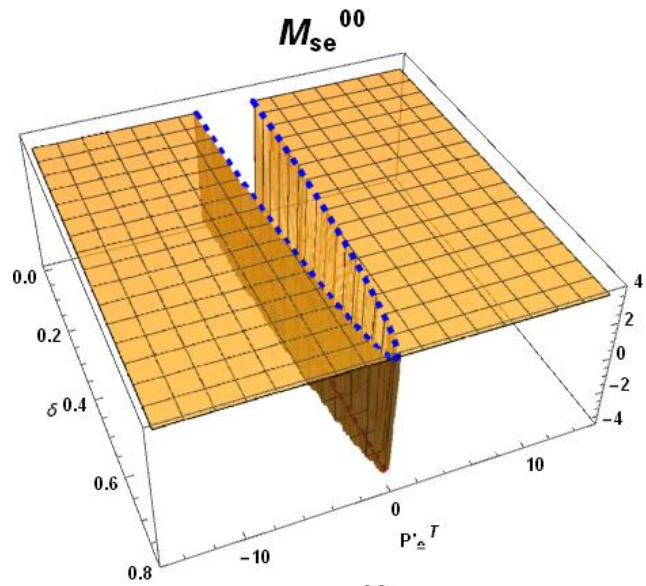


# Seagull channel with longitudinal polarization vectors

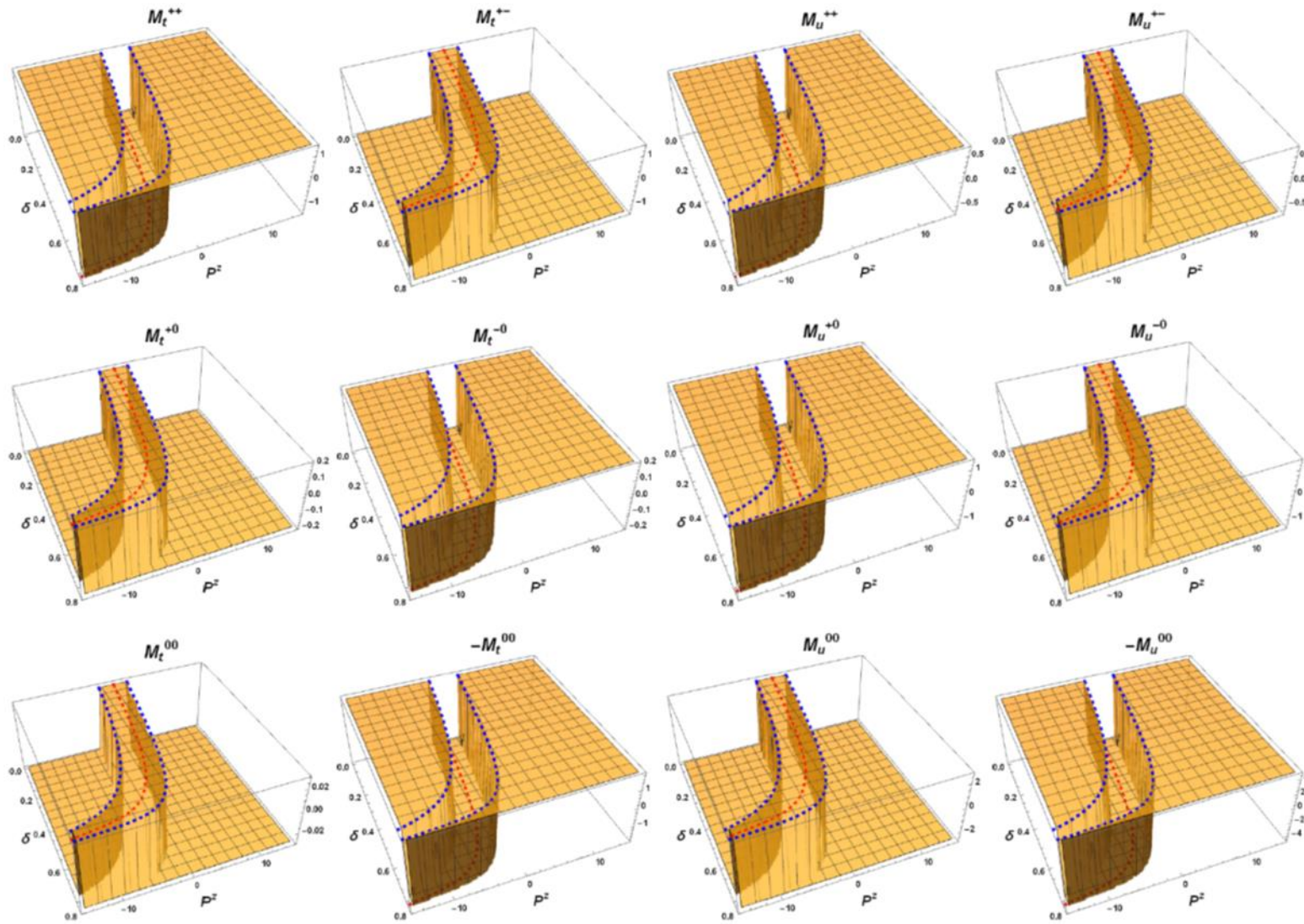


- Phase changes due to the quantum correlation  $\epsilon^{\hat{\mu}}(P'_{-} > 0, 0) \Rightarrow -\epsilon^{\hat{\mu}}(P'_{-} > 0, 0)$

# Seagull channel with longitudinal polarization vectors



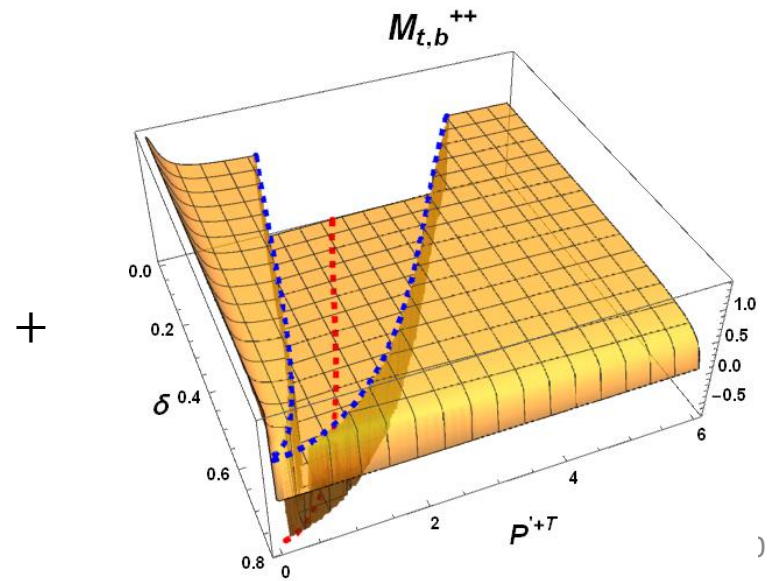
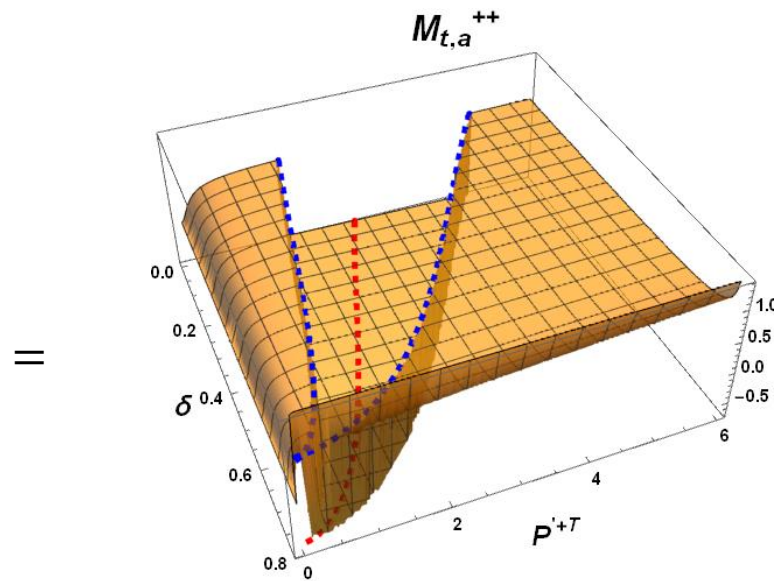
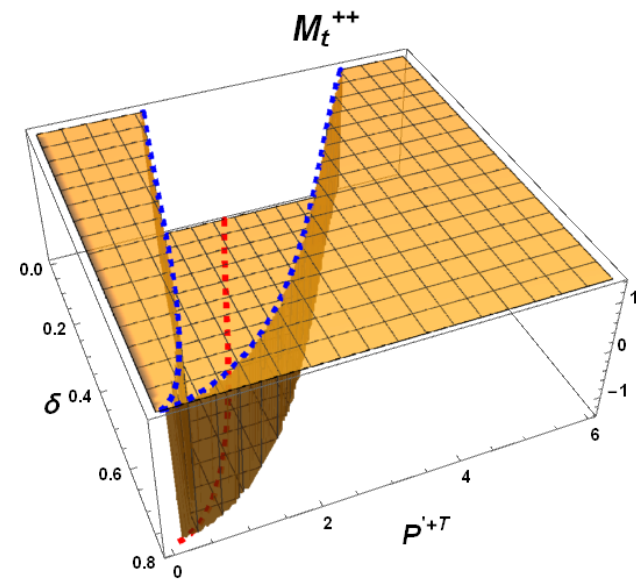
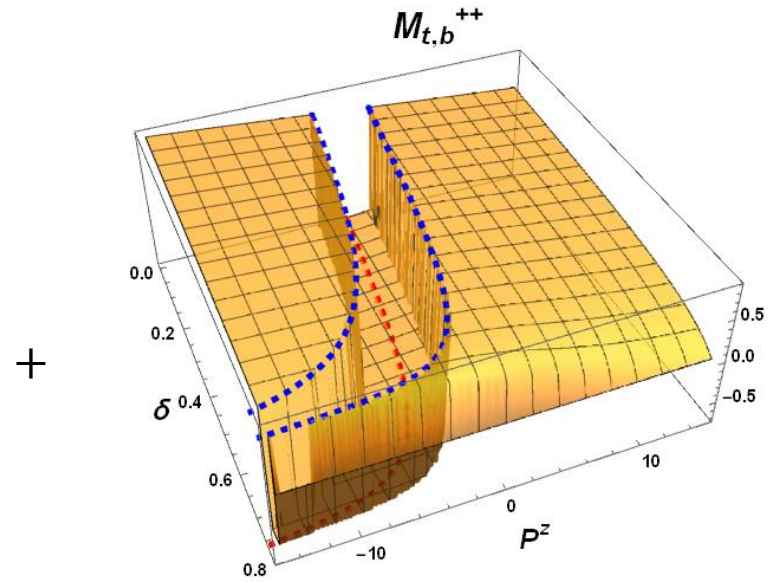
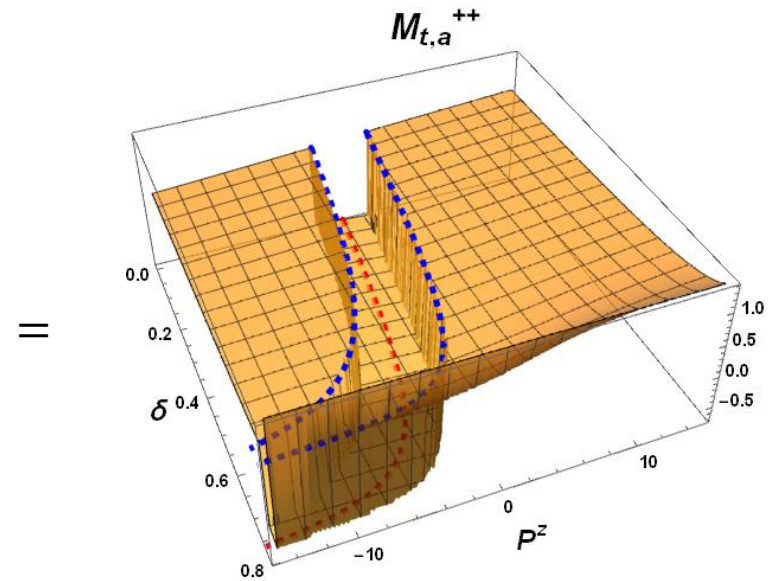
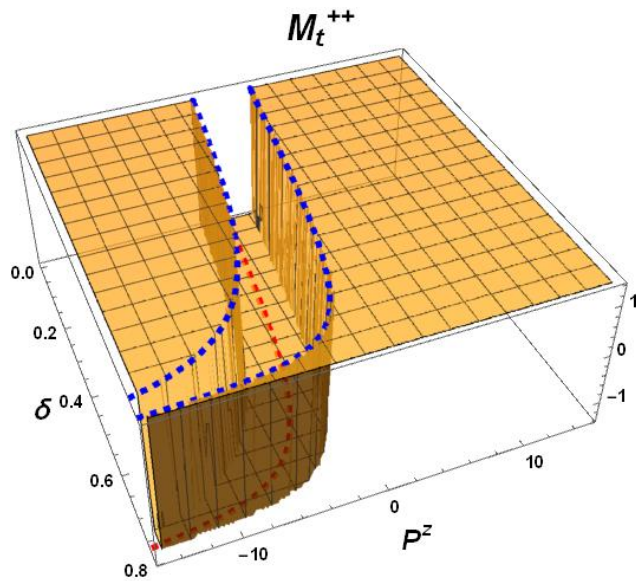
# T and U channels helicity amplitudes



- Depend on orbital angular momentum involving an impact parameter.
- Time-ordered helicity amplitudes exhibit the same features.

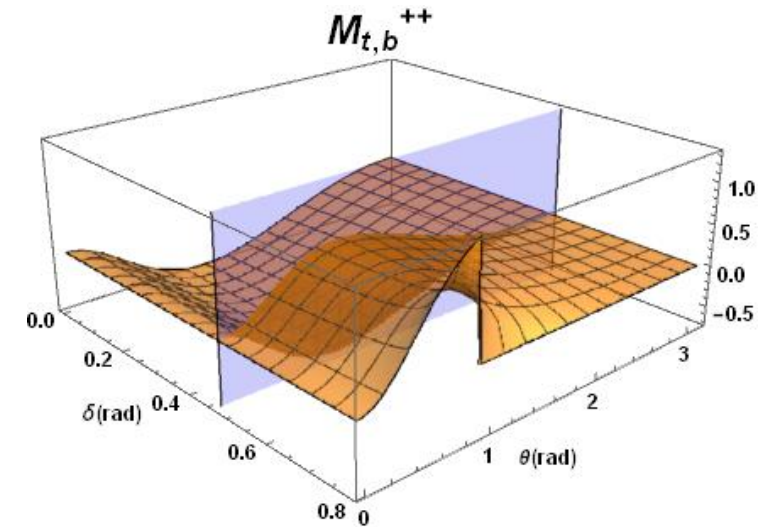
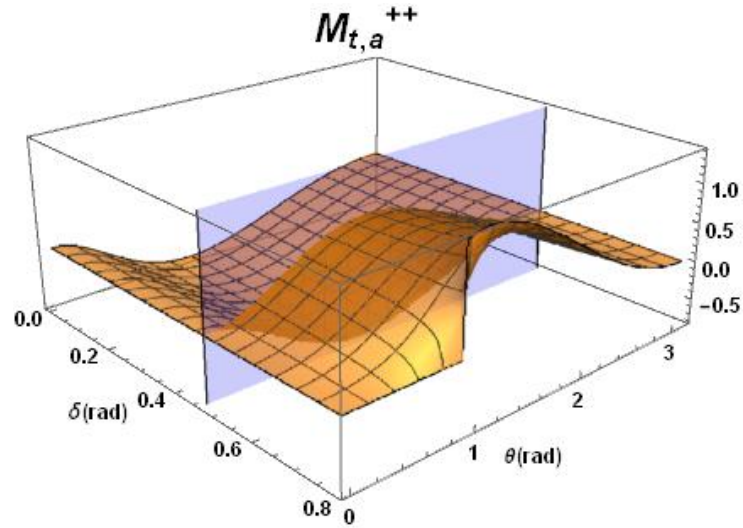
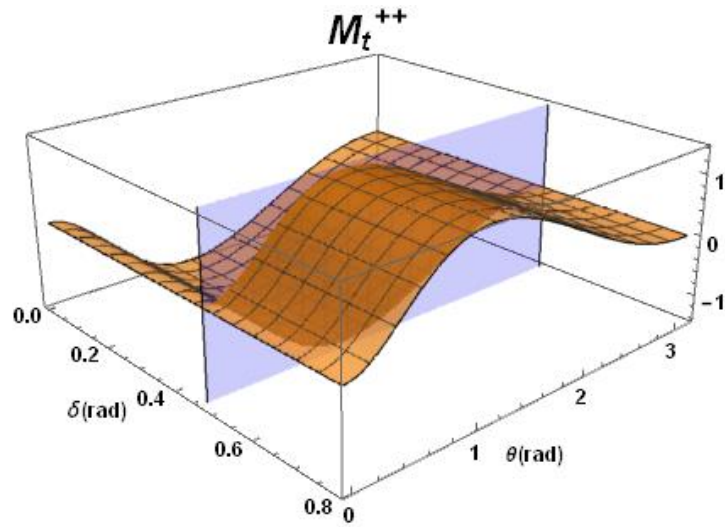
# Time-ordered Helicity Amplitudes

$$M_t^{++} = M_{t,a}^{++} + M_{t,b}^{++} \quad \theta = \frac{\pi}{3}$$



# Angular distribution in the rest frame

$$M_t^{++} = M_{t,a}^{++} + M_{t,b}^{++}$$



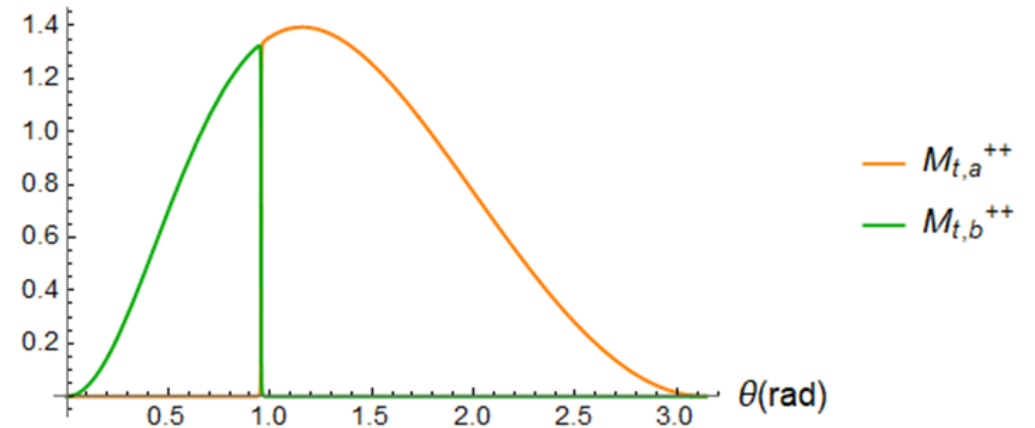
- Decomposition of the covariant scalar propagator exactly at the LF

Critical scattering angle

$$\theta_c = \text{Cos}^{-1} \left( \frac{P_v}{P_s} \right) = 0.955$$

- Critical interpolation angle

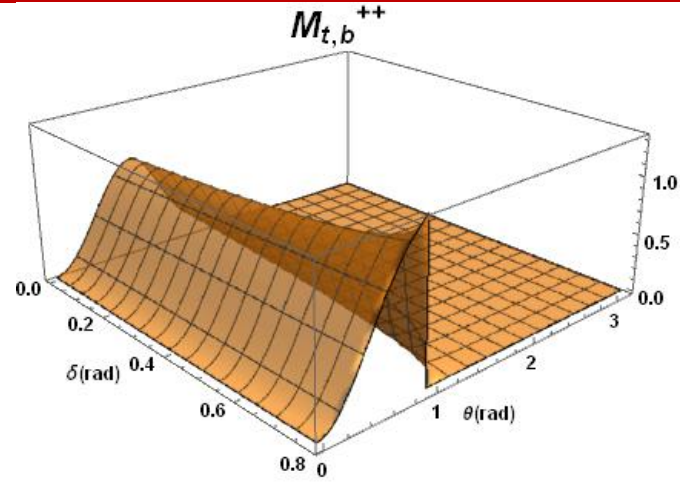
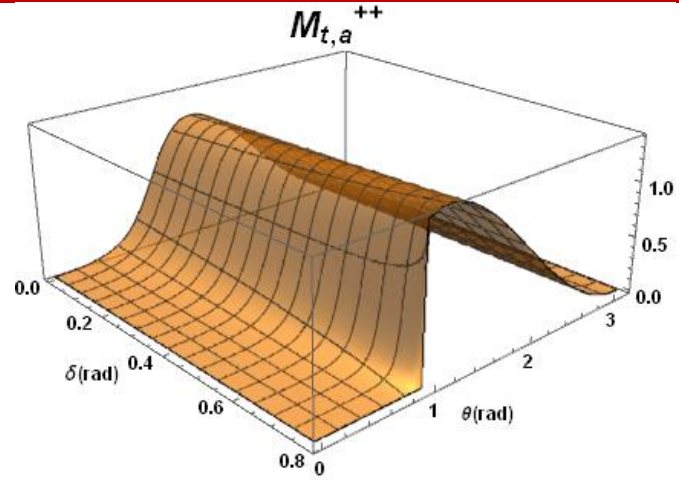
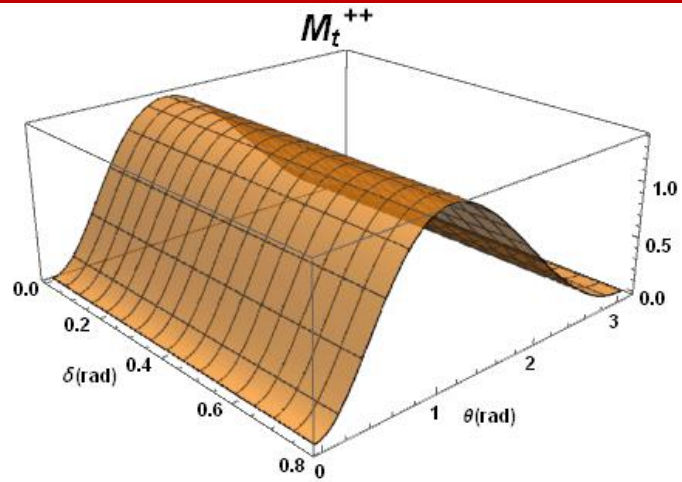
$$\delta_c = \text{tan}^{-1} \left( \frac{P_v}{E_0} \right) = 0.464$$



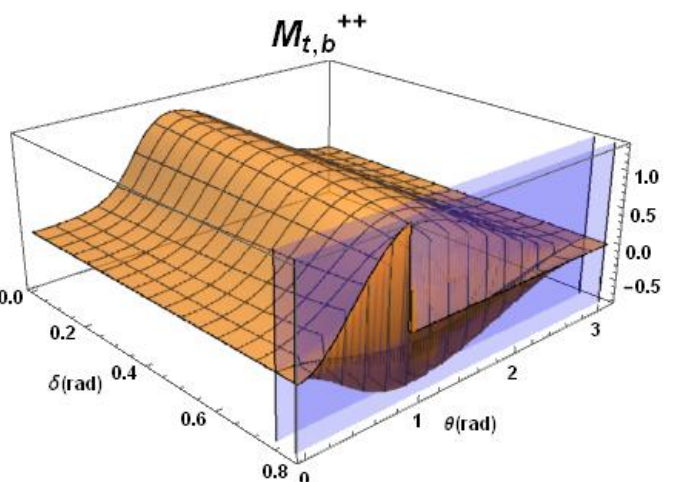
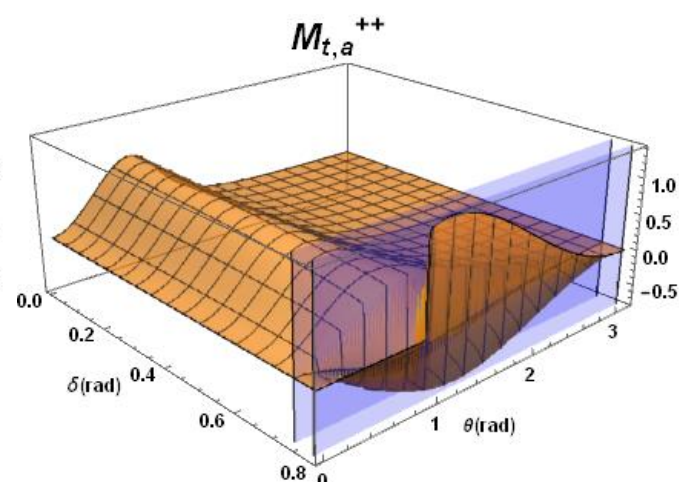
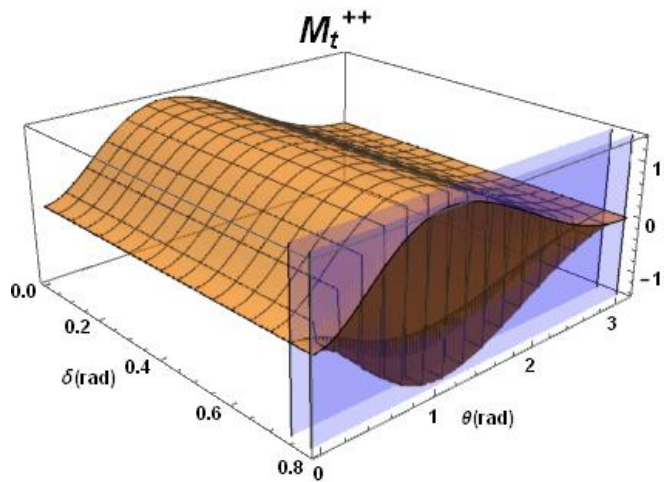
Cross section of the light-front end helicity amplitude

# Angular distribution of helicity amplitudes in the boosted frame

$$P^z = 15$$



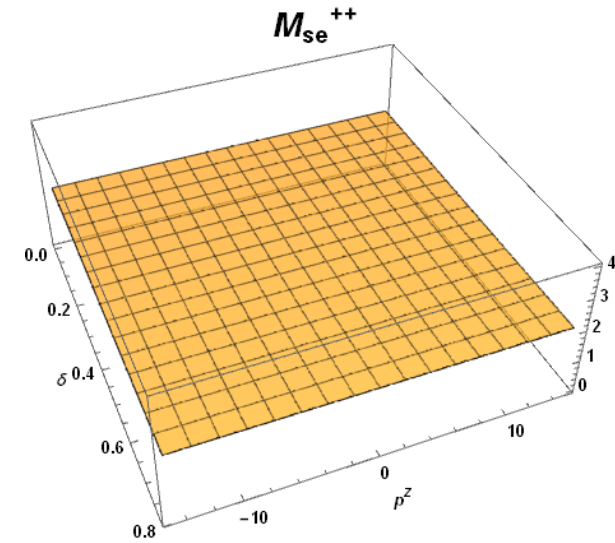
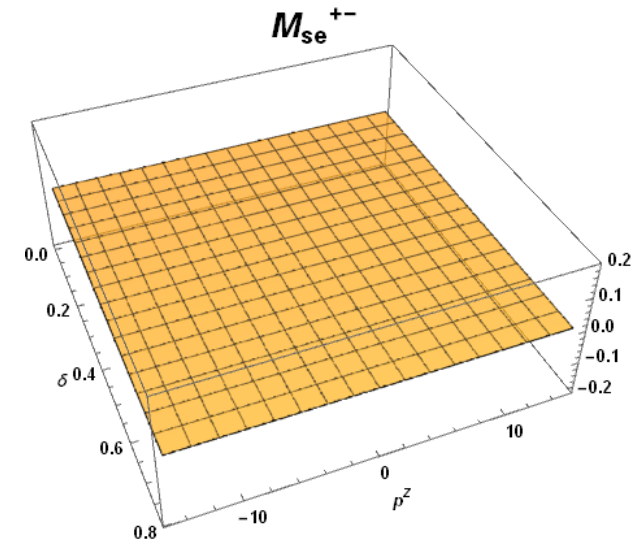
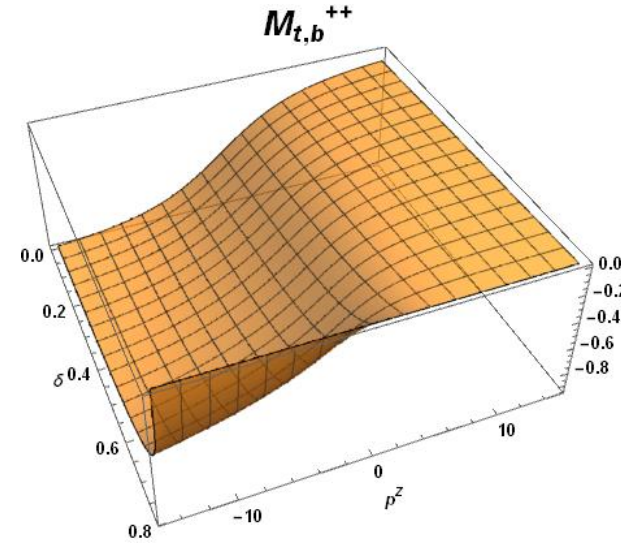
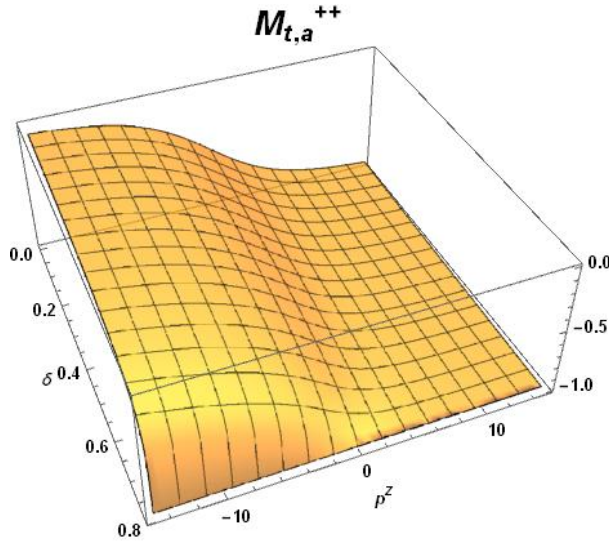
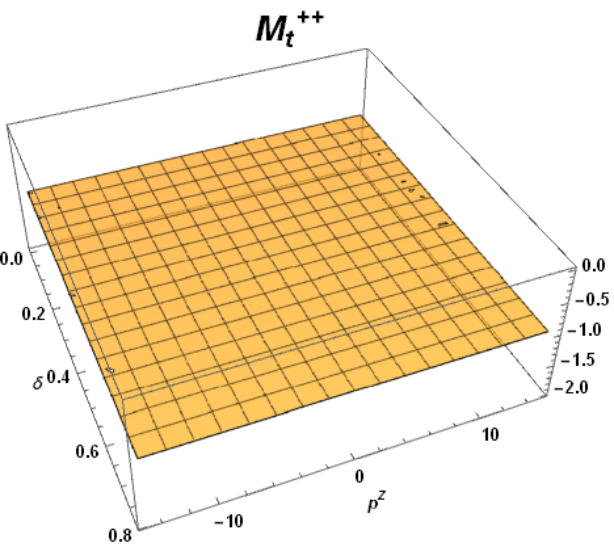
$$P^z = -15$$



- Positive higher momentum helicity amplitudes results in IFD can mimic the light-front results.
- Negative high momentum time-ordered helicity amplitudes in IFD can not mimic the light-front results.
- This clarifies any conceivable confusion between the Infinite momentum frame in IFD and LFD.
- To compare with LaMET results , one may have to select only positive higher momentum.

# Scalar particle and its anti-particle production by two real photons ( $\gamma\gamma \rightarrow SS$ )

- The helicity of the real photon is relativistic invariant that matches its chirality.  
⇒ No boundaries corresponding to the quantum correlation in this process
- All the amplitudes satisfy symmetry based on parity conservation.
- T and U channels have time-ordered helicity amplitudes with the same features.



- Only forward time-ordered processes are allowed at the LF end.

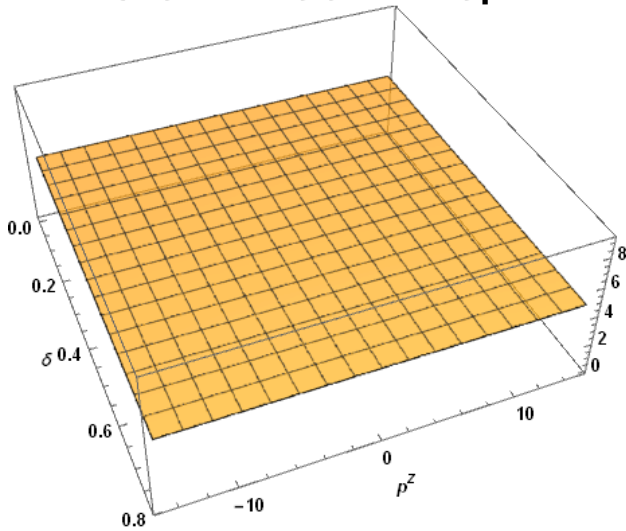
# Total Probabilities

$$|M|^2 = \sum_{\lambda_1, \lambda_2} |M_t^{\lambda_1 \lambda_2} + M_u^{\lambda_1 \lambda_2} + M_{se}^{\lambda_1 \lambda_2}|^2$$

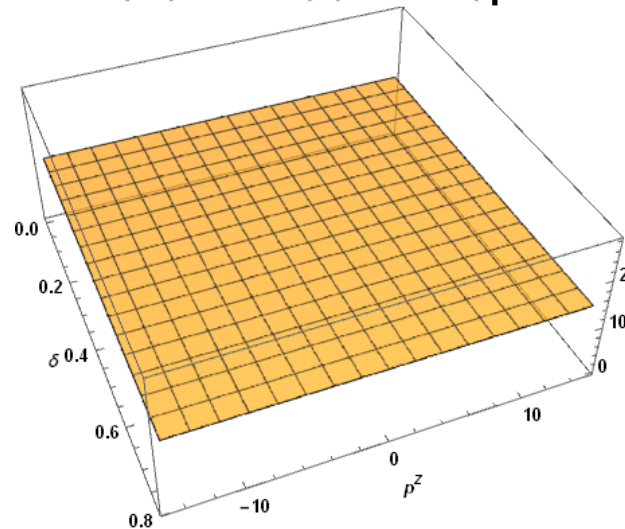
$$|M(\gamma\gamma \rightarrow SS/SS \rightarrow \gamma\gamma)|^2 = 4 \left[ \left[ \frac{t_1 + m_s^2}{t_1 - m_s^2} \right]^2 + \left[ \frac{u_1 + m_s^2}{u_1 - m_s^2} \right]^2 + 4 \left[ \frac{(t_1 + m_s^2)(u_1 + m_s^2) - 2t_1 u_1}{(t_1 - m_s^2)(u_1 - m_s^2)} \right] + 4 \right]$$

$$|M(VV \rightarrow SS/SS \rightarrow VV)|^2 = \left[ \left[ \frac{2(t_2 + m_s^2) - m_v^2}{(t_2 - m_s^2)} \right]^2 + \left[ \frac{2(u_2 + m_s^2) - m_v^2}{(u_2 - m_s^2)} \right]^2 + 2 \left[ \frac{(t_2 + u_2 + 2m_s^2 - 3m_v^2)^2}{(t_2 - m_s^2)(u_2 - m_s^2)} \right] \right] - 2 \left[ \left[ \frac{5t_2 + u_2 + 2m_s^2 - 4m_v^2}{t_2 - m_s^2} \right] + \left[ \frac{5u_2 + t_2 + 2m_s^2 - 4m_v^2}{u_2 - m_s^2} \right] \right] + 16$$

$$|M(\gamma\gamma \rightarrow SS)/(SS \rightarrow \gamma\gamma)|^2$$



$$|M(VV \rightarrow SS)/(SS \rightarrow VV)|^2$$



- $t$  and  $u$  are Mandelstam variables of the corresponding processes.
- Analytical results are symmetric under the  $t \leftrightarrow u$  exchange.
- Total probabilities are independent of the interpolation angle and the frame.

- So far, we discussed the Mikowski four-dimensional relativistic dynamics, we realized that we could extend interpolating dynamics even to the curve space-time.

## Interpolating Five-Dimensional Space-Time

### Einstein Field Equation

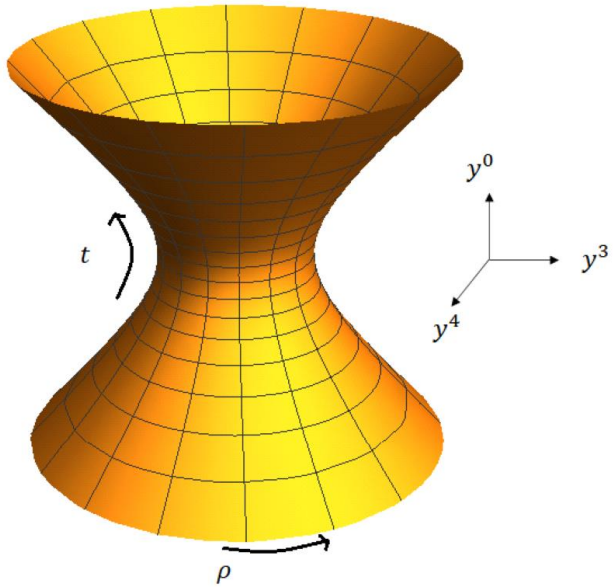
$$R_{ab} - \frac{1}{2}Rg_{ab} - \Lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab}$$

- De Sitter and anti-de Sitter spaces are the maximally symmetric vacuum solutions of Einstein's field equation with positive and negative vacuum energy densities, respectively.

# De Sitter Space ( $\Lambda > 0$ )

# Anti-de Sitter Space ( $\Lambda < 0$ )

- can be visualized as the hyperboloid in flat five-dimensional space.
  - Two dimensions are suppressed in the figures.

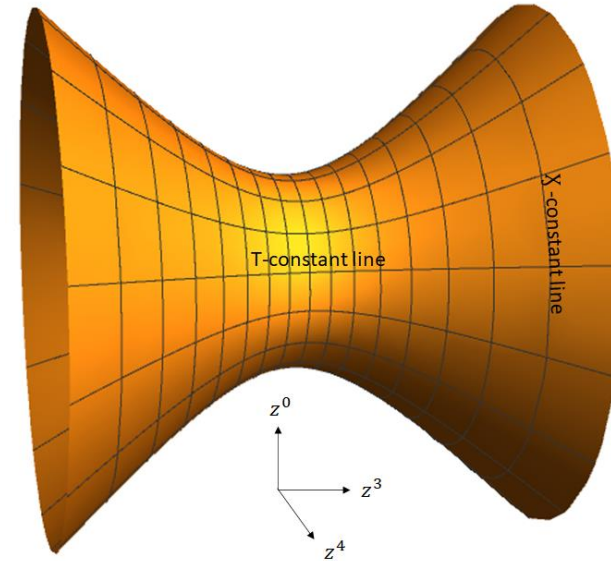


Space-time invariant

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = -l_1^2$$

$$l_1 = \sqrt{\frac{3}{\Lambda_1}}$$

$l_1$  = "De Sitter radius"



Space-time invariant

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 + z_4^2 = l_2^2$$

$$l_2 = \sqrt{\frac{-3}{\Lambda_2}}$$

$l_2$  = "Anti de Sitter radius"

- $y^0 = l_1 \sinh\left(\frac{t}{l_1}\right)$
- $y^1 = l_1 \cosh\left(\frac{t}{l_1}\right) \sin(\rho) \sin(\theta) \cos(\varphi)$
- $y^2 = l_1 \cosh\left(\frac{t}{l_1}\right) \sin(\rho) \sin(\theta) \sin(\varphi)$
- $y^3 = l_1 \cosh\left(\frac{t}{l_1}\right) \sin(\rho) \cos(\theta)$
- $y^4 = l_1 \cosh\left(\frac{t}{l_1}\right) \cos(\rho)$

$$ds_1^2 = dt^2 - l_1^2 \cosh^2\left(\frac{t}{l_1}\right) [d\rho^2 + \sin^2 \rho (d\theta^2 + \sin^2 \theta d\varphi^2)]$$

- $z^0 = l_2 \cosh(X) \sin\left(\frac{T}{l_2}\right)$
- $z^1 = l_2 \sinh(X) \sin(\theta) \cos(\varphi)$
- $z^2 = l_2 \sinh(X) \sin(\theta) \sin(\varphi)$
- $z^3 = l_2 \sinh(X) \cos(\theta)$
- $z^4 = l_2 \cosh(X) \cos\left(\frac{T}{l_2}\right)$

$$ds_2^2 = \cosh^2 X dT^2 - l_2^2 [dX^2 + \sinh^2 X (d\theta^2 + \sin^2 \theta d\varphi^2)]$$

# De Sitter Group ( $SO(4, 1)$ )

# Anti-de Sitter Group ( $SO(3, 2)$ )

□ Ten homogenous transformations

Matrix of operators

De sitter space-time matrix tensor

Matrix of operators

Anti-de sitter space-time matrix tensor

$$R^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix}$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$J^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Pi^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Pi^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Pi^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Pi^3 \\ \Pi^0 & \Pi^1 & \Pi^2 & \Pi^3 & 0 \end{pmatrix}$$

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Infinitesimal operators of the group =  $m_{ab}$

$m_{ab} \Rightarrow$  6 cyclic operators  
4 hyperbolic operators

$m_{ab} \Rightarrow$  4 cyclic operators  
6 hyperbolic operators

Lie-Algebra

$$[R^{\alpha\beta}, R^{\gamma\delta}] = i(\eta^{\beta\gamma} R^{\alpha\delta} - \eta^{\beta\delta} R^{\alpha\gamma} - \eta^{\alpha\gamma} R^{\beta\delta} + \eta^{\alpha\delta} R^{\beta\gamma})$$

Lie-Algebra

$$[J^{\alpha\beta}, J^{\gamma\delta}] = i(g^{\beta\gamma} J^{\alpha\delta} - g^{\beta\delta} J^{\alpha\gamma} - g^{\alpha\gamma} J^{\beta\delta} + g^{\alpha\delta} J^{\beta\gamma})$$

De Sitter translation operators

$$\alpha = \beta = \gamma = \delta = 0, 1, 2, 3, 4$$

$$\Gamma^\mu = R^{4\mu},$$

Anti de Sitter translation operators

$$\Pi^\mu = J^{4\mu}$$

$$[R^{\mu\nu}, R^{\rho\lambda}] = i(\eta^{\nu\rho} R^{\mu\lambda} - \eta^{\nu\lambda} R^{\mu\rho} - \eta^{\mu\rho} R^{\nu\lambda} + \eta^{\mu\lambda} R^{\nu\rho})$$

$$[R^{\mu\nu}, \Gamma^\rho] = i(\eta^{\nu\rho} \Gamma^\mu - \eta^{\mu\rho} \Gamma^\nu)$$

$$[\Gamma^\mu, \Gamma^\nu] = iR^{\mu\nu}$$

$$\mu = \nu = \lambda = \rho = 0, 1, 2, 3$$

$$[J^{\mu\nu}, J^{\rho\lambda}] = i(g^{\nu\rho} J^{\mu\lambda} - g^{\nu\lambda} J^{\mu\rho} - g^{\mu\rho} J^{\nu\lambda} + g^{\mu\lambda} J^{\nu\rho})$$

$$[J^{\mu\nu}, \Pi^\rho] = i(g^{\nu\rho} \Pi^\mu - g^{\mu\rho} \Pi^\nu)$$

$$[\Pi^\mu, \Pi^\nu] = -iJ^{\mu\nu}$$

L. Jaffe J. Math. Phys. **12**, 882 (1971)

- Translation operators in five-dimensional spaces do not commute with each other.

# Interpolating de Sitter and anti-de Sitter space-time between IFD and LFD

- Interpolating de Sitter coordinates

$$y^{\hat{\alpha}} = G_{\hat{\beta}}^{\hat{\alpha}} y^{\beta}$$

$$G_{\hat{\beta}}^{\hat{\alpha}} = \begin{pmatrix} \cos \delta & 0 & 0 & \sin \delta & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin \delta & 0 & 0 & -\cos \delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$y^{\hat{t}} = l_1 \sinh(t/l_1) \cos(\delta) + l_1 \cosh(t/l_1) \sin(\rho) \cos(\theta) \sin(\delta)$$

$$y^{\hat{1}} = l_1 \cosh(t/l_1) \sin(\rho) \sin(\theta) \cos(\phi)$$

$$y^{\hat{2}} = l_1 \cosh(t/l_1) \sin(\rho) \sin(\theta) \sin(\phi)$$

$$y^{\hat{-}} = l_1 \sinh(t/l_1) \sin(\delta) - l_1 \cosh(t/l_1) \sin(\rho) \cos(\theta) \cos(\delta)$$

$$y^{\hat{4}} = l_1 \cosh(t/l_1) \cos(\rho)$$

$$l_1 = \sqrt{\frac{3}{\Lambda_1}}$$

- Interpolating de Sitter space-time matrix tensor

$$\eta^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\hat{\alpha}\hat{\beta}}$$

$$ds_1^2 = \eta_{\hat{\alpha}\hat{\beta}} dy^{\hat{\alpha}} dy^{\hat{\beta}} = (2dy^+ dy^- - dy_{\perp}^2 - dy_4^2)$$

- Interpolating anti-de Sitter coordinates

$$z^{\hat{\alpha}} = G_{\hat{\beta}}^{\hat{\alpha}} z^{\beta}$$

$$z^{\hat{t}} = l_2 \cosh(\chi) \sin(T/l_2) \cos(\delta) + l_2 \sinh(\chi) \cos(\theta) \sin(\delta)$$

$$z^{\hat{1}} = l_2 \sinh(\chi) \sin(\theta) \cos(\phi)$$

$$z^{\hat{2}} = l_2 \sinh(\chi) \sin(\theta) \sin(\phi)$$

$$z^{\hat{-}} = l_2 \cosh(\chi) \sin(T/l_2) \sin(\delta) - l_2 \sinh(\chi) \cos(\theta) \cos(\delta)$$

$$z^{\hat{4}} = l_2 \cosh(\chi) \cos(T/l_2)$$

$$l_2 = \sqrt{\frac{-3}{\Lambda_2}}$$

- Interpolating anti-de Sitter space-time matrix tensor

$$g^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = g_{\hat{\alpha}\hat{\beta}}$$

$$ds_2^2 = g_{\hat{\alpha}\hat{\beta}} dz^{\hat{\alpha}} dz^{\hat{\beta}} = (2dz^+ dz^- - dz_{\perp}^2 + dz_4^2)_{38}$$

- Matrix of interpolating de Sitter operators

$$R^{\hat{\alpha}\hat{\beta}} = G_{\hat{\gamma}}^{\hat{\alpha}} R^{\hat{\gamma}\hat{\delta}} G_{\hat{\delta}}^{\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{+}} & \Gamma^1 & \Gamma^2 & \Gamma^{\hat{-}} & 0 \end{pmatrix}$$

$$\Gamma^{\hat{+}} = \Gamma^0 \cos \delta + \Gamma^3 \sin \delta, \quad \Gamma^{\hat{-}} = \Gamma^0 \sin \delta - \Gamma^3 \cos \delta$$

## Lie-Algebra

$$[R^{\hat{\alpha}\hat{\beta}}, R^{\hat{\gamma}\hat{\delta}}] = i(\eta^{\hat{\beta}\hat{\gamma}} R^{\hat{\alpha}\hat{\delta}} - \eta^{\hat{\beta}\hat{\delta}} R^{\hat{\alpha}\hat{\gamma}} - \eta^{\hat{\alpha}\hat{\gamma}} R^{\hat{\beta}\hat{\delta}} + \eta^{\hat{\alpha}\hat{\delta}} R^{\hat{\beta}\hat{\gamma}})$$

$$\hat{\alpha} = \hat{\beta} = \hat{\gamma} = \hat{\delta} = \hat{+}, \hat{1}, \hat{2}, \hat{-}, \hat{4}$$

Interpolating de Sitter translation operators

$$\Gamma^{\hat{\mu}} = R^{\hat{4}\hat{\mu}}$$

$$[R^{\hat{\mu}\hat{\nu}}, R^{\hat{\rho}\hat{\lambda}}] = i(\eta^{\hat{\nu}\hat{\rho}} R^{\hat{\mu}\hat{\lambda}} - \eta^{\hat{\nu}\hat{\lambda}} R^{\hat{\mu}\hat{\rho}} - \eta^{\hat{\mu}\hat{\rho}} R^{\hat{\nu}\hat{\lambda}} + \eta^{\hat{\mu}\hat{\lambda}} R^{\hat{\nu}\hat{\rho}})$$

$$[R^{\hat{\mu}\hat{\nu}}, \Gamma^{\hat{\rho}}] = i(\eta^{\hat{\nu}\hat{\rho}} \Gamma^{\hat{\mu}} - \eta^{\hat{\mu}\hat{\rho}} \Gamma^{\hat{\nu}})$$

$$[\Gamma^{\hat{\mu}}, \Gamma^{\hat{\nu}}] = iR^{\hat{\mu}\hat{\nu}}$$

$$\hat{\mu} = \hat{\nu} = \hat{\rho} = \hat{\lambda} = \hat{+}, \hat{1}, \hat{2}, \hat{-}$$

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin \delta + K^1 \cos \delta \\ E^{\hat{2}} &= -J^1 \sin \delta + K^2 \cos \delta \\ F^{\hat{1}} &= K^1 \sin \delta - J^2 \cos \delta \\ F^{\hat{2}} &= K^3 \sin \delta + J^1 \cos \delta \end{aligned}$$

- Matrix of interpolating anti-de Sitter operators

$$J^{\hat{\alpha}\hat{\beta}} = G_{\hat{\gamma}}^{\hat{\alpha}} J^{\hat{\gamma}\hat{\delta}} G_{\hat{\delta}}^{\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^1 \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^2 \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{+}} & \Pi^1 & \Pi^2 & \Pi^{\hat{-}} & 0 \end{pmatrix}$$

$$\Pi^{\hat{+}} = \Pi^0 \cos \delta + \Pi^3 \sin \delta, \quad \Pi^{\hat{-}} = \Pi^0 \sin \delta - \Pi^3 \cos \delta$$

## Lie-Algebra

$$[J^{\hat{\alpha}\hat{\beta}}, J^{\hat{\gamma}\hat{\delta}}] = i(g^{\hat{\beta}\hat{\gamma}} R^{\hat{\alpha}\hat{\delta}} - g^{\hat{\beta}\hat{\delta}} J^{\hat{\alpha}\hat{\gamma}} - g^{\hat{\alpha}\hat{\gamma}} J^{\hat{\beta}\hat{\delta}} + g^{\hat{\alpha}\hat{\delta}} J^{\hat{\beta}\hat{\gamma}})$$

Interpolating anti-de Sitter translation operators

$$\Pi^{\hat{\mu}} = J^{\hat{4}\hat{\mu}}$$

$$[J^{\hat{\mu}\hat{\nu}}, J^{\hat{\rho}\hat{\lambda}}] = i(g^{\hat{\nu}\hat{\rho}} J^{\hat{\mu}\hat{\lambda}} - g^{\hat{\nu}\hat{\lambda}} J^{\hat{\mu}\hat{\rho}} - g^{\hat{\mu}\hat{\rho}} J^{\hat{\nu}\hat{\lambda}} + g^{\hat{\mu}\hat{\lambda}} J^{\hat{\nu}\hat{\rho}})$$

$$[J^{\hat{\mu}\hat{\nu}}, \Pi^{\hat{\rho}}] = i(g^{\hat{\nu}\hat{\rho}} \Pi^{\hat{\mu}} - g^{\hat{\mu}\hat{\rho}} \Pi^{\hat{\nu}})$$

$$[\Pi^{\hat{\mu}}, \Pi^{\hat{\nu}}] = -iJ^{\hat{\mu}\hat{\nu}}$$

# Contraction of Interpolating de Sitter and anti-de Sitter spaces into Interpolating Poincare space (vector representation)

•The curvature of five-dimensional space-time is due to the vacuum energy density.

$$\Lambda \rightarrow 0$$

Four-dimensional flat space time

(Eliminating curvature  $\Rightarrow l_1 \rightarrow \infty, l_2 \rightarrow \infty$ )

$$(ds_1^2)_{l_1 \rightarrow \infty} = (ds_2^2)_{l_2 \rightarrow \infty} = ds^2 = \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\nu}} dx^{\hat{\nu}} = (2dx^+ dx^- - dx_{\perp}^2)$$

Interpolating de Sitter translation

$$y'^{\hat{\alpha}} = G_{\hat{\beta}}^{\hat{\alpha}} \left( e^{i \eta_{\hat{\mu}\hat{\nu}} \frac{a^{\hat{\mu}}}{l_1} \Gamma^{\hat{\nu}}} \right)_{\gamma}^{\beta} (G^{-1})_{\hat{\beta}}^{\gamma} y^{\hat{\beta}}$$

$(y'^{\hat{\alpha}})_{l_1 \rightarrow \infty}$

Interpolating anti-de Sitter translation

$$z'^{\hat{\alpha}} = G_{\hat{\beta}}^{\hat{\alpha}} \left( e^{i \eta_{\hat{\mu}\hat{\nu}} \frac{a^{\hat{\mu}}}{l_2} \Pi^{\hat{\nu}}} \right)_{\gamma}^{\beta} (G^{-1})_{\hat{\beta}}^{\gamma} z^{\hat{\beta}}$$

$(z'^{\hat{\alpha}})_{l_2 \rightarrow \infty}$

$$\begin{pmatrix} x'^{\hat{+}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ x'^{\hat{-}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & a^{\hat{+}} \\ 0 & 1 & 0 & 0 & a^{\hat{1}} \\ 0 & 0 & 1 & 0 & a^{\hat{2}} \\ 0 & 0 & 0 & 1 & a^{\hat{-}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \\ 1 \end{pmatrix}$$

Interpolating Poincare translation

$$[P^{\hat{\mu}}, P^{\hat{\nu}}] = 0$$

“On the contraction of groups and their representation”  
By E. Inonu and E.P Wigner (1958)

# Interpolating de Sitter and anti-de Sitter operators in spinor representation

$$\gamma^0 \Rightarrow \text{Time-like Hermitian matrix} \quad \gamma^i \Rightarrow \text{Space-like anti-Hermitian matrices} \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$\gamma^5 \Rightarrow \text{Hermitian matrix} \quad \gamma^4 = -i\gamma^5 \Rightarrow \text{anti-Hermitian matrix}$$

- Interpolating Gamma matrices in chiral basis

$$\gamma^{\hat{\dagger}} = \gamma^0 \cos \delta + \gamma^3 \sin \delta, \quad \gamma^{\hat{1}} = \gamma^1, \quad \gamma^{\hat{2}} = \gamma^2, \quad \gamma^{\hat{-}} = \gamma^0 \sin \delta - \gamma^3 \cos \delta, \quad \gamma^{\hat{4}} = \gamma^4, \quad \gamma^{\hat{5}} = \gamma^5$$

## De Sitter operators $SO(4,1)$

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{\dagger}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{\dagger}} & \Gamma^{\hat{1}} & \Gamma^{\hat{2}} & \Gamma^{\hat{-}} & 0 \end{pmatrix} = \frac{i}{4} \begin{pmatrix} 0 & [\gamma^{\hat{\dagger}}, \gamma^1] & [\gamma^{\hat{\dagger}}, \gamma^2] & [\gamma^{\hat{\dagger}}, \gamma^{\hat{-}}] & [\gamma^{\hat{\dagger}}, \gamma^4] \\ [\gamma^1, \gamma^{\hat{\dagger}}] & 0 & [\gamma^1, \gamma^2] & [\gamma^1, \gamma^{\hat{-}}] & [\gamma^1, \gamma^4] \\ [\gamma^2, \gamma^{\hat{\dagger}}] & [\gamma^2, \gamma^1] & 0 & [\gamma^2, \gamma^{\hat{-}}] & [\gamma^2, \gamma^4] \\ [\gamma^{\hat{-}}, \gamma^{\hat{\dagger}}] & [\gamma^{\hat{-}}, \gamma^1] & [\gamma^{\hat{-}}, \gamma^2] & 0 & [\gamma^{\hat{-}}, \gamma^4] \\ [\gamma^4, \gamma^{\hat{\dagger}}] & [\gamma^4, \gamma^1] & [\gamma^4, \gamma^2] & [\gamma^4, \gamma^{\hat{-}}] & 0 \end{pmatrix}$$

$$\eta^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\hat{\alpha}\hat{\beta}}$$

Clifford algebra  $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}} I$

$$\hat{\alpha} = \hat{\beta} = \hat{\dagger}, 1, 2, \hat{-}, 4$$

## Anti-de Sitter operators $SO(3,2)$

$$J^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{\dagger}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{\dagger}} & \Pi^{\hat{1}} & \Pi^{\hat{2}} & \Pi^{\hat{-}} & 0 \end{pmatrix} = \frac{i}{4} \begin{pmatrix} 0 & [\gamma^{\hat{\dagger}}, \gamma^1] & [\gamma^{\hat{\dagger}}, \gamma^2] & [\gamma^{\hat{\dagger}}, \gamma^{\hat{-}}] & [\gamma^{\hat{\dagger}}, \gamma^5] \\ [\gamma^1, \gamma^{\hat{\dagger}}] & 0 & [\gamma^1, \gamma^2] & [\gamma^1, \gamma^{\hat{-}}] & [\gamma^1, \gamma^5] \\ [\gamma^2, \gamma^{\hat{\dagger}}] & [\gamma^2, \gamma^1] & 0 & [\gamma^2, \gamma^{\hat{-}}] & [\gamma^2, \gamma^5] \\ [\gamma^{\hat{-}}, \gamma^{\hat{\dagger}}] & [\gamma^{\hat{-}}, \gamma^1] & [\gamma^{\hat{-}}, \gamma^2] & 0 & [\gamma^{\hat{-}}, \gamma^5] \\ [\gamma^5, \gamma^{\hat{\dagger}}] & [\gamma^5, \gamma^1] & [\gamma^5, \gamma^2] & [\gamma^5, \gamma^{\hat{-}}] & 0 \end{pmatrix}$$

$$g^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = g_{\hat{\alpha}\hat{\beta}}$$

Clifford algebra  $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2g^{\hat{\alpha}\hat{\beta}} I$

$$\hat{\alpha} = \hat{\beta} = \hat{\dagger}, 1, 2, \hat{-}, 5$$

- Interpolating operators in the spinor representation can be summarized as commutation among the interpolating gamma matrices, and they all satisfy the corresponding Lie algebra.

# Contracting de Sitter and anti-de Sitter spaces into Poincare space

Five-dimensional de Sitter coordinates

( $\Lambda > 0$ )

Group algebra  $SO(4,1)$

${}^5C_2$

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Gamma^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Gamma^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Gamma^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Gamma^{\hat{-}} \\ \Gamma^{\hat{+}} & \Gamma^{\hat{1}} & \Gamma^{\hat{2}} & \Gamma^{\hat{-}} & 0 \end{pmatrix}$$

$[\Gamma^{\hat{\mu}}, \Gamma^{\hat{\nu}}] = iR^{\hat{\mu}\hat{\nu}}$

Five-dimensional anti-de Sitter coordinates

( $\Lambda < 0$ )

Group algebra  $SO(3,2)$ ,

${}^5C_2$

$$J^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 & -\Pi^{\hat{+}} \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} & -\Pi^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} & -\Pi^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 & -\Pi^{\hat{-}} \\ \Pi^{\hat{+}} & \Pi^{\hat{1}} & \Pi^{\hat{2}} & \Pi^{\hat{-}} & 0 \end{pmatrix}$$

$[\Pi^{\hat{\mu}}, \Pi^{\hat{\nu}}] = -iJ^{\hat{\mu}\hat{\nu}}$

Four-dimensional

Poincare space coordinates

Poincare group algebra  $ISO(3,1)$

${}^4C_2$

${}^4C_1$

- Six homogenous operators

- Four inhomogeneous operators

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & E^{\hat{1}} & E^{\hat{2}} & -K^3 \\ -E^{\hat{1}} & 0 & J^3 & -F^{\hat{1}} \\ -E^{\hat{2}} & -J^3 & 0 & -F^{\hat{2}} \\ K^3 & F^{\hat{1}} & F^{\hat{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} P^{\hat{+}} \\ P^{\hat{1}} \\ P^{\hat{2}} \\ P^{\hat{-}} \end{pmatrix}$$

Lie algebra

$$[M^{\hat{\mu}\hat{\nu}}, M^{\hat{\rho}\hat{\lambda}}] = i(g^{\hat{\nu}\hat{\rho}} M^{\hat{\mu}\hat{\lambda}} - g^{\hat{\nu}\hat{\lambda}} M^{\hat{\mu}\hat{\rho}} - g^{\hat{\mu}\hat{\rho}} M^{\hat{\nu}\hat{\lambda}} + g^{\hat{\mu}\hat{\lambda}} M^{\hat{\nu}\hat{\rho}})$$

$$[P^{\hat{\rho}}, M^{\hat{\mu}\hat{\nu}}] = i(g^{\hat{\rho}\hat{\mu}} P^{\hat{\nu}} - g^{\hat{\rho}\hat{\nu}} P^{\hat{\mu}})$$

$[P^{\hat{\mu}}, P^{\hat{\nu}}] = 0$

□ Contraction process does not change the kinematic and dynamic characteristics of interpolating operators

## Future Projects

- Scaled Interpolating Basis and Applications
- Implementation of the Relativistic QFT in Quantum Computing

# Scaled Interpolating Coordinates

The interpolating parameter space  $0 \leq \delta < \frac{\pi}{4}$  can be covered by the unified scheme that introduces scaled interpolating coordinates.

$$x^\mu x_\mu = 0 \Rightarrow \left( \frac{x^\dagger}{\sqrt{\mathcal{C}}} \right)^2 - \left( \frac{x_\dagger}{\sqrt{\mathcal{C}}} \right)^2 = \mathbf{x}_\perp^2$$

➤ Instant form perspective

$$x^N = H.x$$

$$\begin{pmatrix} \frac{x^\dagger}{\sqrt{\mathcal{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_\dagger}{\sqrt{\mathcal{C}}} \end{pmatrix} = \begin{pmatrix} \frac{\cos \delta}{\sqrt{\mathcal{C}}} & 0 & 0 & \frac{\sin \delta}{\sqrt{\mathcal{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin \delta}{\sqrt{\mathcal{C}}} & 0 & 0 & \frac{\cos \delta}{\sqrt{\mathcal{C}}} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{aligned} \frac{x^\dagger}{\sqrt{\mathcal{C}}} &= \frac{x^0 \cos \delta + x^3 \sin \delta}{\sqrt{\mathcal{C}}} \\ \frac{x_\dagger}{\sqrt{\mathcal{C}}} &= \frac{x^0 \cos \delta - x^3 \sin \delta}{\sqrt{\mathcal{C}}} \end{aligned} \Rightarrow (x^0)^2 - (x^3)^2$$

➤ Light-front form perspective

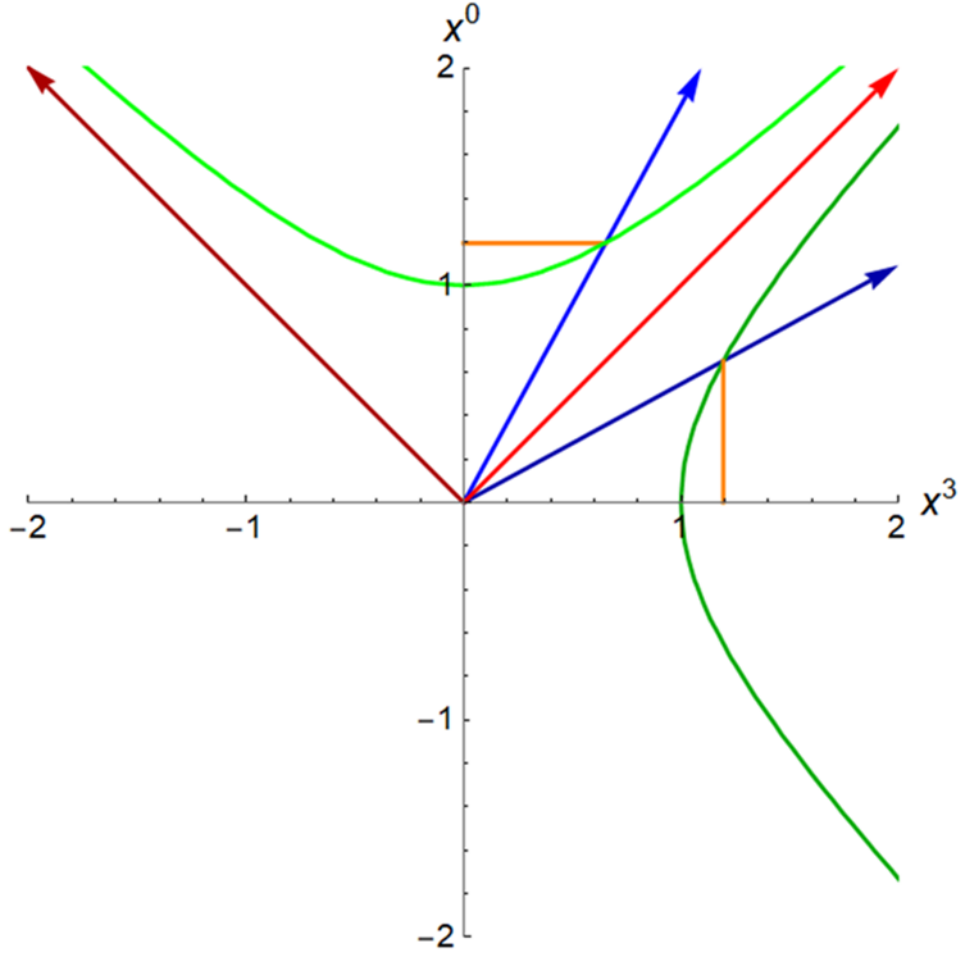
$$x^\nu = L_\mu^\nu . x^\mu$$

$$\begin{pmatrix} x^{\bar{+}} \\ x^{\bar{1}} \\ x^{\bar{2}} \\ x^{\bar{-}} \end{pmatrix} = \begin{pmatrix} \frac{\cos \delta + \sin \delta}{\sqrt{\mathcal{C}}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\cos \delta - \sin \delta}{\sqrt{\mathcal{C}}} \end{pmatrix} \begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix}$$

$$\begin{aligned} x^{\bar{+}} &= \frac{\cos \delta + \sin \delta}{\sqrt{\mathcal{C}}} x^+ = \frac{1}{\sqrt{2}} \left( \frac{x^\dagger}{\sqrt{\mathcal{C}}} + \frac{x_\dagger}{\sqrt{\mathcal{C}}} \right) \\ x^{\bar{-}} &= \frac{\cos \delta - \sin \delta}{\sqrt{\mathcal{C}}} x^- = \frac{1}{\sqrt{2}} \left( \frac{x^\dagger}{\sqrt{\mathcal{C}}} - \frac{x_\dagger}{\sqrt{\mathcal{C}}} \right) \end{aligned} \Rightarrow 2x^{\bar{+}}x^{\bar{-}} = 2x^+x^-$$

# Scaled Interpolating Coordinates

## Instant-Form Scaled Interpolating Basis

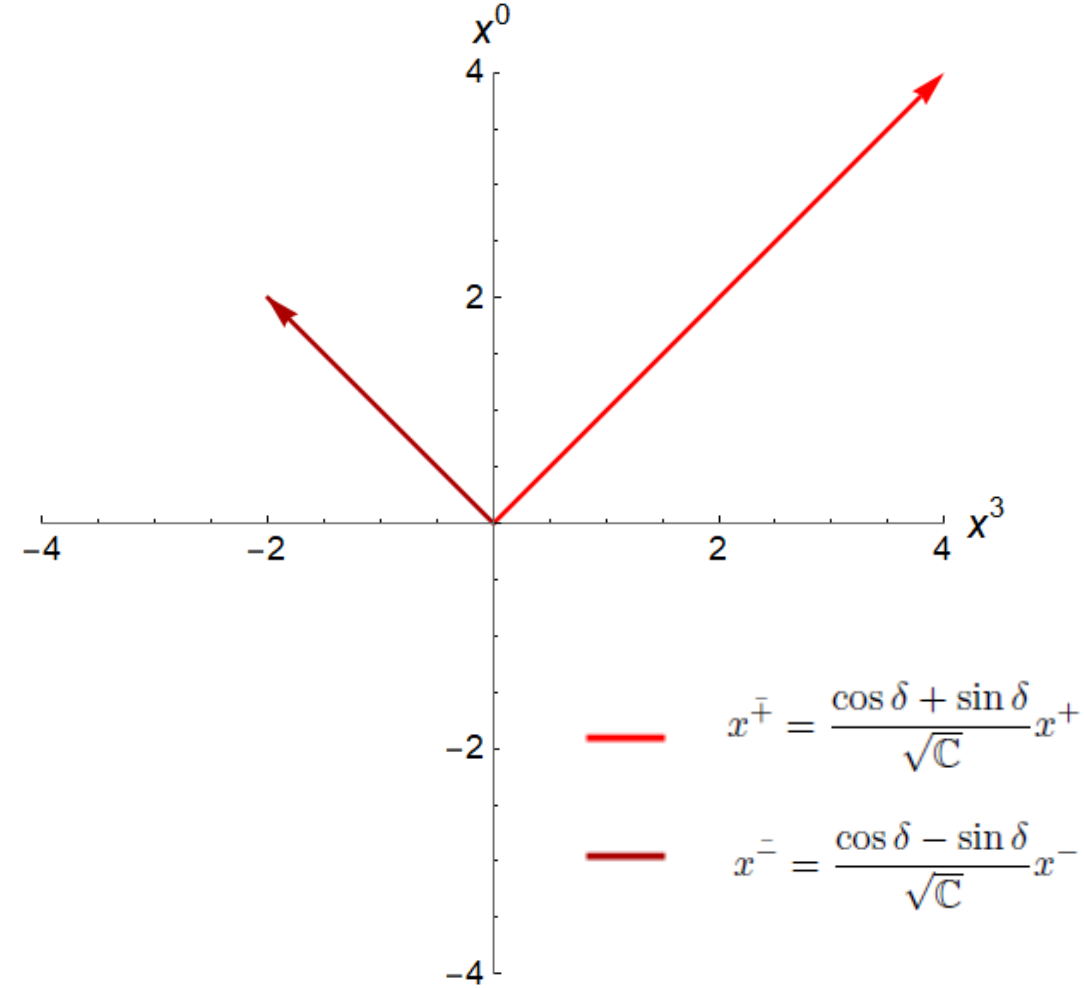


- $\frac{x^{\hat{+}}}{\sqrt{\text{Cos}(2\delta)}}$
- $\frac{x^{\hat{-}}}{\sqrt{\text{Cos}(2\delta)}}$
- $x^+$
- $x^-$
- $(x^0)^2 - (x^3)^2 = 1$
- $(x^0)^2 - (x^3)^2 = -1$

$$(x^0)^2 - (x^3)^2 = \left(\frac{x^{\hat{+}}}{\sqrt{\mathbb{C}}}\right)^2 - \left(\frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}}\right)^2$$

Reduction in degrees of freedom.

## Light-Front Scaled Interpolating Basis



- $x^{\bar{+}} = \frac{\cos \delta + \sin \delta}{\sqrt{\mathbb{C}}} x^+$
- $x^{\bar{-}} = \frac{\cos \delta - \sin \delta}{\sqrt{\mathbb{C}}} x^-$

$$2x^+ x^- = 2x^{\bar{+}} x^{\bar{-}}$$

# Scaled Interpolating Operators

## Scaled Interpolating Instant Form Poincaré Operators

$$M^N = \begin{pmatrix} 0 & \frac{E^1}{\sqrt{C}} & \frac{E^2}{\sqrt{C}} & K^3 \\ -\frac{E^1}{\sqrt{C}} & 0 & J^3 & \frac{\mathcal{K}^1}{\sqrt{C}} \\ -\frac{E^2}{\sqrt{C}} & -J^3 & 0 & \frac{\mathcal{K}^2}{\sqrt{C}} \\ -K^3 & -\frac{\mathcal{K}^1}{\sqrt{C}} & -\frac{\mathcal{K}^2}{\sqrt{C}} & 0 \end{pmatrix}$$

## Scaled Interpolating Light-Front Poincaré Operators

$$M^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} 0 & \frac{E^1(\cos\delta + \sin\delta)}{\sqrt{C}} & \frac{E^2(\cos\delta + \sin\delta)}{\sqrt{C}} & -K^3 \\ -\frac{E^1(\cos\delta + \sin\delta)}{\sqrt{C}} & 0 & J^3 & -\frac{F^1(\cos\delta - \sin\delta)}{\sqrt{C}} \\ -\frac{E^2(\cos\delta + \sin\delta)}{\sqrt{C}} & -J^3 & 0 & -\frac{F^2(\cos\delta - \sin\delta)}{\sqrt{C}} \\ K^3 & \frac{F^1(\cos\delta - \sin\delta)}{\sqrt{C}} & \frac{F^2(\cos\delta - \sin\delta)}{\sqrt{C}} & 0 \end{pmatrix}$$

- Reduction of degrees of freedom  $\Rightarrow$  Elimination of dynamical operators
- All the remaining operators are kinematic operators.
- Since kinematic operators leave the time-invariant, their usage is beneficial in describing the characteristics of the motion with a simpler time-variant expression.
- The same observation applies to the scaled interpolating five-dimensional operators

$$\begin{aligned} E^{\hat{1}} &= J^2 \sin\delta + K^1 \cos\delta, & \mathcal{K}^{\hat{1}} &= -K^1 \sin\delta - J^2 \cos\delta, \\ E^{\hat{2}} &= K^2 \cos\delta - J^1 \sin\delta, & \mathcal{K}^{\hat{2}} &= J^1 \cos\delta - K^2 \sin\delta, \\ F^{\hat{1}} &= K^1 \sin\delta - J^2 \cos\delta, & \mathcal{D}^{\hat{1}} &= -K^1 \cos\delta + J^2 \sin\delta, \\ F^{\hat{2}} &= K^2 \sin\delta + J^1 \cos\delta, & \mathcal{D}^{\hat{2}} &= -J^1 \sin\delta - K^2 \cos\delta. \end{aligned}$$

# Possible Applications in Future Developments

□ In the interpolating parameter space between  $0 \leq \delta < \frac{\pi}{4}$  (Space-like region)

- Scaled interpolating instant form operators satisfy exact lie algebra of standard Poincare operators
- Number of kinematic and dynamic operators are equal, and they are foam invariant
- We can correspond scaled interpolating space to the Euclidean space.
- To make the correspondence which depend on the interpolation angle we need the extend the wick rotation

Minkowski space-time  $\rightarrow$  Euclidean Space-time ( $\delta \rightarrow 0$ )

$$x^0 \rightarrow \tilde{x}^0 = ix^0 \quad \frac{x^{\hat{\dagger}}}{\sqrt{\mathbb{C}}} \rightarrow \frac{\tilde{x}^{\hat{\dagger}}}{\sqrt{\mathbb{C}}} = i \frac{x^{\hat{\dagger}}}{\sqrt{\mathbb{C}}}$$

Space-Time interval in the scaled interpolating Euclidean Space

$$s_E^2 = (\tilde{x}^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = \left( \frac{\tilde{x}^{\hat{\dagger}}}{\sqrt{\mathbb{C}}} \right)^2 + (x^1)^2 + (x^2)^2 + \left( \frac{x^{\hat{-}}}{\sqrt{\mathbb{C}}} \right)^2$$

- Large-Momentum Effective Theory ( or LaMET ) advocates a direct approach to simulate Parton Physics in Euclidean lattice QCD theory.
- We may propose simulating Parton Physics in Euclidean lattice QCD theory using low momentum corporates with interpolation angle.

## ➤ Theoretical Foundation

- Scalar particle decays into a fermion and an anti-fermion.
- Lagrangian density,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_\phi^2 \phi^2 + \bar{\psi} (i\not{\partial} - m_\psi) \psi - g \phi \bar{\psi} \psi.$$

- Hamiltonian density,

$$\mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m_\phi^2 \phi^2 - i \psi^\dagger \boldsymbol{\alpha} \cdot \nabla \psi + m \bar{\psi} \psi + g \phi \bar{\psi} \psi.$$

- Lattice formulation makes QFT finite and discrete.
- The lattice version of QFT is the bridge between continuum field theory and a quantum circuit implementable on hardware.

# Theoretical Foundation : Implementation of the Relativistic QFT in Quantum Computing

- Normalized lattice Hamiltonian,

$$H = \sum_n \left[ \frac{\tilde{\pi}_n^2}{2} + \frac{1}{2a^2} \sum_i (\tilde{\phi}_{n+\hat{i}} - \tilde{\phi}_n)^2 + \frac{1}{2} m^2 \tilde{\phi}_n^2 - i \chi_n^\dagger \sum_i \gamma^0 \gamma^i \frac{\chi_{n+\hat{i}} - \chi_{n-\hat{i}}}{2a} + m \chi_n^\dagger \gamma^0 \chi_n + \tilde{g} \tilde{\phi}_n \bar{\chi}_n \chi_n \right].$$

- Light-front Hamiltonian,

$$P^- = \frac{(P^0 - P^3)}{\sqrt{2}}.$$

- Total longitudinal momentum in the normalized discrete form,

$$P^3 = -\frac{1}{2a} \sum_n \tilde{\pi}_n (\tilde{\phi}_{n+\hat{3}} - \tilde{\phi}_{n-\hat{3}}) - i \sum_n \chi_n^\dagger \frac{\chi_{n+\hat{3}} - \chi_{n-\hat{3}}}{2a}.$$

- LF QFT is highly appealing as the starting point for quantum simulation since LF facilitate lower qubit counts.

- Kreshchuk, M., Jia, S., Kirby, W. M., Goldstein, G., Vary, J. P., & Love, P. J. *Simulating hadronic physics on noisy intermediate-scale quantum devices using basis light-front quantization.* *Physical Review A*, 103(6), 062601 (2021).
- M. Kreshchuk, W. M. Kirby, G. Goldstein, H. Beau-chemin, and P. J. Love, Quantum simulation of quantum field theory in the light-front formulation, *Phys. Rev. A* 105, 032418 (2022).
- W. Qian, R. Basili, S. Pal, G. Luecke, and J. P. Vary, Solving hadron structures using the basis light-front quantization approach on quantum computers, *Phys.Rev. Res.* 4, 043193 (2022).

# Summary and Conclusion

- ❑ Even though quantum orientation entanglement of the helicity states in the light-front form dynamics is highly nontrivial as the transverse rotations in LFD become dynamical, we obtained it using the transverse light-front boost, which is kinematical in LFD.
- ❑ We present a unique relationship for the spin orientation changes with the particle's momentum direction between IFD and LFD for spin-1/2 and spin-1 spinors, as well as polarization vectors.
- ❑ To analyze the quantum orientation entanglement, we introduce a novel method of expanding the interpolating helicity states in terms of the Jacob–Wick helicity.
- ❑ The corresponding probabilistic coefficients follow the structure of the Wigner  $d$  matrix elements, which we use for the interpretation of the quantum orientation entanglement manifested in the angular distributions of the interpolating scattering helicity amplitudes
  
- ❖ Quantum correlation between the orientation entanglement and the interpolating Lorentz boost of the helicity states manifests itself as discrete boundaries in the landscape of frame-dependent interpolating helicity amplitudes
- ❖ Quantum correlation boundaries approach to the LF zero-mode in the limit of  $\delta \rightarrow \frac{\pi}{4}$  in the interpolating formulation.
- ❖ We demonstrate that the symmetric helicity “0” state of a vector particle exhibits quantum correlations distinct from those of the orthogonal antisymmetric helicity “0” state of a scalar particle.
- ❖ Time-ordered helicity amplitudes clarify the general misnomer of the equivalence between the infinite momentum frame of IFD and LFD.
  
- The Interpolating dynamic method can be applied to the five-dimensional spaces despite the curvature of their space-time.
- After establishing interpolating group algebra, we confirm interpolating de Sitter and anti-de Sitter groups can be contracted into interpolating Poincaré group in the limit of vacuum energy densities of their spaces go to zero, making the curvature of the spaces vanish.
- Even though the contraction process does not change the kinematic and dynamic characteristics of interpolating operators, it does reduce the number of homogeneous operators

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My parents, all my family members, and friends

Specially, Ms. Rhonda Bennett



## Lagrangian

$$\mathcal{L} = D_\mu \psi (D^\mu \psi)^\dagger - m^2 \psi \psi^\dagger - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu$$

$$D^\mu = \partial^\mu + ieA^\mu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

## Interpolating polarization vector

$$\epsilon_{\hat{\mu}}(P, +) = -\frac{1}{\sqrt{2}\mathbb{P}} \left( \mathbb{S}|\mathbf{P}_\perp|, \frac{P_1 P_\perp - iP_2 \mathbb{P}}{|\mathbf{P}_\perp|}, \frac{P_2 P_\perp + iP_1 \mathbb{P}}{|\mathbf{P}_\perp|}, -\mathbb{C}|\mathbf{P}_\perp| \right),$$

$$\epsilon_{\hat{\mu}}(P, -) = \frac{1}{\sqrt{2}\mathbb{P}} \left( \mathbb{S}|\mathbf{P}_\perp|, \frac{P_1 P_\perp + iP_2 \mathbb{P}}{|\mathbf{P}_\perp|}, \frac{P_2 P_\perp - iP_1 \mathbb{P}}{|\mathbf{P}_\perp|}, -\mathbb{C}|\mathbf{P}_\perp| \right),$$

$$\epsilon_{\hat{\mu}}(P, 0) = \frac{P^\dagger}{M\mathbb{P}} \left( P^\dagger - \frac{M^2}{P^\dagger}, P_1, P_2, P_\perp \right),$$

- The interpolating transverse polarization vectors respect the gauge condition  $A^\dagger = 0$  and  $\partial_\perp + \partial_\perp \cdot A_\perp \mathbb{C} = 0$  which links the Coulomb gauge  $\nabla \cdot A = 0$  in IFD and light-front gauge  $A^+ = 0$  in the LFD.
- The real photon's helicity  $\lambda$  takes only  $+$  or  $-$ , but not  $0$  as  $M \rightarrow 0$  limit,  $\epsilon_{\hat{\mu}}(P, 0)$  doesn't exist

# Interpolating Helicity Amplitudes

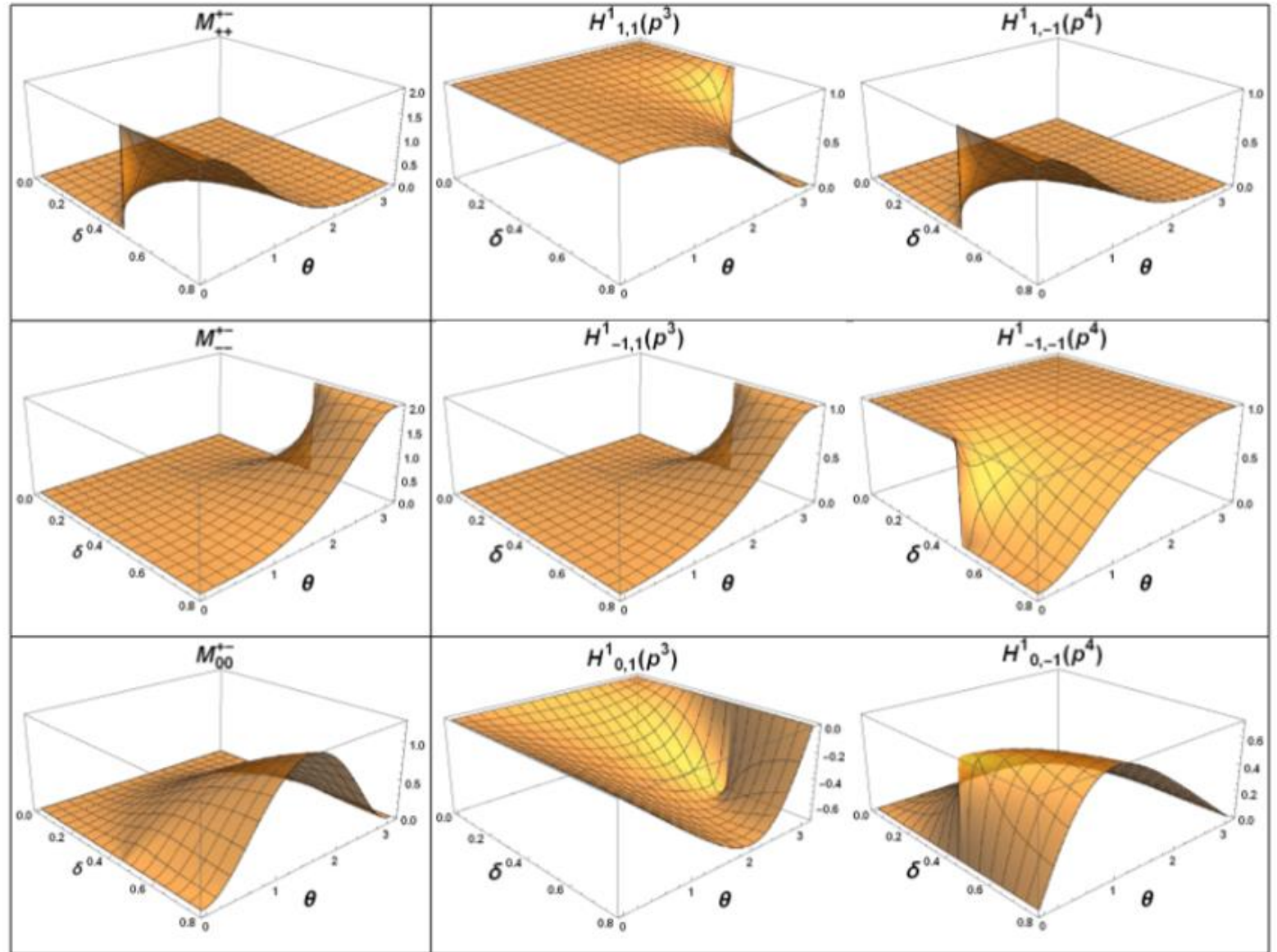
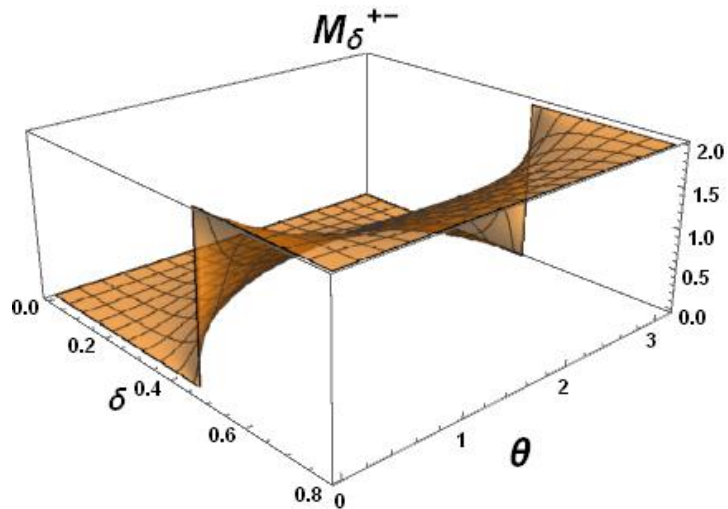
Example:

$$M_{\delta}^{++} = M_{++}^{++} + M_{--}^{++} + M_{00}^{++},$$

$$M_{++}^{++} = 2(H_{1,1}^1(p^3)H_{1,1}^1(p^4)),$$

$$M_{--}^{++} = 2(H_{-1,-1}^1(p^3)H_{-1,-1}^1(p^4)),$$

$$M_{00}^{++} = \frac{-2(E_0^2 + P_v^2)}{m_v^2} (H_{0,1}^1(p^3)H_{0,1}^1(p^4)).$$



# Contracting de Sitter and anti-de Sitter spaces into Poincare space (vector representation)

•The curvature of five-dimensional space-time is due to the vacuum energy density.

$$\Lambda \rightarrow 0$$

Four-dimensional flat space time

(Eliminating curvature  $\Rightarrow l_1 \rightarrow \infty, l_2 \rightarrow \infty$ )

Translation operator in de Sitter space-time

$$\Gamma = e^{i \eta_{\mu\nu} \frac{a^\mu}{l_1} \Gamma^\nu}$$

Translation operator in anti-de Sitter space-time

$$\Pi = e^{i \eta_{\mu\nu} \frac{a^\mu}{l_2} \Pi^\nu}$$

$$\begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} a^2 - (a^0)^2 \omega_1 & a^0 a^1 \omega_1 & a^0 a^2 \omega_1 & a^0 a^3 \omega_1 & a a^0 \omega_2 \\ -a^0 a^1 \omega_1 & a^2 + (a^1)^2 \omega_1 & a^1 a^2 \omega_1 & a^1 a^3 \omega_1 & a a^1 \omega_2 \\ -a^0 a^2 \omega_1 & a^1 a^2 \omega_1 & a^2 + (a^2)^2 \omega_1 & a^2 a^3 \omega_1 & a a^2 \omega_2 \\ -a^0 a^3 \omega_1 & a^1 a^3 \omega_1 & a^2 a^3 \omega_1 & a^2 + (a^3)^2 \omega_1 & a a^3 \omega_2 \\ a a^0 \omega_2 & -a a^1 \omega_2 & -a a^2 \omega_2 & -a a^3 \omega_2 & a^2 (1 - \omega_1) \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}$$

$$\begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} a^2 - (a^0)^2 \omega_3 & a^0 a^1 \omega_3 & a^0 a^2 \omega_3 & a^0 a^3 \omega_3 & a a^0 \omega_4 \\ -a^0 a^1 \omega_3 & a^2 + (a^1)^2 \omega_3 & a^1 a^2 \omega_3 & a^1 a^3 \omega_3 & a a^1 \omega_4 \\ -a^0 a^2 \omega_3 & a^1 a^2 \omega_3 & a^2 + (a^2)^2 \omega_3 & a^2 a^3 \omega_3 & a a^2 \omega_4 \\ -a^0 a^3 \omega_3 & a^1 a^3 \omega_3 & a^2 a^3 \omega_3 & a^2 + (a^3)^2 \omega_3 & a a^3 \omega_4 \\ -a a^0 \omega_4 & a a^1 \omega_4 & a a^2 \omega_4 & a a^3 \omega_4 & a^2 (1 - \omega_3) \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix}$$

$$(y^{\alpha'})_{l_1 \rightarrow \infty}$$

$$(z^{\alpha'})_{l_2 \rightarrow \infty}$$

Translation in the Poincare space

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix}$$

$$P = e^{i \eta_{\mu\nu} a^\mu P^\nu}$$

“On the contraction of groups and their representation”  
By E. Inonu and E.P Wigner (1958)

# De Sitter and anti-de Sitter operators in spinor representation

- Gamma matrices in chiral basis

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\gamma^0 \Rightarrow$  Time- like Hermitian matrix

$\gamma^i \Rightarrow$  Space- like anti-Hermitian matrices

W.A. Hepner. IL NUOVO CIMENTO VOL XXVI,N.2 (1962)

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

□ De Sitter Group ,  $S_0(4,1)$

$\gamma^4 = -i\gamma^5 \Rightarrow$  anti-Hermitian matrix

Matrix of operators

$$R^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 0 & \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^0\gamma^4 \\ \gamma^1\gamma^0 & 0 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^1\gamma^4 \\ \gamma^2\gamma^0 & \gamma^2\gamma^1 & 0 & \gamma^2\gamma^3 & \gamma^2\gamma^4 \\ \gamma^3\gamma^0 & \gamma^3\gamma^1 & \gamma^3\gamma^2 & 0 & \gamma^3\gamma^4 \\ \gamma^4\gamma^0 & \gamma^4\gamma^1 & \gamma^4\gamma^2 & \gamma^4\gamma^3 & 0 \end{pmatrix}$$

□ Anti-de Sitter Group,  $S_0(3,2)$

$\gamma^5 \Rightarrow$  Hermitian matrix

Matrix of operators

$$J^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Pi^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Pi^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Pi^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Pi^3 \\ \Pi^0 & \Pi^1 & \Pi^2 & \Pi^3 & 0 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 0 & \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^0\gamma^5 \\ \gamma^1\gamma^0 & 0 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^1\gamma^5 \\ \gamma^2\gamma^0 & \gamma^2\gamma^1 & 0 & \gamma^2\gamma^3 & \gamma^2\gamma^5 \\ \gamma^3\gamma^0 & \gamma^3\gamma^1 & \gamma^3\gamma^2 & 0 & \gamma^3\gamma^5 \\ \gamma^5\gamma^0 & \gamma^5\gamma^1 & \gamma^5\gamma^2 & \gamma^5\gamma^3 & 0 \end{pmatrix}$$

- Satisfy corresponding Lie algebra

Clifford algebra

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} I$$

$$\alpha, \beta = 0,1,2,3,4$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Clifford algebra

$$\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta} I$$

$$\alpha, \beta = 0,1,2,3,5$$

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

# Contracting de Sitter and anti-de Sitter spaces into Poincare space (matrix representation)

## De Sitter Space-time

- Coordinates  $\underline{y} = y^\alpha \gamma_\alpha$
- Space-time invariant  $Det[\underline{y}] = (l_1)^4$
- Translation operator in de sitter space-time  $\underline{\Gamma} = e^{i \eta_{\mu\nu} \frac{a^\mu}{l_1} \left( \frac{i\gamma^4 \gamma^\nu}{2} \right)}$
- Translated coordinates

$$\underline{y}' = \underline{\Gamma} \underline{y} \underline{\Gamma}^{-1}$$

$$\left( \underline{y}' \right)_{l_1 \rightarrow \infty}$$

Translated coordinates in Poincare space-time

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x^0 - x^3 + a^0 - a^3 & -x^1 + ix^2 - a^1 + ia^2 \\ -x^1 - ix^2 - a^1 - ia^2 & x^0 + x^3 + a^0 + a^3 \end{pmatrix}$$

## Anti-de Sitter Space-time

- Coordinates  $\underline{z} = z^\alpha \gamma_\alpha$
- Space-time invariant  $Det[\underline{z}] = (l_2)^4$
- Translation operator in anti-de sitter space-time  $\underline{\Pi} = e^{i \eta_{\mu\nu} \frac{a^\mu}{l_2} \left( \frac{i\gamma^5 \gamma^\nu}{2} \right)}$
- Translated coordinates

$$\underline{z}' = \underline{\Pi} \underline{z} \underline{\Pi}^{-1}$$

$$\left( \underline{z}' \right)_{l_2 \rightarrow \infty}$$

$$\begin{pmatrix} x^0 + x^3 + a^0 + a^3 & x^1 - ix^2 + a^1 - ia^2 \\ y^1 + iy^2 & y^0 - y^3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Contracting de Sitter and anti-de Sitter spaces into Poincare space

Five-dimensional de Sitter coordinates  
( $\Lambda > 0$ )

Group algebra  $SO(4,1)$   ${}^5C_2$

$$R^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Gamma^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Gamma^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Gamma^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Gamma^3 \\ \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^3 & 0 \end{pmatrix}$$

$$[\Gamma^\mu, \Gamma^\nu] = iR^{\mu\nu}$$

Five-dimensional anti-de Sitter coordinates  
( $\Lambda < 0$ )

Group algebra  $SO(3,2)$   ${}^5C_2$

$$J^{\alpha\beta} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 & -\Pi^0 \\ -K^1 & 0 & J^3 & -J^2 & -\Pi^1 \\ -K^2 & -J^3 & 0 & J^1 & -\Pi^2 \\ -K^3 & J^2 & -J^1 & 0 & -\Pi^3 \\ \Pi^0 & \Pi^1 & \Pi^2 & \Pi^3 & 0 \end{pmatrix}$$

$$[\Pi^\mu, \Pi^\nu] = -iJ^{\mu\nu}$$

Four-dimensional  
Poincare space coordinates  
Poincare group algebra  $ISO(3,1)$

${}^4C_2$

${}^4C_1$

- Six homogenous operators

- Four inhomogeneous operators

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}$$

Lie algebra

$$[M^{\mu\nu}, M^{\rho\lambda}] = i(g^{\nu\rho}M^{\mu\lambda} - g^{\nu\lambda}M^{\mu\rho} - g^{\mu\rho}M^{\nu\lambda} + g^{\mu\lambda}M^{\nu\rho})$$

$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$$

$$[P^\mu, P^\nu] = 0$$

$$\mu = \nu = \lambda = \rho = 0, 1, 2, 3$$

# Scaled Interpolating Coordinates

Complete Space-time invariant

$$s^2 = x^\mu x_\mu = x^{\hat{\mu}} x_{\hat{\mu}} = \left(\frac{x^{\hat{+}}}{\sqrt{c}}\right)^2 - (x^1)^2 - (x^2)^2 - \left(\frac{x^{\hat{-}}}{\sqrt{c}}\right)^2 = 2x^+ x^- - (x^1)^2 - (x^2)^2 = 2x^{\bar{+}} x^{\bar{-}} - (x^1)^2 - (x^2)^2$$

- Space-time matrix tensor in scaled interpolating dynamic

$$g^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Even though  $g^N$  is equal to the Makowski space-time matrix, this space-time matrix valid for any  $\delta$

- Space-time matrix tensor in new scaled light-front interpolating dynamic

$$g^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

# Scaled Interpolating Operators

- Infinitesimal transformation matrices of scaled interpolating operators in the scaled interpolating basis coincide with the Infinitesimal transformation operators of rotation and boost operators in the standard basis

$$\begin{aligned}
 H \cdot \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} \cdot H^{-1} &= K^1 & H \cdot \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} \cdot H^{-1} &= -J^2 & \frac{E^{\hat{1}}}{\sqrt{\mathbb{C}}} &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_3 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
 H \cdot \frac{E^{\hat{2}}}{\sqrt{\mathbb{C}}} \cdot H^{-1} &= K^2 & H \cdot \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} \cdot H^{-1} &= J^1 & & & & & & & \\
 H \cdot K^3 \cdot H^{-1} &= K^3 & H \cdot J^3 \cdot H^{-1} &= J^3 & \frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathbb{C}}} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, & \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathbb{C}}} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & J_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

- Even though scaled interpolating operators seems to depend on the interpolation angle , infinitesimal transformation matrices do not depend on the interpolation angle

- Satisfy same Lie algebra

$$[M^{\mu\nu}, M^{\rho\lambda}] = i(g^{\nu\rho}M^{\mu\lambda} - g^{\nu\lambda}M^{\mu\rho} - g^{\mu\rho}M^{\nu\lambda} + g^{\mu\lambda}M^{\nu\rho})$$

Poincare Matrix

$$M^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}$$

# Scaled Interpolating Operators

## Translation operators

Infinitesimal translation operators in scaled interpolating basis

$$\begin{pmatrix} \frac{x'^{\hat{+}}}{\sqrt{\mathcal{C}}} \\ x'^{\hat{1}} \\ x'^{\hat{2}} \\ \frac{x'_{\hat{-}}}{\sqrt{\mathcal{C}}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{a^{\hat{+}}}{\sqrt{\mathcal{C}}} \\ 0 & 1 & 0 & 0 & a^{\hat{1}} \\ 0 & 0 & 1 & 0 & a^{\hat{2}} \\ 0 & 0 & 0 & 1 & \frac{a_{\hat{-}}}{\sqrt{\mathcal{C}}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x^{\hat{+}}}{\sqrt{\mathcal{C}}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ \frac{x_{\hat{-}}}{\sqrt{\mathcal{C}}} \\ 1 \end{pmatrix}$$

$$H \cdot \frac{P^{\hat{+}}}{\sqrt{\mathcal{C}}} \cdot H^{-1} = P^0$$

$$H \cdot P^2 \cdot H^{-1} = P^2$$

$$H \cdot P^1 \cdot H^{-1} = P^1$$

$$H \cdot \frac{P_{\hat{-}}}{\sqrt{\mathcal{C}}} \cdot H^{-1} = P^3$$

Table 01: Kinematic and dynamic generators in scaled interpolating basis

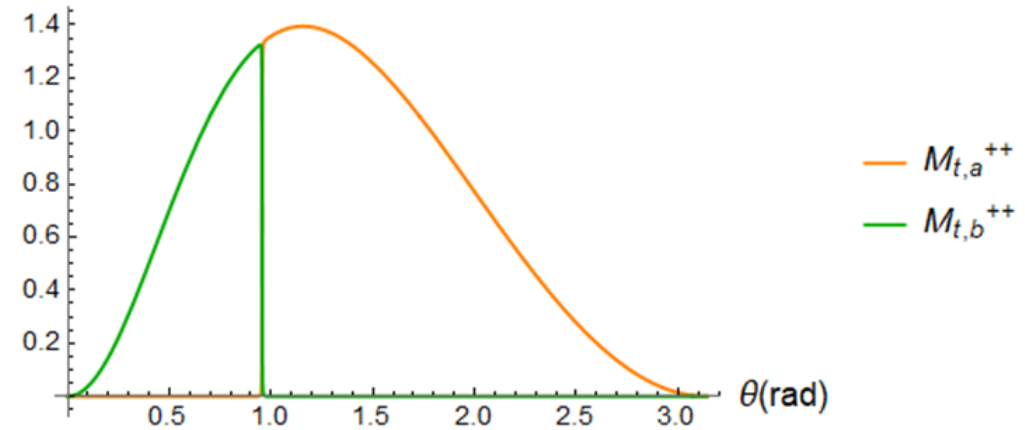
	Kinematic	Dynamic
$\delta = 0$	$\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathcal{C}}} = -J^2, \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathcal{C}}} = J^1, J^3, P^1, P^2, \frac{P_{\hat{-}}}{\sqrt{\mathcal{C}}} = P^3$	$\frac{E^{\hat{1}}}{\sqrt{\mathcal{C}}} = K^1, \frac{E^{\hat{2}}}{\sqrt{\mathcal{C}}} = K^2, K^3, \frac{P^{\hat{+}}}{\sqrt{\mathcal{C}}} = P^0$
$0 \leq \delta < \pi/4$	$\frac{\mathcal{K}^{\hat{1}}}{\sqrt{\mathcal{C}}}, \frac{\mathcal{K}^{\hat{2}}}{\sqrt{\mathcal{C}}}, J^3, P^1, P^2, \frac{P_{\hat{-}}}{\sqrt{\mathcal{C}}}$	$\frac{E^{\hat{1}}}{\sqrt{\mathcal{C}}}, \frac{E^{\hat{2}}}{\sqrt{\mathcal{C}}}, K^3, \frac{P^{\hat{+}}}{\sqrt{\mathcal{C}}}$

# LFD and Large Momentum Frame in the IFD

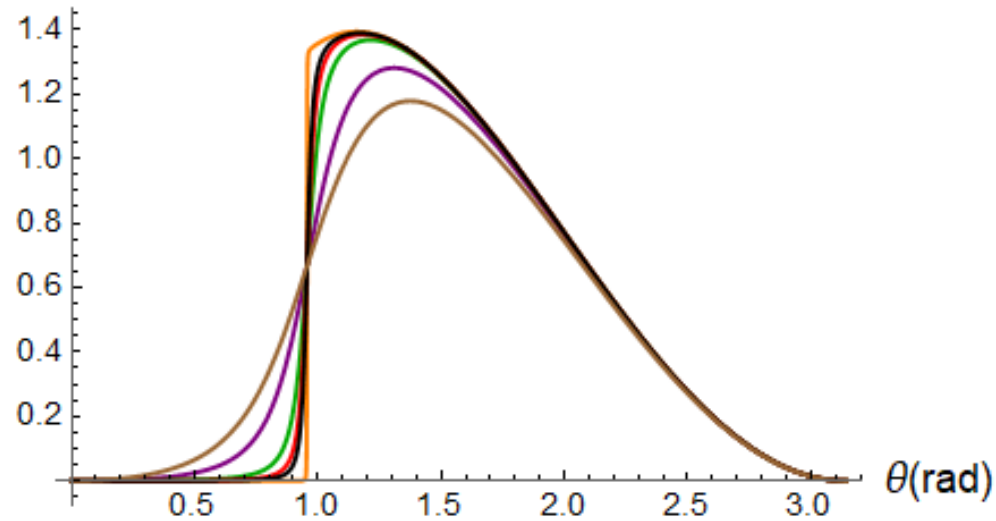
- Decomposition of the covariant scalar propagator exactly at the LF

Ex:  $M_t^{++} = M_{t,a}^{++} + M_{t,b}^{++}$

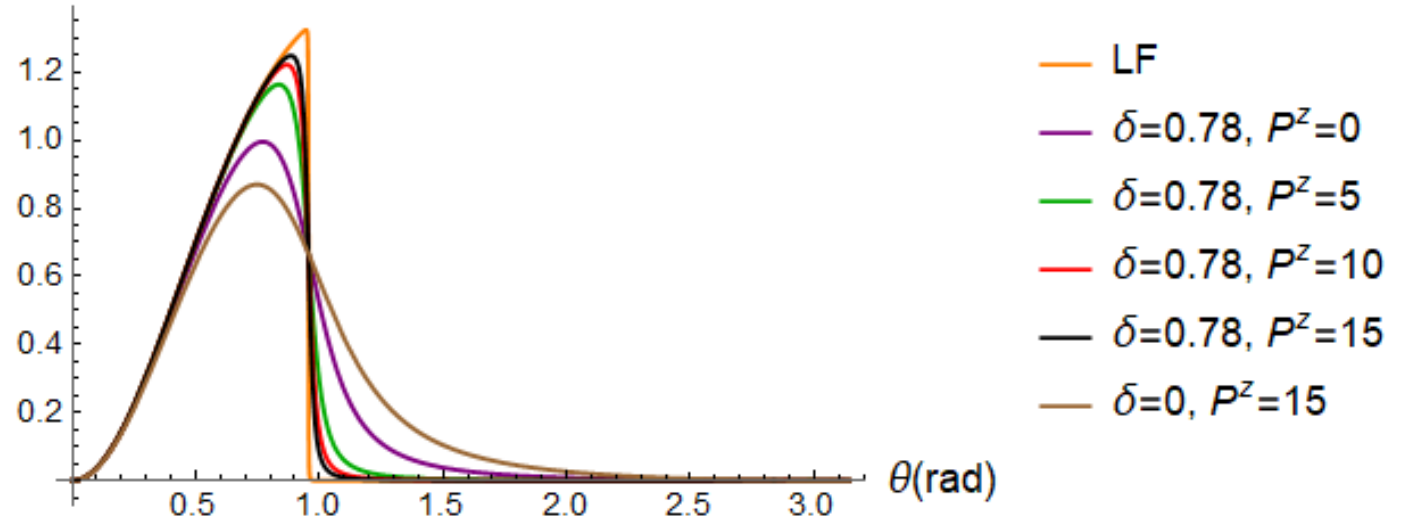
“Forward” and “Backward” helicity amplitudes



$M_{t,a}^{++}$



$M_{t,b}^{++}$



# Light-front Scaled Interpolating Operators

- Infinitesimal transformation matrix of scaled light-front interpolating operators in the scaled light-front interpolating basis

$$E^{\bar{1}} = \frac{E^1(\cos \delta + \sin \delta)}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad E^{\bar{2}} = \frac{E^2(\cos \delta + \sin \delta)}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \quad K^{\bar{3}} = K^3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$F^{\bar{1}} = \frac{F^1(\cos \delta - \sin \delta)}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F^{\bar{2}} = \frac{F^2(\cos \delta - \sin \delta)}{\sqrt{\mathbb{C}}} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J^{\bar{3}} = J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Light –front scaled interpolating operators seems to depend on the interpolation angle, but infinitesimal transformation matrices do not depend on the interpolation angle

- Satisfy same light-front Lie algebra

$$M^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} 0 & E^{\bar{1}} & E^{\bar{2}} & -K^3 \\ -E^{\bar{1}} & 0 & J^3 & -F^{\bar{1}} \\ -E^{\bar{2}} & -J^3 & 0 & -F^{\bar{1}} \\ K^3 & F^{\bar{1}} & F^{\bar{2}} & 0 \end{pmatrix}$$

# Light-front Scaled Interpolating Operators

Table 01: Kinematic and dynamic generators in scaled interpolating Light font basis

	Kinematic	Dynamic
$\delta = 0$	$E^{\bar{1}} = E^1, E^{\bar{2}} = E^2, K^3, J^3, P^1, P^2, P^{\bar{+}} = P^+$	$F^{\bar{1}} = F^1, F^{\bar{2}} = F^2, P^{\bar{-}} = P^-$
$0 \leq \delta < \pi/4$	$E^{\bar{1}}, E^{\bar{2}}, K^3, J^3, P^1, P^2, P^{\bar{+}}$	$F^{\bar{1}}, F^{\bar{2}}, P^{\bar{-}}$

- In the interpolating parameter space between  $0 \leq \delta < \frac{\pi}{4}$ 
  - Scaling light-front won't change the number of kinematic generators and dynamic generators in the light-front.
  - Even though basis and operators depend on the interpolation angle matrices are independent on interpolation angle, and they are form-invariant with the light-front form

## Remark

In original interpolation formalism,  $\delta = 0$  symbolizes the IFD and  $0 \leq \delta < \frac{\pi}{4}$  is the space-like region. Here we look at the same region from the perspective of light-front . When we do so ,the light-front coordinate appears to be scaled and  $\delta$  becomes scaling parameter.

# 5-dimensional Scale interpolating Anti-de Sitter space-time S0(3,2)

## Scaled interpolating operators

$$R^N = \begin{pmatrix} 0 & \frac{E^{\hat{1}}}{\sqrt{C}} & \frac{E^{\hat{2}}}{\sqrt{C}} & K^3 & -\frac{\Pi^{\hat{+}}}{\sqrt{C}} \\ -\frac{E^{\hat{1}}}{\sqrt{C}} & 0 & J^3 & \frac{\kappa^{\hat{1}}}{\sqrt{C}} & -\Pi^{\hat{1}} \\ -\frac{E^{\hat{2}}}{\sqrt{C}} & -J^3 & 0 & \frac{\kappa^{\hat{2}}}{\sqrt{C}} & -\Pi^{\hat{2}} \\ -K^3 & -\frac{\kappa^{\hat{1}}}{\sqrt{C}} & -\frac{\kappa^{\hat{2}}}{\sqrt{C}} & 0 & -\frac{\Pi_{\hat{-}}}{\sqrt{C}} \\ \frac{\Pi^{\hat{+}}}{\sqrt{C}} & \Pi^{\hat{1}} & \Pi^{\hat{2}} & \frac{\Pi_{\hat{-}}}{\sqrt{C}} & 0 \end{pmatrix}$$

Satisfy Anti-de sitter standard lie algebra

## Light-front scaled interpolating operators

$$J^{\bar{\alpha}\bar{\beta}} = \begin{pmatrix} 0 & E^{\bar{1}} & E^{\bar{2}} & -K^3 & -\Pi^{\bar{+}} \\ -E^{\bar{1}} & 0 & J^3 & -F^{\bar{1}} & -\Pi^{\bar{1}} \\ -E^{\bar{2}} & -J^3 & 0 & -F^{\bar{2}} & -\Pi^{\bar{2}} \\ K^3 & F^{\bar{1}} & F^{\bar{2}} & 0 & -\Pi^{\bar{-}} \\ \Pi^{\bar{+}} & \Pi^{\bar{1}} & \Pi^{\bar{2}} & \Pi^{\bar{-}} & 0 \end{pmatrix}$$

Satisfy Anti-de sitter light-front lie algebra

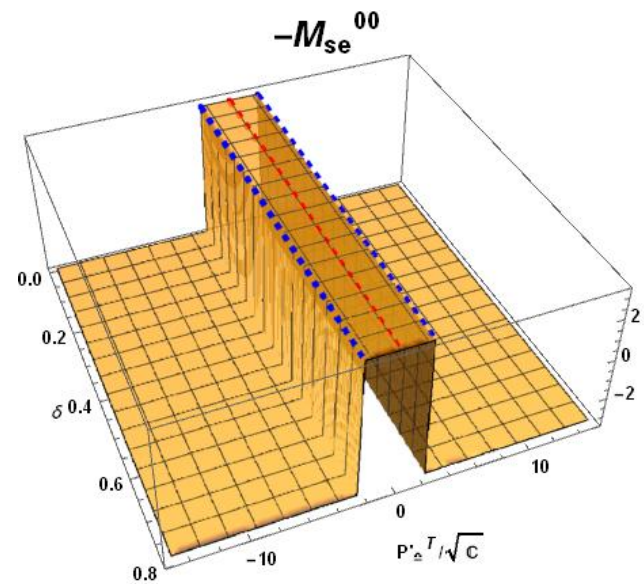
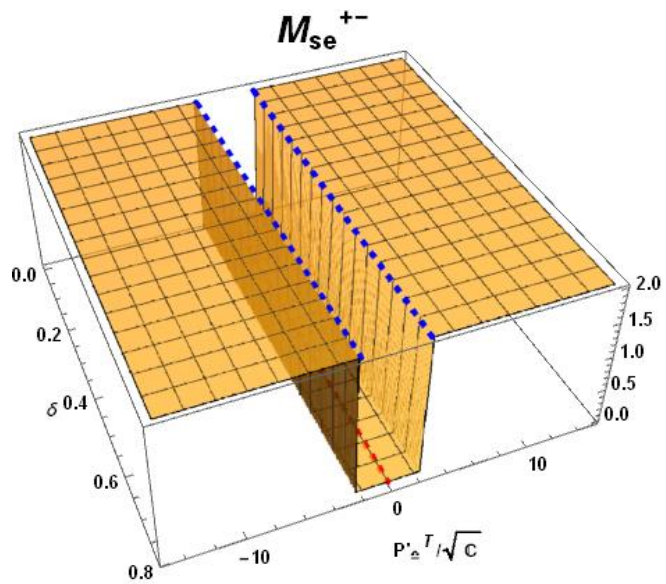
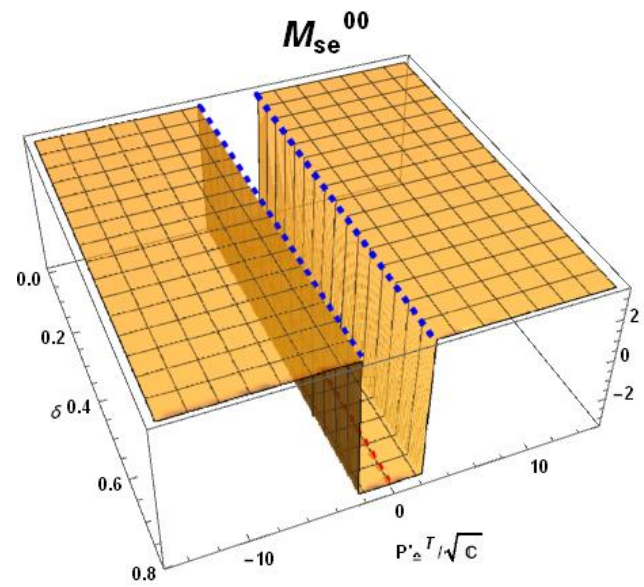
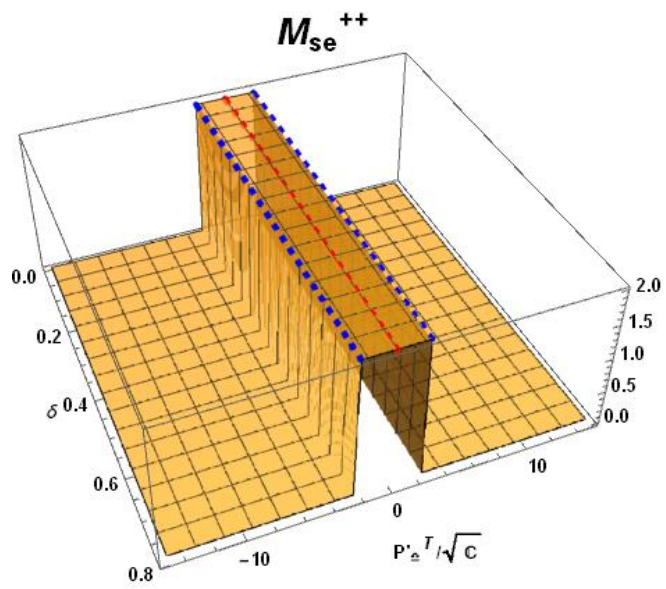
$$ds_5^2 = \text{Cosh}^2 \chi dT^2 - l^2 [d\chi^2 + \text{Sinh}^2 \chi (d\theta^2 + \text{Sin}^2 \theta d\phi^2)]$$

$$ds_4^2 = dT^2 - [dr^2 + r^2 (d\theta^2 + \text{Sin}^2 \theta d\phi^2)]$$

- Ten homogenous operators  ${}^5C_2$
- When vacuum energy density becomes zero
  - Reduction in number of degrees of freedom.
  - Operators  $\rightarrow$  6 homogenous operators + 4 inhomogeneous operators ISO (3,1)

$${}^4C_2$$

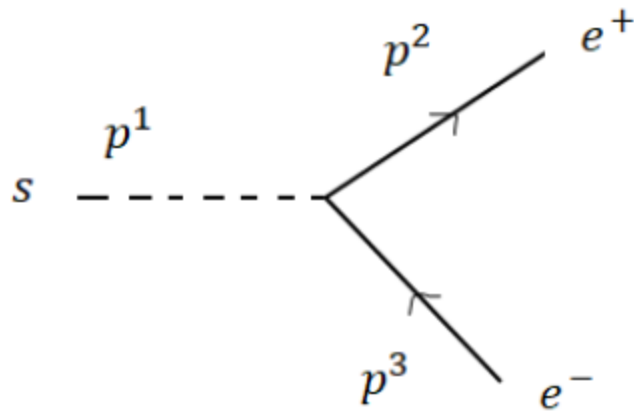
$${}^4C_1$$



$\frac{(P_{\pm}^T)^T}{\sqrt{C}} \longrightarrow$  Finite Value

## Scalar particle decays into a fermion and an anti-fermion

$$\phi \longrightarrow \psi + \bar{\psi}$$



Feynman digram

$$\begin{aligned}
 H_{\text{int}} &= g \int d^3x \phi(\vec{x}) \bar{\psi}(\vec{x}) \psi(\vec{x}) \\
 &= g \sum_{r,s} \int \frac{d^3\vec{k} d^3\vec{p} d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{2E_{\vec{k}} 2E_{\vec{p}} 2E_{\vec{q}}}} \times \\
 &\quad \left\{ a_{\vec{k}} b_{\vec{q}}^{r\dagger} b_{\vec{p}}^s \bar{u}^r(\vec{q}) u^s(\vec{p}) \delta^{(3)}(\vec{k} + \vec{p} - \vec{q}) \right. \\
 &\quad + a_{\vec{k}} b_{\vec{q}}^{r\dagger} c_{\vec{p}}^{s\dagger} \bar{u}^r(\vec{q}) v^s(\vec{p}) \delta^{(3)}(\vec{k} - \vec{p} - \vec{q}) \\
 &\quad + a_{\vec{k}} c_{\vec{q}}^r b_{\vec{p}}^s \bar{v}^r(\vec{q}) u^s(\vec{p}) \delta^{(3)}(\vec{k} + \vec{p} + \vec{q}) \\
 &\quad + a_{\vec{k}} c_{\vec{q}}^r c_{\vec{p}}^{s\dagger} \bar{v}^r(\vec{q}) v^s(\vec{p}) \delta^{(3)}(\vec{k} + \vec{q} - \vec{p}) \\
 &\quad + a_{\vec{k}}^\dagger b_{\vec{q}}^{r\dagger} b_{\vec{p}}^s \bar{u}^r(\vec{q}) u^s(\vec{p}) \delta^{(3)}(-\vec{k} + \vec{p} - \vec{q}) \\
 &\quad + a_{\vec{k}}^\dagger b_{\vec{q}}^{r\dagger} c_{\vec{p}}^{s\dagger} \bar{u}^r(\vec{q}) v^s(\vec{p}) \delta^{(3)}(-\vec{k} - \vec{p} - \vec{q}) \\
 &\quad + a_{\vec{k}}^\dagger c_{\vec{q}}^r b_{\vec{p}}^s \bar{v}^r(\vec{q}) u^s(\vec{p}) \delta^{(3)}(-\vec{k} + \vec{p} + \vec{q}) \\
 &\quad \left. + a_{\vec{k}}^\dagger c_{\vec{q}}^r c_{\vec{p}}^{s\dagger} \bar{v}^r(\vec{q}) v^s(\vec{p}) \delta^{(3)}(-\vec{k} + \vec{q} - \vec{p}) \right\}.
 \end{aligned}$$

# Decay Amplitude

Consider the relativistically normalized initial and final states for the scalar particle decays into a fermion and an anti-fermion

$$|i\rangle = \sqrt{2E_{\vec{k}}} a_{\vec{k}}^{\dagger} |0\rangle$$

$$|f\rangle = \sqrt{4E_{\vec{q}}E_{\vec{p}}} b_{\vec{q}}^{\dagger} c_{\vec{p}}^{\dagger} |0\rangle$$

$$H_{\text{decay}} = g \sum_{r,s} \int \frac{d^3p d^3q d^3k}{(2\pi)^6} \frac{1}{\sqrt{2E_{\vec{p}} 2E_{\vec{q}} 2E_{\vec{k}}}} \delta^{(3)}(\vec{k} - \vec{q} - \vec{p}) a_{\vec{k}} b_{\vec{q}}^{r\dagger} c_{\vec{p}}^{s\dagger} \bar{u}^r(\vec{q}) v^s(\vec{p}).$$

$$M^{\lambda_1, \lambda_2} = g \cdot \bar{u}(p^2, \lambda_1) v(p^3, \lambda_2).$$

“qumodes” (short for *quantum modes*) are the basic units of information in continuous-variable (CV) quantum computing, analogous to qubits in standard (discrete-variable) quantum computing.

# Lattice formulation makes QFT finite and discrete

- Continuum field theories have infinitely many degrees of freedom.
- The lattice discretization replaces space(-time) by a finite grid, turning the field into a finite set of variables (one per site).

This makes it possible to:

- Store the field values in a finite number of qubits
- Represent the Hamiltonian as a finite matrix acting on this quantum register.

**The lattice version of QFT is the bridge between continuum field theory and a quantum circuit implementable on hardware.**

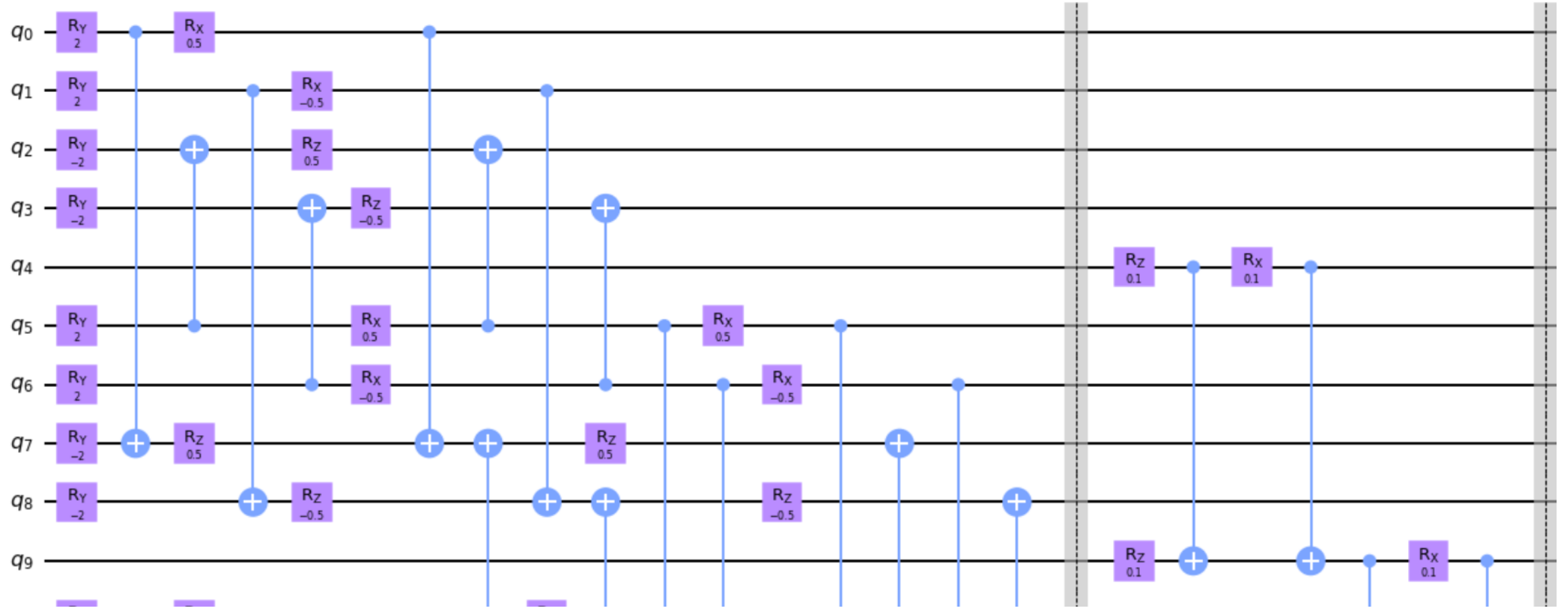
## Lattice formulation makes QFT finite and discrete

In quantum algorithms for simulating a Hamiltonian  $H = \sum_j H_j$ , we use Trotterization to approximate:

$$U(t) = e^{-iHt} \approx \left( \prod_j e^{-iH_j \Delta t} \right)^n, \quad \text{where } \Delta t = t/n$$

Each exponential  $e^{-iH_j \Delta t}$  becomes a **quantum gate sequence** that acts on a small number of qubits.

# Quantum circuit in progress



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